ON THE BASE LOCUS OF THE GENERALIZED THETA DIVISOR

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Abstract. In response to a question of Beauville, we give a new class of examples of base points for the linear system $|\Theta|$ on the moduli space $SU_X(r)$ of semistable rank r vector bundles of trivial determinant on a curve X and we prove that for sufficiently large r the base locus is positive dimensional.

1. INTRODUCTION

Let X be a compact Riemann surface of genus g. In his survey [2], A. Beauville raises a few questions about the base locus of the linear system $|\Theta|$, where Θ is the theta (or determinant) bundle on the moduli space $SU_X(r)$ of semistable rank r vector bundles on X of trivial determinant. It is known (see [2], §3) that $E \in SU_X(r)$ is a base point for $|\Theta|$ if and only if $H^0(E \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^{g-1}(X)$[1]. When $r = 2$, or $r = 3$ and X is of genus 2 or generic of any genus, it is known that $|\Theta|$ is base point free. However, M. Raynaud constructs in [9] examples of bundles which lead to the existence of base points of $|\Theta|$ for $r = n^g$, where n is an integer ≥ 2 dividing g. His construction gives finitely many base points in each genus. Among other things, Beauville asks in [2] if one could find new examples of such base points and if the base locus is actually of strictly positive dimension.

The purpose of this note is to give at least a partial answer to Beauville's question. From a qualitative point of view, our results can be summarized in the following:

Theorem. *(a) For every $g \geq 2$, there exists a rank $\rho(g)$ such that for all $r \geq \rho(g)$ the linear system $|\Theta|$ on $SU_X(r)$ has base points. Also, for every $g \geq 2$ there exist ranks where some base points are stable.*

(b) Moreover, for every $g \geq 2$ and every $k \geq 2$, there exists an integer $\rho(k,g)$ such that for all $r \geq \rho(k,g)$, the base locus of $|\Theta|$ on $SU_X(r)$ has dimension at least $(k-1)g$.

For the examples and for more precise numerical statements see Section 2.

Another question raised by Beauville adresses the freeness of $|2\Theta|$. In the spirit of [4], one could also look at other moduli spaces, not necessarily of trivial determinant. We remark, by using a theorem of Lange and Mukai-Sakai (see [6],[8]), that the global

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[1]When E is semistable, this should be interpreted as a statement about its equivalence class: by a common argument using the Jordan-Hölder filtration of E and the fact that $\chi(E \otimes L) = 0$, it is easy to see that $H^0(E \otimes L) \neq 0, \forall L \in \mathrm{Pic}^{g-1}(X)$ iff $H^0(gr(E) \otimes L) \neq 0, \forall L \in \mathrm{Pic}^{g-1}(X)$. So this property does not depend on the choice of a bundle in the class of E.

generation of low multiples of the theta divisor on such moduli spaces cannot go hand in hand with the strange duality conjecture (see [2], §8).

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2. The examples and a lower bound for the dimension of the base locus

Throughout the paper we will denote by $SU_X(r, A)$ the moduli space of semistable bundles of rank r and fixed determinant A and by $U_X(r, d)$ the moduli space of semistable bundles of rank r and degree d on X.

Consider a line bundle L on X of degree $d \geq 2g + 1$. We will restrict to the case $g \geq 2$, since for $g \leq 1$ the space $SU_X(r)$ is well understood. Denote by M_L the kernel of the evaluation map:

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

and let $Q_L = M_L^*$. These vector bundles are well known for their importance in the study of the minimal resolution of X in the embedding defined by L (see [7], §1 for a survey).

