

EFFECTIVE VERY AMPLENESS FOR GENERALIZED THETA DIVISORS

EDUARDO ESTEVES AND MIHNEA POPA

Abstract

Given a smooth projective curve X , we give effective very ampleness bounds for generalized theta divisors on the moduli spaces $SU_X(r, d)$ and $U_X(r, d)$ of semistable vector bundles of rank r and degree d on X with fixed, respectively arbitrary, determinant.

1. Introduction

In this paper we address the problem of effective very ampleness for linear series on the moduli spaces of vector bundles of arbitrary rank on smooth projective curves. We improve and combine techniques used in [Es] and [Po1], and use the dimension estimate for Quot-schemes obtained in [PR]. The problem has been solved before only in the case of the determinant line bundle on moduli spaces of rank-2 vector bundles with fixed determinant, where optimal results have been proved in [BV1], [BV2] and [vGI].

Let X be a smooth projective complex curve of genus $g \geq 2$, and fix integers r and d , with $r > 0$. Let $U_X(r, d)$ denote the moduli space of equivalence classes of semistable vector bundles on X of rank r and degree d , and $SU_X(r, L)$ the moduli space of semistable rank- r vector bundles of fixed determinant L . The isomorphism class of $SU_X(r, L)$ depends only on the degree of L , say d , so we will often use the notation $SU_X(r, d)$ for $SU_X(r, L)$. It is well known (see [DN], Thm. B, p. 55) that $\text{Pic}(SU_X(r, d)) \cong \mathbf{Z}$, and the ample generator, henceforth denoted by \mathcal{L} , is called the *determinant line bundle*.

Theorem A.

For each $m \geq r^2 + r$ the linear series $|\mathcal{L}^m|$ on $SU_X(r, d)$ separates points, and is very ample on the smooth locus $SU_X^s(r, d)$. In particular, if r and d are coprime, then $|\mathcal{L}^m|$ is very ample.

EE was partially supported by CNPq, Processo 202151/90-5 and Processo 300004/95-8, PRONEX, Convênio 41/96/0883/00, and FAPERJ, Processo E-26/170.418/2000-APQ1. MP was partially supported by NSF Grant DMS-0200150.

It was already known that $|\mathcal{L}^m|$ is base-point-free for m in the range of the theorem. In fact, it is stated in [PR], Thm. 8.1, p. 661, and proved using the method developed in [Po1], that $|\mathcal{L}^m|$ is base-point-free for each $m \geq \lceil r^2/4 \rceil$, at least when $d = 0$; see also our Theorem 2.7. So the very ampleness bound is roughly four times larger. Note that these bounds do not depend on g — this is important in applications, as emphasized in [Po1], especially in Sec. 6.

The lack of knowledge about the tangent space to $SU_X(r, d)$ at a strictly semistable point prevents us from giving criteria for the separation of tangent vectors at strictly semistable points when r and d have a common factor.

Our approach consists of finding enough generalized theta divisors. These divisors arise from vector bundles on X as described below. Set $h := \gcd(r, d)$, $r_1 := r/h$ and $d_1 := d/h$. For each vector bundle F on X of rank mr_1 and degree $m(r_1(g-1) - d_1)$ there is an associated subscheme of $SU_X(r, d)$, parameterizing semistable bundles E such that $h^0(E \otimes F) \neq 0$. If the subscheme is not the whole $SU_X(r, d)$, then it is the support of a divisor, called a *generalized theta divisor*, and denoted Θ_F ; see [DN], Sec. 0.2, p. 55. The divisor Θ_F belongs to $|\mathcal{L}^m|$. If $m = 1$, the divisor Θ_F is called *basic*.

It was recognized in [Fa] that these divisors can be used to produce sections of \mathcal{L}^m which do not vanish at any given point, for m sufficiently large. Later, they were also used in [Es] to separate points and tangent vectors, again for m large. However, there is little or no emphasis on effectiveness in these papers.

Since the generalized theta divisors can be defined by the existence of sections or, equivalently, maps between vector bundles, the most efficient way for producing effective results seems to be a counting technique based on dimension estimates for appropriate Quot-schemes. The result we need, Lemma 2.1, was proved in [PR], improving on a result of [Po1].

On $U_X(r, d)$ the situation is slightly different. As above, we can define analogous divisors Θ_F , but this time they will only move in the same linear series if their determinant, $\det F$, is kept fixed. Nevertheless, the same bound is obtained.

Theorem B.

Let Θ be a basic generalized theta divisor on $U_X(r, d)$. For each $m \geq r^2 + r$, the linear series $|m\Theta|$ on $U_X(r, d)$ separates points, and is very ample on the smooth locus $U_X^s(r, d)$. In particular, if r and d are coprime, then $|m\Theta|$ is very ample.

Theorem B is proved using Theorem A, but is not an immediate corollary. Also, methods analogous to those used in the proof of Theorem A would not yield *the same* bound on m ; see the beginning of Section 4. We emphasize this since we have encountered a quite widespread, but unfounded opinion that such results on $U_X(r, d)$ should follow immediately from those on $SU_X(r, d)$ by using the fact that

$U_X(r, d)$ admits an étale cover from $SU_X(r, d) \times J(X)$, where $J(X)$ is the Jacobian of X .

To prove Theorem B we use, as in [Po2], the Verlinde bundles on $J(X)$. This time however we need a more refined cohomological criterion for the global generation of arbitrary (as opposed to locally free) coherent sheaves on an abelian variety. This result has been proved recently in [PP] via techniques relying on the Fourier–Mukai transform.

Finally, combining this technique with the sharp results on rank-2 bundles with fixed determinant mentioned in the first paragraph, we obtain better, and presumably optimal, results.

Theorem C.

Let Θ be a basic generalized theta divisor on $U_X(2, d)$. If d is odd, then $|3\Theta|$ is very ample. If d is even, and C is not hyperelliptic, then $|5\Theta|$ is very ample.

