GENERIC VANISHING AND SINGULARITIES OF THETA DIVISORS

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Abstract. We study the connection between the generic vanishing conjecture for subvarieties of principally polarized abelian varieties and various predictions about the singularities of theta divisors. We deduce new cases of a conjecture on the multiplicities of points on irreducible theta divisors, as well as Beauville's conjecture characterizing Jacobians by the size of their theta singularity loci.

A. Introduction

All throughout we work over $\mathbb{C}$. The main technical result of the paper is a stronger version of a conjecture made by Pareschi and the second author, about the equivalence of two notions of regularity for ideal sheaves on abelian varieties, designed according to [PP1] to correspond to the notion of Castelnuovo-Mumford 2-regularity in projective space, namely the implication (3) $\Longrightarrow$ (2) in [PP3, Conjecture A].[1] Note that while geometric nondegeneracy was an assumption in the conjecture, here we see that it is not needed, and in fact follows from the hypothesis.

Theorem A. *Let (A, Θ) be an indecomposable principally polarized abelian variety, and $X \subset A$ a proper closed subscheme. If $\mathcal{I}_X(2\Theta)$ satisfies IT_0, then $\mathcal{I}_X(\Theta)$ is a GV-sheaf. Moreover, X is geometrically nondegenerate and integral.*

The case when X is a divisor was obtained earlier in [Me]. For the technical notions in the theorem above, please see §1. If a coherent sheaf $\mathcal{F}$ on an abelian variety is GV, then $\mathcal{F}(\Theta)$ satisfies IT_0; see [PP4, Proposition 3.1]. The converse however is not true for arbitrary $\mathcal{F}$; see [PP3, Example 3.2]. Therefore the conjecture/theorem above is a property that is specific to twisted ideal sheaves. The (a priori stronger) GV property for $\mathcal{I}_X(\Theta)$ is more useful for applications, as it is related to a Schottky-type conjecture formulated in [PP3] (namely (2) $\Longrightarrow$ (5) in that same Conjecture A in *loc. cit.*), which has been verified in a few cases in [PP3], [CMPS], [Ch].

Conjecture B. *Let (A, Θ) be an indecomposable principally polarized abelian variety of dimension g. If $X \subset A$ is a geometrically nondegenerate closed subscheme of pure dimension[2] $1 \leq d \leq g-2$, then $\mathcal{I}_X(\Theta)$ is a GV-sheaf if and only if one of the following holds:*

(1) *(A, Θ) is the Jacobian of a smooth curve C, and X is $+$ or $-$ a translate of $W_d(C)$.*
(2) *(A, Θ) is the intermediate Jacobian of a smooth cubic threefold Y, and X is $+$ or $-$ a translate of the Fano surface F of lines on Y.*

Although not directly relevant to this paper, it is worth mentioning that both items in this conjecture should also be equivalent to X having minimal class, i.e. $[X] = \theta^{g-d}/(g-d)!$, by a well-known conjecture of Debarre [De1], which is the most significant conjecture in this direction.

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[1] It was already shown there that (2) implies (3), and is equivalent to (4), a condition in terms of M-regularity.

[2] Note that the geometric nondegeneracy and pure dimension assumptions can now be removed by Theorem A.

It is shown in [PP3] that every GV-subscheme, i.e. one satisfying the conditions in Conjecture B, has minimal class. Therefore the conjecture on minimal classes can be considered as input in the results below; note that besides the Matsusaka-Ran criterion, which implies that it holds for curves, it is also known to hold in dimension up to four, by [Ra]. For more in this direction see e.g. the Introduction of [CMPS].

While results about Conjecture B do rely specifically on the GV property, the IT_0 property for $I_X(2\Theta)$ is easier to verify in practice, and as a consequence, Theorem A unlocks a number of significant applications regarding the singularities of theta divisors. The main one is related to the following well-known conjecture of the multiplicities of points on irreducible theta divisors. The first part has been folklore for some time, see for instance [Gr, Conjecture 5.12], while the stronger second statement is formulated as [Ca1, Question 4.7].

Conjecture C. *Let (A, Θ) be an indecomposable principally polarized abelian variety of dimension g. Then:*

(1) *for every $x \in \Theta$ we have* $\mathrm{mult}_x(\Theta) \leq \frac{g+1}{2}$.

(2) *more generally, we have*

$$\dim \Sigma_m(\Theta) \leq g - 2m + 1,$$

where $\Sigma_m(\Theta) := \{x \in \Theta \mid \mathrm{mult}_x(\Theta) \geq m\}$.