Among the properties of Q_L, we quote from [7], §1.4, the following: if $x_1, \ldots, x_d$ are the points of a generic hyperplane section of $X \subset \mathbb{P}(H^0(L))$, then Q_L sits in an extension:

$$0 \longrightarrow \bigoplus_{i=1}^{d-g-1} \mathcal{O}_X(x_i) \longrightarrow Q_L \longrightarrow \mathcal{O}_X(x_{d-g} + \ldots + x_d) \longrightarrow 0$$

This induces for every integer p an inclusion:

$$(1) \qquad 0 \longrightarrow \bigoplus_{1 \leq i_1 < \ldots < i_p \leq d-g-1} \mathcal{O}_X(x_{i_1} + \ldots + x_{i_p}) \longrightarrow \bigwedge^p Q_L$$

Recall also from [5], §3 that Q_L is stable, and so $\bigwedge^p Q_L$ is poly-stable (i.e. a direct sum of stable bundles of the same slope).

Definition. Similarly to a definition in [9], we say that a vector bundle E satisfies property $(*)$ if and only if:

$$H^0(E \otimes \xi) \neq 0, \forall \xi \in \mathrm{Pic}^0(X).$$

In all that follows we will denote $\gamma := [\frac{g+1}{2}]$.

Proof of Theorem.(a). Notice first that to find a base point for $|\Theta|$ it is enough to exhibit a semistable bundle E of integral slope $0 \leq \mu(E) \leq g - 1$ satisfying $(*)$, since we could then twist by a suitable line bundle.

Claim: *For every line bundle L on X of degree $d \geq 2g + 1$, the bundle $\bigwedge^\gamma Q_L$ satisfies property $(*)$.*

Proof of claim. From (1) it is clear that for $x_1, \ldots, x_p$ general points on X we have $H^0(\bigwedge^p Q_L(-x_1 - \ldots -x_p)) \neq 0$. So for any p, a generic line bundle $\xi \in \mathrm{Pic}^0(X)$ of the form $\xi = \mathcal{O}_X(A_p - B_p)$, with A_p, B_p generic effective divisors of degree p, satisfies $H^0(\bigwedge^p Q_L \otimes \xi) \neq 0$.

On the other hand it is well known (see [1]) that every $\xi \in \mathrm{Pic}^0(X)$ can be written in the form $\xi = \mathcal{O}_X(A_\gamma - B_\gamma)$ with A_γ, B_γ effective divisors of degree γ. Hence $H^0(\bigwedge^\gamma Q_L \otimes \xi) \neq 0$ for a general ξ and by semicontinuity the same must hold for every $\xi \in \mathrm{Pic}^0(X)$, which proves the claim.

So, as noted above, it is enough to get integral slopes for $\bigwedge^\gamma Q_L$ for suitable choices of d. Since the computations differ from case to case and tend to get messy, we will restrict to giving examples that work uniformly rather then trying to find the smallest possible rank for each genus.

We can obtain an uniform answer by choosing $d = g(\gamma + 1)$, when we get $\mu(\bigwedge^\gamma Q_L) = \gamma + 1$. The corresponding rank will be $rk(\bigwedge^\gamma Q_L) = \binom{g \cdot \gamma}{\gamma}$ (actually for most g's this is by no means the best answer).

It is easy to see that since the bundles $\bigwedge^\gamma Q_L$ are poly-stable and satisfy $(*)$, at least one of their stable summands (which have the same slope) must also satisfy $(*)$. Thus the constructions above also give us examples of stable base points in each genus. On the other hand, the existence of a base point $E \in SU_X(r)$ induces the existence of decomposable base points for every rank $r' \geq r$: simply take $E \oplus \mathcal{O}_X^{\oplus(r'-r)}$. $\square$

Remark: There are many versions of this construction that give additional examples. Let us just mention them without getting into numerology. One could look at $\bigwedge^p Q_L$ for $p > \gamma$ such that $\mu(\bigwedge^p Q_L) \leq g-1$ or work with $S^p Q_L$ instead of $\bigwedge^p Q_L$. It is probably most interesting though to replace Q_L by Q_E, where E is a semistable bundle of slope $\mu(E) > 2g$ (so automatically very ample) and Q_E is defined exactly as Q_L. By [3] Q_E is known to be semistable and a closer analysis shows that a result analogous to the claim above holds. Using this construction one can check that by good numerical choices we can make $\bigwedge^\gamma Q_E$ have any integral slope $[\frac{g+1}{2}] < \mu \leq g - 1$.