The paper is organized as follows. In Section 2 we prove a very general existence result, Lemma 2.3, using counting arguments. In Section 3 we discuss the problem of separating effectively points, Theorem 3.2, and tangent vectors, Theorem 3.4, and prove Theorem A. Finally, in Section 4 we use Verlinde bundles and global generation results for coherent sheaves on abelian varieties to show Theorems B and C.

Acknowledgments. The first author would like to thank MIT for its hospitality while this work was initiated. The second author would like to thank R. Lazarsfeld and M. Roth for general discussions related to this topic, and IMPA for its hospitality and support during part of the work on the paper.

2. The Existence Lemma

The key computational tool used in the proof of Lemma 2.3, to which this section is devoted, is the main result of [PR], stated below. First, a piece of notation: for each bundle E on X , and each $k = 0, \dots, r$, let $d_k(E)$ be the minimum degree of a rank- k quotient of E .

Lemma 2.1 ([PR], Thm. 4.1, p. 637).

Let E be a vector bundle of rank r and degree d . Let $\text{Quot}_{k,e}(E)$ be the scheme of quotients of E of rank k and degree e . Then

$$\dim \text{Quot}_{k,e}(E) \leq k(r - k) + (e - d_k)r \text{ for each } e \geq d_k, \text{ where } d_k := d_k(E).$$

Remark 2.2.

At first sight, the appearance of d_k in the bound for $\dim \text{Quot}_{k,e}(E)$ might seem undesirable, as it is not clear how d_k depends on E and k . However, we will see that it is the very presence of d_k that allows for the proof of Lemma 2.3.

It is convenient to look at $d_k(E)$ also from a different perspective. For each k with $0 \leq k \leq r$, put $s_k(E) := rd_{r-k}(E) - (r-k)d$, where r and d are the rank and degree of E . (An equivalent and more usual definition is $s_k(E) := kd - rf_k(E)$, where $f_k(E)$ is the maximum degree of a rank- k subbundle of E ; see [La], p. 450. Also, the usual definition does not contemplate the cases $k = 0$ and $k = r$.) Note that $\gcd(r, d)$ divides $s_k(E)$. Also, E is semistable if and only if $s_k(E) \geq 0$ for every k .

Lemma 2.3.

Let E be a vector bundle of rank r and degree d . Fix $\epsilon \in \mathbf{Z}$. Let $m \in \mathbf{Z}$ such that

$$m \geq \max(h, [(r + \epsilon)^2/4]).$$

Let V be the general stable bundle of rank mr_1 and degree $m(r_1(g-1) - d_1) + \epsilon$. If k is the rank of a map $V^* \rightarrow E$, then $s_k(E) \leq 0$, with equality only if $k = 0$ or $k < r + \epsilon$.

(Note that, if $\epsilon \leq 0$, the lemma says that there are no maps $V^* \rightarrow E$ of rank r .)

Proof.

We will count moduli. Let

$$U := U_X^s(mr_1, mr_1(g-1) - md_1 + \epsilon).$$

Recall that $\dim U = (mr_1)^2(g-1) + 1$. For each integers k and ℓ , with $k > 0$, put: $U_{k,\ell} := \{[V] \in U \mid \text{there is a map } \psi : V^* \rightarrow E \text{ with image of rank } k \text{ and degree } \ell\}$.

The lemma will follow once we prove that

$$\dim U_{k,\ell} \leq (mr_1)^2(g-1) \text{ if } s_k(E) > 0, \text{ or if } s_k(E) = 0 \text{ and } k \geq r + \epsilon.$$

So fix k and ℓ , with $k > 0$. Let V run among the stable bundles of rank mr_1 and degree $mr_1(g-1) - md_1 + \epsilon$, and $\psi : V^* \rightarrow E$ among the maps whose image has rank k and degree ℓ . Let $F := \text{Im}(\psi)$ and $G := \text{Ker}(\psi)$. In other terms, we have the following diagram of maps, where the horizontal sequence is exact, the vertical map is an inclusion, and the maps in the triangle commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & G & \longrightarrow & V^* & \longrightarrow & F & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \psi & & \\ & & & & & & \downarrow & & \\ & & & & & & E & & \end{array}$$

The bundles F , or rather the quotients E/F , are parameterized by the scheme Q of quotients of E of degree $d - \ell$ and rank $r - k$. By Lemma 2.1, letting $d_{r-k} := d_{r-k}(E)$,

$$\dim Q \leq k(r-k) + (d - \ell - d_{r-k})r.$$

The bundles G are subbundles of fixed rank, equal to $mr_1 - k$, and fixed degree of the stable bundles V^* . So they are parameterized by a scheme T — a subset of a relative Quot-scheme over an étale cover of U . For counting purposes, as is probably well known, and observed in [BGN], Rmk. 4.2(i), p. 656, we may assume that $\dim T$ is at most the dimension of the moduli space of stable bundles of same rank, $mr_1 - k$. So

$$\dim T \leq (mr_1 - k)^2(g - 1) + 1.$$

The bundles V^* are extensions of the bundles F by the bundles G . These extensions are parameterized by the projectivization of $H^1(F^* \otimes G)$. Not all extensions of F by G are stable. In fact, none will be unless $\text{Hom}(F, G) = 0$, as a stable bundle admits only the trivial endomorphisms. If $\text{Hom}(F, G) = 0$ then, by the Riemann–Roch Theorem,

$$h^1(F^* \otimes G) = -\chi(F^* \otimes G) = -(mr_1 - k)\chi(F^*) - k \deg G = p + 1,$$

where

$$p := (mr_1 - k)(\ell + k(g - 1)) + k(mr_1(g - 1) - md_1 + \epsilon + \ell) - 1.$$

Let A be the open subset of $Q \times T$ parameterizing the pairs (F, G) satisfying $h^0(F^* \otimes G) = 0$. Then the bundles V^* are parameterized by a projective bundle P over A of relative dimension p .