Remark (Equality case)**.** Note that (2) implies that in (1) we can also add that the upper bound $\mathrm{mult}_x(\Theta) = \frac{g+1}{2}$, if attained at all, can only be attained at finitely many points. In fact we will see that we can conjecture something more precise: the upper bound is attained at exactly one point, and only on the Jacobian of a hyperelliptic curve of odd genus, at the multiple of a g^1_2, or at the origin of the intermediate Jacobian of a smooth cubic threefold.

Conjecture C is known in dimension up to 5 by Casalaina-Martin's work on singularities of theta divisors on Prym varieties [Ca2], and it is also known when Θ has isolated singularities by [MP, Theorem I]. Using a strategy outlined by the second author in a 2018 Oberwolfach lecture [Po], based on the use of Hodge ideals as in [MP], and which was conditional on a result like Theorem A, we show:

Theorem D. *Conjecture B implies Conjecture C.*

Thanks to results in [PP3], [CMPS] and [Ch], we then get:

Corollary E. *Conjecture C holds unconditionally for $g \leq 6$, or for arbitrary g when* $\dim \mathrm{Sing}(\Theta) \leq 2$ *or* $\dim \mathrm{Sing}(\Theta) = g - 3$.

Recall that an irreducible theta divisor is normal, hence in any case

$$\dim \mathrm{Sing}(\Theta) \leq g - 3.$$

For a more precise statement about the unconditional cases, please see Corollary 8.3. This also gives an alternative proof for the case $g \leq 5$.

An upper bound on the dimension of $\Sigma_m(\Theta)$ that goes in the direction of Conjecture C(2) is given in Theorem 8.1 (which is independent of Theorem A or Conjecture B) and Corollary 8.2. A special case of the latter that is easy to state is:

Theorem F. *Conjecture C(2) holds in arbitrary dimension for $m \leq 3$, or when Θ is 1-Du Bois.*[3]

[3]This means that its minimal exponent satisfies $\widetilde{\alpha}(\Theta) \geq 2$, or equivalently its first Hodge ideal satisfies $I_1(\Theta) = \mathcal{O}_A$.

The method we use for the results above also leads to a proof of the following Schottky-type conjecture originating with Beauville [Be]; see also Debarre's [De2, Conjecture 3] and the surrounding discussion. It was verified in dimension at most 5 in [Be, Proposition 6.6], using the theory of generalized Prym varieties.

Theorem G. *If a ppav (A, Θ) of dimension g satisfies $\dim \mathrm{Sing}(\Theta) = g - 3$, then (A, Θ) is the Jacobian of a hyperelliptic curve.*

It is of course known that all hyperelliptic Jacobians satisfy this property. It is also not hard to show that if Θ is reducible, then $\dim \mathrm{Sing}(\Theta) = g - 2$. (See [EL, Corollary 2] for a converse to this statement, and more.) Therefore the conjecture/theorem fully describes the case of maximal dimension for singular loci of irreducible theta divisors. As shown in [De2, §4.3] (see also [HJ] for more details), it also implies a conjecture of Debarre [De2, Conjecture 2] on the base loci of polarizations; cf. also [ACP] for related results.[4]

Corollary H. *If (A, L) be a polarized abelian variety, then $\mathrm{Bs}(L)$ is generically reduced along any component of codimension 2.*

The main technical tool used in the paper is the zero-scheme of the first Hodge ideal $I_1(\Theta)$ of [MP]. The method applies however to the center of minimal exponent of [SY] as well, answering a problem raised there (and shown in dimension one) in a few additional cases; see Corollary 10.1 for the precise statement.

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B. Preliminaries

1. **Vanishing properties.** Let (A, Θ) be a complex principally polarized abelian variety (ppav), and let $\widehat{A} = \mathrm{Pic}^0(A)$. The cohomological support loci of a coherent sheaf $\mathcal{F}$ on A are

$$V^i(\mathcal{F}) := \{\alpha \in \widehat{A} \mid H^i(A, \mathcal{F} \otimes \alpha) \neq 0\}.$$

We say that $\mathcal{F}$ *satisfies* IT_0 if

$$H^j(A, \mathcal{F} \otimes \alpha) = 0 \quad \text{for all} \quad \alpha \in \widehat{A} \quad \text{and all} \quad j > 0,$$

i.e. $V^i(\mathcal{F}) = \emptyset$ for all $i > 0$. We say that $\mathcal{F}$ is a *GV-sheaf* if

$$\mathrm{codim}_{\widehat{A}} V^i(\mathcal{F}) \geq i \quad \text{for all} \quad i > 0.$$

A closed subscheme $X \subseteq A$ is called a GV-subscheme if $\mathcal{I}_X(\Theta)$ is a GV-sheaf.