The additional feature that makes these examples interesting is that they come in positive dimensional families (roughly speaking by varying L), so in the range covered by them the base locus is indeed positive dimensional.

Proof of Theorem.(b). The following is the more precise statement referred to in the introduction:

Claim: *Fix $g \geq 2$, $d \geq 2g + 1$, $k \geq 2$ and let L be any line bundles of degree kd. Then there exists a $(k-1)g$ dimensional family of (equivalence classes of) semistable bundles of rank $\binom{k(d-g)}{\gamma}$ and fixed determinant $L^{\otimes\binom{k(d-g)-1}{\gamma-1}}$ satisfying property $(*)$.*

Proof of claim. Fix L of degree kd. To every $k-1$ line bundles $L_1, \ldots, L_{k-1} \in \mathrm{Pic}^d(X)$ associate $L_k := L \otimes L_1^* \otimes \ldots \otimes L_{k-1}^* \in \mathrm{Pic}^d(X)$, so that $L_1 \otimes \ldots \otimes L_k = L$. Set $F_{L_1,\ldots,L_{k-1}} := L_1 \oplus \ldots \oplus L_k$. Thus $\det(F_{L_1,\ldots,L_{k-1}}) = L$ and clearly $Q_{F_{L_1,\ldots,L_{k-1}}} = Q_{L_1} \oplus \ldots \oplus Q_{L_k}$ (cf. the remark above for the definition).

It is enough to prove that the morphism:

$$\psi: \ \mathrm{Pic}^d(X) \times \ldots \times \mathrm{Pic}^d(X) \ \longrightarrow \ SU_X\left(\binom{k(d-g)}{\gamma}, L^{\otimes\binom{k(d-g)-1}{\gamma-1}}\right)$$

$$(L_1, \ldots, L_{k-1}) \qquad\qquad \rightsquigarrow \qquad\qquad \bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$$

is finite. We have:

$$\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}} \cong \bigoplus_{i_1+\ldots+i_k=\gamma} \left(\bigwedge^{i_1} Q_{L_1} \otimes \ldots \otimes \bigwedge^{i_k} Q_{L_k}\right)$$

In particular $\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$ is poly-stable of slope $\gamma \cdot \frac{d}{d-g}$.

Assume now that:

$$\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}} \cong \bigwedge^\gamma Q_{F_{L_1',\ldots,L_{k-1}'}}$$

for some other $L_1', \ldots, L_k'$ as before. By the previous formula one has inclusions:

$$\bigwedge^\gamma Q_{L_i'} \hookrightarrow \bigwedge^\gamma Q_{F_{L_1',\ldots,L_{k-1}'}}.$$

As noted before, all the bundles above are poly-stable (of the same slope), so $\bigwedge^\gamma Q_{L_i'}$ is a direct sum of some collection of the stable summands of $\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$. There are finitely many ways in which this can occur, so it is enough then to notice that the morphism:

$$\phi: \ \mathrm{Pic}^d(X) \ \longrightarrow \ U_X\left(\binom{d-g}{\gamma}, \gamma \cdot \frac{d}{d-g} \cdot \binom{d-g}{\gamma}\right)$$
$$M \qquad \rightsquigarrow \qquad \bigwedge^\gamma Q_M$$

is finite. This is clear since $\det(Q_M)=M$ and the claim is proved.

Taking in particular $d = g(\gamma+1)$ as in part (a), we get that the bundles $\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$ have integral slope $\gamma + 1$ and so they lead to base points as before. Hence we can take $\rho(k,g) = \binom{k \cdot g \cdot \gamma}{\gamma}$.