Clearly, $\dim U_{k,\ell} \leq \dim P$. Now,

$$\begin{aligned} \dim P &= \dim(P/A) + \dim Q + \dim T \\ &\leq (mr_1 - k)(\ell + k(g - 1)) + k(mr_1(g - 1) - md_1 + \epsilon + \ell) - 1 \\ &\quad + k(r - k) + (d - \ell - d_{r-k})r + (mr_1 - k)^2(g - 1) + 1 \\ &= (mr_1)^2(g - 1) + k(r + \epsilon - k) + mr_1\ell - kmd_1 + (d - \ell - d_{r-k})r. \end{aligned}$$

Using $s_k(E) = kd - r(d - d_{r-k})$, we have

$$mr_1\ell - kmd_1 + (d - \ell - d_{r-k})r = -r_1(d - \ell - d_{r-k})(m - h) - ms_k(E)/h.$$

This gives

$$(2.3.1) \quad \begin{aligned} \dim U_{k,\ell} &\leq (mr_1)^2(g - 1) + k(r + \epsilon - k) \\ &\quad - r_1(d - \ell - d_{r-k})(m - h) - ms_k(E)/h. \end{aligned}$$

Now, $d - \ell \geq d_{r-k}$, since $d - \ell$ is the degree of the quotient E/F , which has rank $r - k$. Since $m \geq h$, we get

$$\dim U_{k,\ell} \leq (mr_1)^2(g - 1) + k(r + \epsilon - k) - ms_k(E)/h.$$

If $s_k(E) > 0$ then $s_k(E) \geq h$, and hence $k(r + \epsilon - k) - ms_k(E)/h \leq 0$ from the hypothesis on m . The same holds if $s_k(E) = 0$ and $k \geq r + \epsilon$. In either case, $\dim U_{k,\ell} \leq (mr_1)^2(g - 1)$, as asserted. \square

Remark 2.4.

As it can be seen from the last three lines in the above proof, it is enough to assume that $m \geq h$ and $m \geq hi(r + \epsilon - i)/s_i(E)$ for every i such that $s_i(E) > 0$ for the conclusion of Lemma 2.3 to hold. In addition, if E is semistable, and if $m \geq h + i(r + \epsilon - i)/r_1$ for every i such that $s_i(E) = 0$, then the image of each map $V^* \rightarrow E$ is semistable of slope d/r . Indeed, let $\phi: V^* \rightarrow E$ be a map, say of rank k . Then $s_k(E) \geq 0$ because E is semistable, and $s_k(E) \leq 0$ by Lemma 2.3. So $s_k(E) = 0$. Assume, by contradiction, that $F := \text{Im}(\phi)$ is not semistable of slope d/r . Since $s_k(E) = 0$ and E is semistable, the degree ℓ of F satisfies $d - \ell > d_{r-k}$. Now, $m \geq h + k(r + \epsilon - k)/r_1$ by hypothesis, since $s_k(E) = 0$. So Formula (2.3.1) yields $\dim U_{k,\ell} \leq (mr_1)^2(g - 1)$. Since $[V] \in U_{k,\ell}$, the bundle V is not general, a contradiction.

Remark 2.5.

The upper bound for $\dim U_{k,\ell}$ in Formula (2.3.1) may not be sharp in particular cases, as we are using a bound on the dimension of the space of quotients, rather than on the number of moduli. However, it does not seem possible to improve the bound in a uniform way; see [PR], §8.

Proposition 2.6.

Let E be a stable bundle of rank r and degree d . If $m \geq \max(\lceil r^2/4 \rceil, h)$, the general stable bundle V of rank mr_1 and degree $m(r_1(g - 1) - d_1)$ satisfies $H^0(V \otimes E) = 0$.

Proof.

Let k be the rank of a map $V^* \rightarrow E$. Apply Lemma 2.3 with $\epsilon := 0$. Then $s_k(E) \leq 0$, with equality only if $k < r$. However, since E is stable, $s_i(E) > 0$ for each i with $0 < i < r$, and $s_r(E) = 0$. So $k = 0$, whence $H^0(V \otimes E) = 0$. \square

Note that $\lceil r^2/4 \rceil \geq r \geq h$ if $r \geq 4$.

Theorem 2.7 (cf. [PR], Thm. 8.1, p. 661).

The linear series $|\mathcal{L}^m|$ on $SU_X(r, d)$ has no base points for each $m \geq \max(h, \lceil r^2/4 \rceil)$.

Proof.

Let E be a direct sum of stable bundles E_i corresponding to a point $[E]$ of $SU_X(r, d)$. Let k_i and e_i be the rank and degree of E_i , and set $t_i := \gcd(k_i, e_i)$. Since $e_i/k_i = d/r$, we have $r_1 = k_i/t_i$ and $d_1 = e_i/t_i$. Also, $t_i \leq h$.

Apply Proposition 2.6 to each E_i . Since $m \geq \max(t_i, \lceil k_i^2/4 \rceil)$, the general stable bundle V of rank mr_1 and degree $m(r_1(g - 1) - d_1)$ satisfies $H^0(V \otimes E_i) = 0$. So $H^0(V \otimes E) = 0$, and thus Θ_V is a divisor of $|\mathcal{L}^m|$ that does not contain $[E]$ in its support. \square

3. Very ampleness on $SU_X(r, d)$

The existence of generalized theta divisors gives a very simple sufficient criterion for a pluritheta linear series to separate points.

Lemma 3.1.

Let E_1 and E_2 be semistable bundles representing distinct points of $SU_X(r, d)$. If there exists a bundle F of rank mr_1 and degree $m(r_1(g-1) - d_1)$ such that $h^0(E_1 \otimes F) \neq 0$ and $h^0(E_2 \otimes F) = 0$, then the linear series $|\mathcal{L}^m|$ separates the points corresponding to E_1 and E_2 .

Proof.

The statement simply says that Θ_F is a divisor in $|\mathcal{L}^m|$ passing through the point corresponding to E_1 , but not through that corresponding to E_2 . \square

For our purposes, an *elementary transformation* of a vector bundle E at a point P of X is the vector bundle E' sitting in an exact sequence,

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \mathbf{C}_P \longrightarrow 0,$$

where \mathbf{C}_P is the skyscraper sheaf at P . The map $E \rightarrow \mathbf{C}_P$ corresponds to a linear functional on $E(P)$, and E' depends only on the hyperplane of zeros of this functional. We will use elementary transformations to distinguish between maps from a given bundle to E .