[4] A stronger version, unfortunately with an incorrect proof, was first formulated in [ACP, Theorem 4]. See also [Pa] for further comments and work in this direction.

2. **Theta-dual.** For a reduced closed subscheme $Y \subseteq A$, we denote its theta-dual by $V(Y)$. Set-theoretically,

$$V(Y) = \{a \in A \mid Y \subseteq \Theta + a\}.$$

In particular $Y - V(Y) \subseteq \Theta$. For a detailed study of theta-duals, please see [PP3]; in particular, they are endowed there with a scheme structure, using the Fourier-Mukai transform.

3. **Pontrjagin product.** Let $p_1, p_2 : A \times A \to A$ be the natural projections, and let

$$m\colon A \times A \to A, \quad m(x,y) = x + y$$

be the addition map. For coherent sheaves $\mathcal{E}, \mathcal{G}$ on A, denote $\mathcal{E} \boxtimes \mathcal{G} = p_1^*\mathcal{E} \otimes p_2^*\mathcal{G}$, and define

$$\mathcal{E} * \mathcal{G} = m_*(\mathcal{E} \boxtimes \mathcal{G})$$

to be their *Pontrjagin product*, with the derived version $\mathcal{E} *^{\mathbf{R}} \mathcal{G}$. Thus the ordinary and derived Pontrjagin products agree whenever $R^j m_*(\mathcal{E} \boxtimes \mathcal{G}) = 0$ for $j > 0$. More about sheaf-theoretic Pontrjagin products can be found in [PP2, §5].

By interchanging the factors via $\tau(x,y) = (y,x)$ and the equality $m \circ \tau = m$, we get a natural isomorphism

$$\varphi_{\mathcal{E},\mathcal{G}} : \mathcal{E} * \mathcal{G} \xrightarrow{\sim} \mathcal{G} * \mathcal{E}.$$

If $a : \mathcal{E} \to \mathcal{E}'$ and $b : \mathcal{G} \to \mathcal{G}'$ are morphisms, it is immediate to check that

$$\varphi_{\mathcal{E}',\mathcal{G}'} \circ (a * b) = (b * a) \circ \varphi_{\mathcal{E},\mathcal{G}}. \tag{3.1}$$

Example 3.2. (1) Let $a \in A$ and $M = \mathcal{O}(\Theta_a)$. We note that

$$R^i m_*(M \boxtimes M) = 0 \quad \text{for all} \quad i > 0,$$

which implies that $M * M \simeq M *^{\mathbf{R}} M$. (This will be useful later on.) Indeed, for $t \in A$ we identify $m^{-1}(t)$ with A by $x \mapsto (x, t-x)$, and we write $\sigma_t(x) = t - x$. Under this identification we have

$$(M \boxtimes M)|_{m^{-1}(t)} \simeq M \otimes \sigma_t^* M.$$

This is an ample line bundle on A for each t, hence it has vanishing higher cohomology, and therefore the cohomology and base change theorem applies to imply the claim.

(2) In addition to the notation in (1), let now X be a closed subscheme of A, and let $\mathcal{F} = \mathcal{I}_X \otimes M$. Assume in addition that $\mathcal{I}_X(2\Theta)$ satisfies IT_0. We again have

$$R^i m_*(\mathcal{F} \boxtimes M) = 0 \quad \text{for all} \quad i > 0,$$

and consequently $\mathcal{F} * M \simeq \mathcal{F} *^{\mathbf{R}} M$. This time we have

$$(\mathcal{F} \boxtimes M)|_{m^{-1}(t)} \simeq \mathcal{F} \otimes \sigma_t^* M.$$

Since this is of the form $\mathcal{I}_X(2\Theta_b)$ for some b, it has vanishing higher cohomology by our assumption, and therefore the same argument applies.

4. **Hodge ideals.** We continue to consider a complex ppav (A, Θ). Recall from [MP] that there is a Hodge filtration $F_\bullet \mathcal{O}_A(*\Theta)$ on the $\mathcal{D}_A$-module $\mathcal{O}_A(*\Theta)$ of meromorphic functions on A with poles of arbitrary order. In each degree it is contained in the pole order filtration $P_k\mathcal{O}_A(*\Theta) = \mathcal{O}_A\big((k+1)\Theta\big)$, hence for each $k \geq 0$ we have

$$F_k\mathcal{O}_A(*\Theta) = \mathcal{O}_A((k+1)\Theta) \otimes I_k(\Theta),$$

where $I_k(\Theta)$ is the k-th Hodge ideal of Θ. For $k = 0$ we have

$$I_0(\Theta) \simeq \mathcal{I}\big((1-\epsilon)\Theta\big)$$

the multiplier ideal of the $\mathbb{Q}$-divisor $(1-\epsilon)\Theta$, where $0 < \epsilon \ll 1$ is a rational number; see [MP, Proposition 10.1]. This holds for any reduced divisor on a smooth variety, but in our case a well known result of Kollár [Ko, Theorem 17.13] says then that $I_0(\Theta) \simeq \mathcal{O}_A$, i.e. the pair (A, Θ) is log canonical.