This argument actually gives a statement about equivalence classes, since the bundles in the family that we have constructed are all poly-stable. Again, by adding trivial bundles we get the same statement in all ranks $r \geq \rho(k,g)$. $\qquad\square$

One could conjecture that the base locus is at least g-dimensional whenever it is non empty. Perhaps in view of the remark above an even more optimistic guess could be made.

3. Strange duality versus freenes of low multiples of theta

In connection with Beauville's question about low multiples of Θ, we show that the strange duality conjecture implies the existence of base points on $|k\Theta|$ for small k, on suitable moduli spaces.

Consider first, in general, the moduli space $SU_X(r, A)$ for some $A \in \operatorname{Pic}^m(X)$, $m \in \mathbb{N}$, $m \leq g - 1$. Consider also $F \in U_X(k, k(g - 1 - m))$ and define Θ_F on $SU_X(r, A)$ to be $\Theta_F = \tau_F^* \Theta$, where τ_F is the map :

$$\begin{array}{rcl} \tau_F : SU_X(r, A) & \longrightarrow & U_X(kr, kr(g - 1)) \\ E & \rightsquigarrow & E \otimes F \end{array}$$

and Θ is the canonical theta divisor on $U_X(kr, kr(g - 1))$. Set theoretically of course $\Theta_F = \{E \mid H^0(E \otimes F) \neq 0\}$. The famous strange duality conjecture, discussed at length in [4], or more precisely its geometric formulation (see [2], §8), asserts that the linear system $|k\Theta|$ on $SU_X(r, A)$ is spanned by the divisors Θ_F as F varies in $U_X(k, k(g - 1 - m))$.

Let us consider in particular E to be one of Raynaud's examples (see [9], §3) i.e. a bundle E with $rk(E) = n^g$, $\mu(E) = \frac{g}{n}$, $n|g$ which satisfies $(*)$. A theorem of Lange and Mukai-Sakai (see [6], [8]) implies that every $F \in U_X(k, k(g - 1 - \frac{g}{n}))$ has a subbundle $M \hookrightarrow F$ of degree $\deg(M) \geq \frac{k(g-1-\frac{g}{n})-(k-1)g}{k}$. Assume further that $\frac{g}{k} \geq 1 + \frac{g}{n}$, which can be achieved for good choices of g and n. Then $m := \deg(M) \geq 0$ and so, by property $(*)$:

$$H^0(E \otimes M) = H^0(E \otimes M(-mp) \otimes O_X(mp)) \neq 0,$$

where p is a point on X. Since $M \hookrightarrow F$, we obtain:

$$H^0(E \otimes F) \neq 0 \text{ for all } F \in U_X\left(k, k\left(g - 1 - \frac{g}{n}\right)\right).$$

By the discussion above, the strange duality conjecture then implies that $|k\Theta|$ on $SU_X(n^g, \det(E))$ has a base point at E.

Remarks: 1. The conclusion above suggests (assuming the strange duality conjecture is true!) that one should expect $|k\Theta|$ to have base points, say for example for k small enough with respect to g or r, even extrapolating to $SU_X(r)$.
2. In the discussion above we cannot use the examples from the previous section instead of Raynaud's examples, since the condition $\deg(M) \geq 0$ is not necessarily satisfied any more.

Let us conclude with another analogous application of the strange duality conjecture:

Consider $F_L = \bigwedge^\gamma Q_L$ from the previous section (of integral slope) and denote $A_L = \det(F_L) = L^{\otimes \binom{d-g-1}{\gamma-1}}$, $r' = rk(F_L) = \binom{d-g}{\gamma}$. Then exactly by the same argument as above, one can check that F_L is a base point for $|\Theta|$ on $SU_X(r', A_L)$ under mild assumptions on d. One can also apply the same argument for at least the Raynaud examples such that $g \geq n$ and $(g, n) \neq 1$ (in particular for those of integral slope).

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