Theorem 3.2.

The linear series $|\mathcal{L}^m|$ on $SU_X(r, d)$ separates points for each $m \geq r^2 + r$.

Proof.

Let G_1 and G_2 be two semistable bundles corresponding to distinct points of $SU_X(r, d)$. We may assume G_1 and G_2 are direct sums of stable bundles. If they have a common stable summand K , say $G_1 = H_1 \oplus K$ and $G_2 = H_2 \oplus K$, then our problem is reduced to that of separating the points corresponding to H_1 and H_2 in a moduli space of bundles of smaller rank, as done in the proof of [Es], Thm. 16, p. 590. So we may assume G_1 and G_2 have no common stable summand. In particular, there is a stable summand E_0 of G_1 distinct from the stable summands E_1, \dots, E_n of G_2 . To apply Lemma 3.1 we need only to find a bundle F of rank mr_1 and degree $m(r_1(g-1) - d_1)$ such that $h^0(E_0 \otimes F) \neq 0$ but $h^0(E_i \otimes F) = 0$ for each $i = 1, \dots, n$.

For each i , let k_i denote the rank of E_i . We first claim that, for m as in the statement of the theorem, the general stable bundle V of rank mr_1 and degree $m(r_1(g-1) - d_1) + 1$ satisfies the following two properties:

- (1) For each $i = 0, \dots, n$, we have $h^1(E_i \otimes V) = 0$ and $h^0(E_i \otimes V) = k_i$.

- (2) For each $i = 1, \dots, n$, and each nonzero maps $\phi: E_0^* \rightarrow V$ and $\psi: E_i^* \rightarrow V$, the sum $(\phi, \psi): E_0^* \oplus E_i^* \rightarrow V$ is injective with saturated image.

Indeed, let us check property 1 first. The bundle $W := V^* \otimes \omega_C$ is the general bundle of rank mr_1 and degree $m(r_1(g-1) + d_1) - 1$. Let k be the rank of a map $W^* \rightarrow E_i^*$. Apply Lemma 2.3 with $\epsilon := -1$. Then $s_k(E_i) \leq 0$, with equality only if $k < k_i - 1$. Since E_i is stable, $k = 0$. Thus $h^0(W \otimes E_i^*) = 0$. By Serre Duality, $h^1(V \otimes E_i) = 0$. Riemann–Roch yields now $h^0(E_i \otimes V) = k_i$.

As for property 2, apply Lemma 2.3 with $\epsilon := 1$ to E_0 , E_i and $E_0 \oplus E_i$. Then ϕ^* has rank k satisfying $s_k(E_0) \leq 0$. Since $\phi \neq 0$ and E_0 is stable, ϕ^* is surjective. Likewise, ψ^* is surjective. The same analysis shows that the rank of $(\phi, \psi)^*$ is k_0 , k_i , or $k_0 + k_i$. Now, the image of $(\phi, \psi)^*$ surjects onto E_0 and E_i , because ϕ^* and ψ^* are surjective. So, if the rank of $(\phi, \psi)^*$ were not $k_0 + k_i$, we would obtain an isomorphism between E_0 and E_i . Hence $(\phi, \psi)^*$ is surjective, and thus (ϕ, ψ) is injective with saturated image.

Fix a nonzero map $\phi: E_0^* \rightarrow V$ and set $Q := \text{Coker}(\phi)$. By property 2 above, Q is a vector bundle. In addition, for each nonzero map $\psi: E_i^* \rightarrow V$ the composition $\tilde{\psi}: E_i^* \rightarrow V \rightarrow Q$ is injective with saturated image. Fix any point P of X . Then the image of $\tilde{\psi}(P)$ is a vector subspace $L_{\psi,i}$ of dimension k_i of $Q(P)$. Letting ψ vary, the $L_{\psi,i}$ move in a subvariety M_i of the Grassmannian $\text{Grass}(k_i, Q(P))$. Note that $\dim M_i \leq k_i - 1$ because $h^0(E_i \otimes V) = k_i$ from property 1 above. Let H be a hyperplane of $Q(P)$, and let Z_i be the closed subset of $\text{Grass}(k_i, Q(P))$ parameterizing subspaces contained in H . The codimension of Z_i is k_i . Choosing H general enough, by [Kl], Cor. 4, (i), p. 291, the intersection $M_i \cap Z_i$ is empty for each $i = 1, \dots, n$.

Let H' be the inverse image of H in $V(P)$, and F the elementary transformation of V at P corresponding to the hyperplane H' . Since H' contains the image of $\phi(P)$, the map ϕ factors through F , whence $h^0(E_0 \otimes F) \neq 0$. On the other hand, H' does not contain the image of $\psi(P)$ for any nonzero map $\psi: E_i^* \rightarrow V$. So $h^0(E_i \otimes F) = 0$ for each $i = 1, \dots, n$. Now, F has rank mr_1 and degree $m(r_1(g-1) - d_1)$, so, by Lemma 3.1, $|\mathcal{L}^m|$ separates the points corresponding to G_1 and G_2 . \square

Let S be the spectrum of the algebra of dual numbers $A := \mathbf{C}[\epsilon]/(\epsilon^2)$. If E is a stable bundle, representing a point $[E]$ of $SU_X(r, d)$, the tangent space to $SU_X(r, d)$ at $[E]$ can be identified with the space of deformations of E over S with constant determinant.

Lemma 3.3.

Let E be a stable bundle representing a point $[E]$ of $SU_X(r, d)$. The linear series $|\mathcal{L}^m|$ separates tangent vectors at $[E]$ if, for each nontrivial deformation \mathcal{E} of E

over S , there exists a bundle F of rank mr_1 and degree $m(r_1(g-1) - d_1)$ such that $h^0(E \otimes F) = 1$ and the unique (modulo \mathbf{C}) section of $E \otimes F$ does not extend over S to a section of $\mathcal{E} \otimes F$.