In this paper we will deal with the Hodge ideal $I_1(\Theta)$. We will need:

Theorem 4.1. *We have:*

(1) *For every irreducible closed subset $W \subset A$ of codimension r, if $m = \mathrm{mult}_W(\Theta)$, then $I_1(\Theta) \subseteq \mathcal{I}_W^{(q)}$, where $q = \min\{m-1, 2m-r\}$. (Here $\mathcal{I}_W^{(q)}$ is the q-th symbolic power of the ideal sheaf $\mathcal{I}_W$.)*

(2) *$H^i\big(A, I_1(\Theta) \otimes \mathcal{O}_A(2\Theta) \otimes \alpha\big) = 0$ for all $i > 0$ and all $\alpha \in \widehat{A}$, i.e. $I_1(\Theta) \otimes \mathcal{O}_A(2\Theta)$ satisfies IT_0.*

The first is a special case of [MP, Theorem E], a local property that holds on any smooth variety, while the second is a special case of [MP, Theorem 28.2], which is in turn the better behaved abelian varieties version of the Hodge ideal analogue of Nadel's vanishing theorem.

C. GV-subschemes

5. **Vanishing away from theta-dual.** The first thing to show is that all the cohomological support loci of $\mathcal{I}_X(\Theta)$ are subsets of the theta-dual of X_{red} (see §2), under the IT_0 assumption. (Eventually we will see that X must in fact be integral.)

Proposition 5.1. *Let (A, Θ) be a ppav, and let X be a closed subscheme of A. If $\mathcal{I}_X(2\Theta)$ satisfies IT_0, then we have*

$$H^i(A, \mathcal{I}_X(\Theta_a)) = 0 \quad \text{for all} \quad a \notin V(X_{\mathrm{red}}) \quad \text{and all} \quad i \geq 0.$$

In other words, for all $i \geq 0$ we have

$$V^i\big(\mathcal{I}_X(\Theta)\big) \subseteq V(X_{\mathrm{red}}).$$

Proof. Let $a \notin V(X_{\mathrm{red}})$, and set

$$M = \mathcal{O}_A(\Theta_a) \quad \text{and} \quad \mathcal{F} = \mathcal{I}_X \otimes M.$$

We are aiming to show that

$$H^i(A, \mathcal{F}) = 0 \quad \text{for all} \quad i \geq 0.$$

By the choice of a, there exist a closed point $p \in X$, and a section $s \in H^0(A, M)$, such that $s(p) \neq 0$. By abuse of notation, we denote by $M(p)$ the fiber of M at p, thought of as a skyscraper sheaf at p or a vector space over $\mathbb{C}$ depending on the specific setting. Consider

$$f\colon \mathcal{F} \hookrightarrow M, \qquad g\colon M \to M(p),$$

i.e. the natural inclusion and evaluation maps. Clearly $g \circ f = 0$.

We consider the composition of Pontrjagin products

$$\mathcal{F} * M \xrightarrow{f*\mathrm{id}} M * M \xrightarrow{\mathrm{id}*g} M * M(p), \tag{5.2}$$

Claim: this composition is equal to 0.

Let us first conclude the proof assuming the Claim. It is immediate to check that the composed morphism above also factors as

$$\mathcal{F} * M \xrightarrow{\mathrm{id}*g} \mathcal{F} * M(p) \xrightarrow{f*\mathrm{id}} M * M(p).$$

Here the second map is injective; it suffices to have the injectivity of the map $\mathcal{F}\boxtimes M(p) \to M\boxtimes M(p)$ on $A \times A$, but this can be identified on $A \times \{p\}$ with the inclusion $\mathcal{F} \hookrightarrow M$ tensored over $\mathbb{C}$ with a vector space, which stays injective. Therefore we deduce that

$$\mathrm{id} * g \colon \mathcal{F} * M \longrightarrow \mathcal{F} * M(p)$$

is the 0 map. On the other hand, by Example 3.2(2), for all i we have

$$H^i(A, \mathcal{F} * M) \simeq H^i(A \times A, \mathcal{F} \boxtimes M) \simeq H^i(A, \mathcal{F}) \otimes H^0(A, M),$$

as the higher cohomology of M vanishes. Similarly,

$$H^i(A, \mathcal{F} * M(p)) \simeq H^i(A \times A, \mathcal{F} \boxtimes M(p)) \simeq H^i(A, \mathcal{F}) \otimes M(p).$$