Proof.

Let $G := \omega_C \otimes F^*$. Let $\iota: SU_X(r, d) \rightarrow SU_X(r, -d)$ be the involution taking a vector bundle to its dual. By Serre Duality, $\iota^*\Theta_G = \Theta_F$. Let $f: S \rightarrow SU_X(r, d)$ be the map induced by \mathcal{E} . Then ιf is induced by \mathcal{E}^* .

By hypothesis, $h^0(E \otimes F) = 1$; so $h^1(E^* \otimes G) = 1$ by Serre Duality. By Riemann–Roch, also $h^0(E^* \otimes G) = 1$. From the hypothesis, the unique (modulo \mathbf{C}) nonzero map $E^* \rightarrow G^* \otimes \omega_C$ does not extend over S to a map $\mathcal{E}^* \rightarrow G^* \otimes \omega_C \otimes \mathcal{O}_S$. By [Es], Lemma 11, p. 582, $(\iota f)^*\Theta_G$ is reduced. Then $f^*\Theta_F$ is reduced as well. So Θ_F is a divisor in $|\mathcal{L}^m|$ containing $[E]$, whose tangent space at $[E]$ does not contain the tangent vector corresponding to \mathcal{E} . \square

Theorem 3.4.

The linear series $|\mathcal{L}^m|$ on $SU_X(r, d)$ separates tangent vectors at stable points for each $m \geq r^2 + r$.

Proof.

Let E be a stable bundle of rank r and degree d . Let \mathcal{E} be a nontrivial deformation of E over S . We need only to find a bundle F satisfying the conditions set forth in Lemma 3.3.

Let V be the general stable bundle of degree $m(r_1(g-1) - d_1) + 1$ and rank mr_1 . Let $\lambda, \mu \in \text{Hom}(V^*, E) - \{0\}$. By Lemma 2.3, since E is stable, both λ and μ are surjective. Consider the sum $(\lambda, \mu): V^* \rightarrow E \oplus E$. Again by Lemma 2.3, and the stability of E , either (λ, μ) is surjective or it has rank r . In the latter case, the image of (λ, μ) projects isomorphically onto both factors, yielding an automorphism τ of E such that $\lambda = \tau\mu$. By stability, $\text{Aut}(E) = \mathbf{C}^*$. Hence, either (λ, μ) is surjective or λ is a multiple of μ .

As in the proof of Theorem 3.2, we also have $h^1(E \otimes V) = 0$ and $h^0(E \otimes V) = r$.

Fix now a nonzero map $\phi: E^* \rightarrow V$. Since ϕ^* is surjective, ϕ is injective with saturated image. Let $\pi: V \rightarrow W$ be the quotient map to its cokernel. Since $h^1(E \otimes V) = 0$, the map ϕ extends over S to an injection $\mu: \mathcal{E}^* \rightarrow V \otimes \mathcal{O}_S$ with flat cokernel. The injection μ corresponds to a tangent vector at $[\pi]$ of the Quot-scheme of V , whence to a map $\nu: E^* \rightarrow W$. All of the extensions of ϕ over S correspond to maps of the form $\nu + \pi\psi$ for $\psi \in \text{Hom}(E^*, V)$. All of these maps are nonzero because \mathcal{E} is nontrivial.

For each nonzero map $\psi: E^* \rightarrow V$ consider the dual map $(\phi, \psi)^*: V^* \rightarrow E \oplus E$. If ψ is not a multiple of ϕ then $(\phi, \psi)^*$ is surjective, and thus $(\phi, \psi): E^* \oplus E^* \rightarrow V$ is injective with saturated image. So either $\pi\psi: E^* \rightarrow W$ is zero, when ψ is a

multiple of ϕ , or injective with saturated image. In particular, the vector subspace $N \subseteq \text{Hom}(E^*, W)$ generated by $\pi\psi$ for $\psi \in \text{Hom}(E^*, V)$ has dimension $r - 1$. Note also that $\nu \notin N$.

Let $\lambda: N \otimes \mathcal{O}_X \rightarrow \text{Hom}(E^*, W)$ be the natural map. Since each nonzero map in N is injective with saturated image, λ is also injective with saturated image. Thus, if $\nu(P)$ belonged to $\text{Im}(\lambda(P))$ for each $P \in X$, the section of $\text{Hom}(E^*, W)$ corresponding to ν would factor through λ . We would get $\nu \in N$, a contradiction.

So pick a point P of X such that $\nu(P) \notin \text{Im}(\lambda(P))$. Let $A := \text{Im}(\lambda(P))$ and $B \subseteq \text{Hom}(E^*(P), W(P))$ the subspace generated by A and $\nu(P)$. Consider the associated projective spaces $\mathbf{P}(A)$ and $\mathbf{P}(B)$ of one-dimensional subspaces of A and B . Then $\mathbf{P}(B)$ has dimension $r - 1$ and $\mathbf{P}(A)$ is a hyperplane in $\mathbf{P}(B)$. For each $Q \in \mathbf{P}(B)$ denote by L_Q the image of the unique (modulo \mathbf{C}^*) map $E^*(P) \rightarrow W(P)$ corresponding to Q . Let $U \subseteq \mathbf{P}(B)$ be the open subset of points Q such that L_Q has maximal dimension r . Then $U \supseteq \mathbf{P}(A)$, and hence the complement of U in $\mathbf{P}(B)$ is a finite set T . For each $Q \in T$ the image L_Q does not have maximal dimension, but is nonzero because $\nu(P) \notin A$.

Let H be a general hyperplane of $W(P)$. So H does not contain any of the finitely many subspaces L_Q for $Q \in T$. Also, let S be the closed subset of $\text{Grass}(r, W(P))$ parameterizing subspaces contained in H . The codimension of S is r . So, by [Kl], Cor. 4, (i), p. 291, since H is general, the intersection of S and $\{[L_Q] \mid Q \in U\}$ is empty. The upshot is that H does not contain L_Q for any $Q \in \mathbf{P}(B)$.

Let G be the elementary transformation of W at P corresponding to the hyperplane H , and F its inverse image under $\pi: V \rightarrow W$. Then F has rank mr_1 and degree $m(r_1(g-1) - d_1)$. In addition, ϕ factors through an injection $\phi': E^* \rightarrow F$ with quotient map $\pi': F \rightarrow G$ induced by π . However, those maps $\psi: E^* \rightarrow V$ that are not multiples of ϕ do not factor through F , since H does not contain the image of $\pi\psi(P)$ for any such ψ . Hence $h^0(E \otimes F) = 1$, and every map $E^* \rightarrow F$ is a multiple of ϕ' .

Finally, ϕ' does not extend over S to a map $\mathcal{E}^* \rightarrow F \otimes \mathcal{O}_S$. In fact, if ϕ' had such an extension, coupling it with the inclusion $i: F \rightarrow V$, we would get an extension $\mathcal{E}^* \rightarrow V \otimes \mathcal{O}_S$ of ϕ over S . So there would be a map $\nu': E^* \rightarrow G$, corresponding to the extension of ϕ' , whose composition with the inclusion $G \subseteq W$ is of the form $\nu + \pi\psi$ for a certain $\psi: E^* \rightarrow V$. Then $\nu(P) + \pi\psi(P)$ would factor through H , reaching a contradiction with our choice of H . \square

Proof. (of Theorem A)

By Theorem 3.2, the complete linear series $|\mathcal{L}^m|$ corresponds to an injective map $\phi: SU_X(r, d) \rightarrow \mathbf{P}^N$. Since ϕ is injective and proper, $\phi(SU_X^s(r, d))$ is open in $\phi(SU_X(r, d))$ and $SU_X^s(r, d) = \phi^{-1}(\phi(SU_X^s(r, d)))$. So the restriction ψ of ϕ to

$SU_X^s(r, d)$ is injective and proper over its image. By Theorem 3.4, ψ is an embedding. \square

4. Very ampleness on $U_X(r, d)$

Let $U_X(r, d)$ be the moduli space of semistable bundles of rank r and degree d on X . Let m be a positive integer. As before, to a vector bundle G of rank mr_1 and degree $m(r_1(g-1) - d_1)$ we can associate a subscheme of $U_X(r, d)$ parameterizing semistable bundles E such that $h^0(E \otimes G) \neq 0$. If the subscheme is not the whole $U_X(r, d)$, then it is the support of a (generalized theta) divisor, denoted Θ_G ; see [DN], §0.2, p. 55.

We search for effective results — bounds on m — towards the very ampleness of the linear series $|m\Theta_G|$. However, in contrast with the case of $SU_X(r, d)$, as G moves among vector bundles of equal rank and degree, the divisors Θ_G will only move in the same linear series if $\det G$ is fixed; see the formula at the bottom of p. 57 in [DN]. As a consequence, the methods presented in Section 3 do not apply immediately. It is still possible to make them work, by considering theta divisors of the form $\Theta_{G_1 \oplus G_2}$, where each G_i is a bundle of rank $m_i r_1$ and degree $m_i(r_1(g-1) - d_1)$, with $m_1 + m_2 = m$. The bundle G_1 is chosen so that Θ_{G_1} has the desired separation properties, and G_2 is added only to have the determinant of $G_1 \oplus G_2$ equal to $(\det F)^m$, and thus $\Theta_{G_1 \oplus G_2} \in |m\Theta_F|$; cf. the end of the proof of Lemma 10 on p. 580 in [Es]. However, this procedure would lead to bounds on m worse than those found for $SU_X(r, d)$.

In this section we show that in fact one can obtain for $U_X(r, d)$ the same bounds as for $SU_X(r, d)$. Our method relies on the use of the Verlinde bundles defined in [Po2]. We begin by recalling the notation.

Let L be a line bundle of degree d on X and π_L the composition:

$$\pi_L: U_X(r, d) \xrightarrow{\det} \text{Pic}^d(X) \xrightarrow{\otimes L^{-1}} J(X).$$

Set $U := U_X(r, d)$ and $J := J(X)$. Let F be a vector bundle on X of rank mr_1 and degree $m(r_1(g-1) - d_1)$ for which there is a generalized theta divisor Θ_F . Put

$$V_m := \pi_{L*} \mathcal{O}_U(m\Theta_F).$$

The fibers of π_L are the $SU_X(r, A)$, for A running over all line bundles of degree d on X , and $\mathcal{O}_U(\Theta_F)|_{SU_X(r, A)} = \mathcal{L}_A$, the determinant line bundle on $SU_X(r, A)$. Since there is an isomorphism $SU_X(r, A) \cong SU_X(r, B)$ for each line bundles A and B of degree d , and the isomorphism carries \mathcal{L}_A to \mathcal{L}_B , the *Verlinde number*, $v_m := h^0(SU_X(r, A), \mathcal{L}_A^m)$, does not depend on A . Hence V_m is a bundle of rank v_m , called the *Verlinde bundle*.

Consider the diagram of maps,

$$\begin{array}{ccc} SU_X(r, L) \times J(X) & \xrightarrow{\tau} & U_X(r, d) \\ p_2 \downarrow & & \downarrow \pi_L \\ J(X) & \xrightarrow{r_J} & J(X) \end{array}$$

where τ is given by tensor product, p_2 is the second projection, and r_J is given by multiplication by r on the Jacobian. The diagram is clearly commutative. Both τ and r_J are étale coverings of degree r^{2g} ; see [TT], Prop. 8, p. 338 for τ . Hence the diagram is a fiber diagram.

By [DT], Cor. 6, p. 350,

$$(4.0.1) \quad \tau^* \mathcal{O}_U(\Theta_F) \cong \mathcal{L} \boxtimes \mathcal{O}_J(rr_1 \Theta_N),$$

where N is any line bundle of degree $g - 1$ on X such that $N^r \cong L \otimes (\det F)^h$. Then the base-change formula for the diagram above yields

$$(4.0.2) \quad r_J^* V_m \cong \bigoplus_{i=1}^{v_m} \mathcal{O}_J(mrr_1 \Theta_N).$$

Let Σ be a finite scheme over \mathbf{C} . A coherent sheaf \mathcal{F} on a scheme Z is called Σ -spanned (resp. Σ -spanned on an open subset $Y \subseteq Z$) if for each map $f: \Sigma \rightarrow Z$ (resp. $f: \Sigma \rightarrow Y$), each section of $\mathcal{F}|_{f(\Sigma)}$ lifts to a global section of \mathcal{F} . The sheaf \mathcal{F} is spanned if and only if it is $\text{Spec}(\mathbf{C})$ -spanned. If \mathcal{F} is a line bundle, then $|\mathcal{F}|$ separates points if and only if \mathcal{F} is $\text{Spec}(\mathbf{C} \times \mathbf{C})$ -spanned. If, in addition, Z is complete, $|\mathcal{F}|$ is very ample if and only if \mathcal{F} is Σ -spanned for each scheme Σ of length 2.

Proposition 4.1.

For each $k > 0$, each $m > kh$, and each scheme Σ of length k , the bundle V_m is Σ -spanned.

Proof.

Let a and b be nonnegative integers such that $a + b \leq k$. We claim that

$$H^i(V_m \otimes \mathcal{I}_{T,J} \otimes \mathcal{O}_J(-\Theta_{N_1}) \otimes \cdots \otimes \mathcal{O}_J(-\Theta_{N_b})) = 0$$

for each $i > 0$, each subscheme $T \subseteq J$ of length a and each line bundles N_1, \dots, N_b of degree $g - 1$. We will prove our claim by induction on a .

Set $G := V_m \otimes \mathcal{O}_J(-\Theta_{N_1}) \otimes \cdots \otimes \mathcal{O}_J(-\Theta_{N_b})$. Suppose first that $a = 0$. Since r_J is a finite étale covering, $r_{J*} \mathcal{O}_J$ is a bundle containing \mathcal{O}_J as a subbundle, and so $H^i(F) \subseteq H^i(r_J^* F)$ for each coherent sheaf F on J . Now, for any line bundle

N of degree $g - 1$ on X , the divisors Θ_{N_j} and Θ_N are algebraically equivalent. In addition, $r_j^* \Theta_N$ is numerically equivalent to $r^2 \Theta_N$. So, by Formula (4.0.2),

$$r_j^* G \cong \bigoplus_{i=1}^{v_m} \mathcal{O}_J(D),$$

where D is numerically equivalent to $(mrr_1 - br^2)\Theta_N$. Since

$$mrr_1 > khrr_1 = kr^2 \geq br^2,$$

the divisor D is ample. Thus $H^i(\mathcal{O}_J(D)) = 0$ for all $i > 0$, and hence $H^i(G) = 0$ for all $i > 0$.

Suppose now that $a > 0$. Let $S \subseteq T$ be a closed subscheme of colength 1. By the induction hypothesis,

$$(4.1.1) \quad H^i(G \otimes \mathcal{I}_{S,J}) = 0 \text{ for each } i > 0.$$

Tensor the natural exact sequence of ideal sheaves $0 \rightarrow \mathcal{I}_{T,J} \rightarrow \mathcal{I}_{S,J} \rightarrow \mathcal{I}_{S,T} \rightarrow 0$ by G , and consider the resulting long exact sequence in cohomology. Since the support of $\mathcal{I}_{S,T}$ is finite, (4.1.1) yields

$$H^i(G \otimes \mathcal{I}_{T,J}) = 0 \text{ for each } i > 1.$$

Furthermore, $H^1(G \otimes \mathcal{I}_{T,J}) = 0$ if and only if each section of $G \otimes \mathcal{I}_{S,T}$ lifts to one of $G \otimes \mathcal{I}_{S,J}$. Since $\mathcal{I}_{S,T}$ has length 1, this will be the case if $G \otimes \mathcal{I}_{S,J}$ is globally generated. Now, it follows from the M -regularity criterion of [PP] (more precisely from the Corollary on p. 286), that $G \otimes \mathcal{I}_{S,J}$ is globally generated if $H^i(G \otimes \mathcal{I}_{S,J} \otimes \mathcal{O}_J(-\Theta_N)) = 0$ for each line bundle N of degree $g - 1$ on X and each $i > 0$. As this is the case by the induction hypothesis, we get $H^1(G \otimes \mathcal{I}_{T,J}) = 0$, thus proving our claim.

Now, let $T \subseteq J$ be a finite subscheme of length at most k . Tensor the natural exact sequence

$$0 \rightarrow \mathcal{I}_{T,J} \rightarrow \mathcal{O}_J \rightarrow \mathcal{O}_T \rightarrow 0$$

by V_m , and consider the resulting exact sequence in cohomology. By the claim, $H^1(V_m \otimes \mathcal{I}_{T,J}) = 0$. So the restriction map $H^0(V_m) \rightarrow H^0(V_m|_T)$ is surjective. \square

The use of Verlinde bundles allows us to reduce the very ampleness problem on $U_X(r, d)$ to that on $SU_X(r, d)$.

Theorem 4.2.

Let Σ be a finite scheme of length k . If \mathcal{L}^m is Σ -spanned on $SU_X(r, d)$, and $m > kh$, then $\mathcal{O}_U(m\Theta_F)$ is Σ -spanned on $U_X(r, d)$.

Proof.

We may assume that $\Sigma \subseteq U$. Put $S := \pi_L(\Sigma)$. Note that S has length at most k , say ℓ . We need only to prove the following two statements.

Statement 1: Each section of $\mathcal{O}_U(m\Theta_F)|_\Sigma$ lifts to one of $\mathcal{O}_U(m\Theta_F)|_{\pi_L^{-1}(S)}$. Indeed, since r_J is an étale covering, $S = r_J(S')$ for some subscheme $S' \subseteq J$ of length ℓ , and thus $\Sigma = \tau(\Sigma')$ for some subscheme $\Sigma' \subseteq SU_X(r, L) \times S'$ of length k . So we need to show that each section of $\tau^*\mathcal{O}_U(m\Theta_F)|_{\Sigma'}$ lifts to one of $\tau^*\mathcal{O}_U(m\Theta_F)|_{SU_X(r, L) \times S'}$. Now, $\tau^*\mathcal{O}_U(m\Theta_F) = \mathcal{L}^m \boxtimes \mathcal{O}_J(mrr_1\Theta_N)$ by Formula (4.0.1). We are left then with proving that the natural map

$$\lambda: H^0(\mathcal{L}^m) \otimes H^0(\mathcal{O}_{S'}) \rightarrow H^0(\mathcal{L}^m \boxtimes \mathcal{O}_{S'}|_{\Sigma'})$$

is surjective. Let T be the image of Σ' in $SU_X(r, L)$ under the projection. Since Σ' is isomorphic to Σ , and \mathcal{L}^m is Σ -spanned, the natural map $H^0(\mathcal{L}^m) \rightarrow H^0(\mathcal{L}^m|_T)$ is surjective. Then $H^0(\mathcal{L}^m) \otimes H^0(\mathcal{O}_{S'})$ surjects onto

$$H^0(\mathcal{L}^m|_T) \otimes H^0(\mathcal{O}_{S'}) = H^0(\mathcal{L}^m \otimes \mathcal{O}_{S'}|_{T \times S'}).$$

Now, since $\Sigma' \subseteq T \times S'$, each section of $\mathcal{L}^m \otimes \mathcal{O}_{S'}|_{\Sigma'}$ lifts to one of $\mathcal{L}^m \otimes \mathcal{O}_{S'}|_{T \times S'}$. So λ is surjective, proving the statement.

Statement 2: Each section of $\mathcal{O}_U(m\Theta_F)|_{\pi_L^{-1}(S)}$ lifts to one of $\mathcal{O}_U(m\Theta_F)$. Indeed,

$$H^0(\mathcal{O}_U(m\Theta_F)) = H^0(V_m) \text{ and } H^0(\mathcal{O}_U(m\Theta_F)|_{\pi_L^{-1}(S)}) = H^0(V_m|_S),$$

the latter because the formation of $\pi_{L*}\mathcal{O}_U(m\Theta_F)$ commutes with base change. Since S has length at most k , the statement follows from Proposition 4.1. \square

The proof of Theorem 4.2 shows that, if \mathcal{L}^m is Σ -spanned on the stable locus $SU_X^s(r, d)$, and if $m > kh$, then $\mathcal{O}_U(m\Theta_F)$ is Σ -spanned on $U_X^s(r, d)$.

Proof. (of Theorem B)

Apply Theorem A and Theorem 4.2, first with $\Sigma := \text{Spec}(\mathbf{C} \times \mathbf{C})$ and then with $\Sigma := \text{Spec}(\mathbf{C}[\epsilon]/(\epsilon^2))$, keeping in mind the observation after Theorem 4.2. \square

Proof. (of Theorem C)

If d is odd, then $|\mathcal{L}|$ is very ample by the main result of [BV2], whereas if d is even and C is nonhyperelliptic, then $|\mathcal{L}|$ is very ample by [vGI], Thm. 1, p. 134. Now apply Theorem 4.2, observing that $h = 1$ in the first case, while $h = 2$ in the second case. \square

REFERENCES

- [BGN] L. Brambila-Paz, I. Grzegarczyk and P. E. Newstead, *Geography of Brill-Noether loci for small slopes*, J. Algebraic Geom. **6** (1997), 645–669.
- [BV1] S. Brivio and A. Verra, *The theta divisor of $SU_C(2, 2d)^s$ is very ample if C is not hyperelliptic*, Duke Math. J. **82** (1996), 503–552.
- [BV2] S. Brivio and A. Verra, *On the theta divisor of $SU(2, 1)$* , Internat. J. Math. **10** (1999), 925–942.

- [DT] R. Donagi and L. Tu, *Theta functions for $SL(n)$ versus $GL(n)$* , Math. Res. Lett. **1** (1994), 345–357.
- [DN] J.-M. Drezet and M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), 53–94.
- [Es] E. Esteves, *Separation properties of theta functions*, Duke Math J. **98** (1999), 565–593.
- [Fa] G. Faltings, *Stable G -bundles and projective connections*, J. Algebraic Geom. **2** (1993), 507–568.
- [vGI] B. van Geemen and E. Izadi, *The tangent space to the moduli space of vector bundles on a curve and the singular locus of the theta divisor of the Jacobian*, J. Algebraic Geom. **10** (2001), 133–177.
- [KI] S. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.
- [La] H. Lange, *Zur Klassifikation von Regelmannigfaltigkeiten*, Math. Ann. **262** (1983), 447–459.
- [PP] G. Pareschi and M. Popa, *Regularity on abelian varieties I*, J. Amer. Math. Soc. **16** (2003), 285–302.
- [Po1] M. Popa, *Dimension estimates for Hilbert schemes and effective base point freeness on moduli spaces of vector bundles on curves*, Duke Math J. **107** (2001), 469–495.
- [Po2] M. Popa, *Verlinde bundles and generalized theta linear series*, Trans. Amer. Math. Soc. **354** (2002), 1869–1898.
- [PR] M. Popa and M. Roth, *Stable maps and Quot schemes*, Invent. Math. **152** (2003), 625–663.
- [TT] M. Teixidor i Bigas and L. Tu, “Theta divisors for vector bundles” in *Curves, Jacobians, and abelian varieties*, Contemp. Math. **136**, Amer. Math. Soc., Providence, 1992, 327–342.

Eduardo Esteves

Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, 22460-320 Rio de Janeiro RJ, Brazil; esteves@impa.br

Mihnea Popa

Department of Mathematics, Harvard University, One Oxford Street, Cambridge MA 02138, USA; mpopa@math.harvard.edu