Moreover, the map on cohomology induced by $\mathrm{id} * g$ is identified with

$$H^i(A, \mathcal{F}) \otimes H^0(A, M) \xrightarrow{1\otimes \mathrm{ev}_p} H^i(A, \mathcal{F}) \otimes M(p),$$

where ev_p is the evaluation of global sections at p. Now $H^0(A, M) = \mathbb{C}s$ and $M(p)$ are both one-dimensional, and $\mathrm{ev}_p(s) = s(p) \neq 0$, so ev_p is an isomorphism. It follows that $H^i(A, \mathcal{F}) = 0$ for all $i \geq 0$, as required.

It remains to prove the Claim, i.e. the vanishing of the composition in (5.2). To this end, we first show that the involution $\varphi_{M,M}$ on $M * M$ induced by the exchange map $(x, y) \to (y, x)$ on $A \times A$ is equal to the identity. To see this, we first note that $E = M * M$ is a simple vector bundle on A. Indeed, by Example 3.2(1) we have $E \simeq M *^{\mathbf{R}} M$, and by [Muk, §3, formula (3.7)] we know that the Fourier-Mukai transform

$$\Phi \colon D^b(A) \longrightarrow D^b(\widehat{A}),$$

interchanges Pontrjagin and tensor product. Therefore

$$\Phi_A(E) \simeq \widehat{M}^{\otimes 2}.$$

Here $\widehat{M}$ is the total Fourier-Mukai transform of M, which is well known to be a line bundle on $\widehat{A}$ (the line bundle associated to a translate of $-\Theta$). The endomorphism group of a line bundle is $\mathbb{C}$, and since Φ is an equivalence [Muk, Theorem 2.2], we obtain $\mathrm{End}_A(E) \simeq \mathbb{C}$.

Since E is simple, we have that $\varphi_{M,M} = \lambda \cdot \mathrm{id}_E$ for some $\lambda \in \mathbb{C}$. Moreover, $\varphi_{M,M}$ fixes the nontrivial section $s \boxtimes s$ of E, since this is preserved by the exchange involution. Hence $\lambda = 1$.

We use this to deduce our claim, namely that the morphism

$$\Psi = (\mathrm{id} * g) \circ (f * \mathrm{id}) \colon \mathcal{F} * M \to M * M(p)$$

is 0. By applying the commutation relation (3.1) to id_M and g, we get

$$\varphi_{M,M(p)} \circ (\mathrm{id} * g) = (g * \mathrm{id}) \circ \varphi_{M,M}.$$

Using $\varphi_{M,M} = \mathrm{id}_E$, we finally get

$$\varphi_{M,M(p)} \circ \Psi = (g * \mathrm{id}_M) \circ (f * \mathrm{id}_M) = (g \circ f) * \mathrm{id}_M = 0.$$

But $\varphi_{M,M(p)}$ is an isomorphism, so $\Psi = 0$. □

Corollary 5.3. *Under the hypothesis of Proposition 5.1, if X is geometrically nondegenerate, then $I_X(\Theta)$ is a GV-sheaf.*

Proof. Note that for $i > 0$ and any $a \in A$ we have

$$H^{i+1}(A, \mathcal{I}_X(\Theta_a)) \simeq H^i(X, \mathcal{O}_X(\Theta_a)),$$

which vanishes for $i \geq d+1$, where $d = \dim X$. Therefore we only need to concern ourselves with $V^i\big(\mathcal{I}_X(\Theta)\big)$ for $i \leq d+1$, and show that their dimension is at most $g - i$.

Now since X is geometrically nondegenerate, so is X_{red} and by [PP3, Lemma 4.4] we have

$$\dim V(X_{\mathrm{red}}) \leq g - d - 1.$$

Combined with Proposition 5.1, this immediately implies what we want. □

Geometric nondegeneracy will be treated in the next section, but it is automatic in the 0-dimensional case, and this already has an interesting consequence.

Corollary 5.4. *Let $Y \subseteq A$ be a 0-dimensional closed subscheme such that $\mathcal{I}_Y(2\Theta)$ satisfies IT_0. Then Y is exactly one reduced point.*

Proof. According to Corollary 5.3, $\mathcal{I}_X(\Theta)$ is a GV-sheaf. Thus for general $a \in A$ we have $H^1(A, \mathcal{I}_X(\Theta_a)) = 0$, and therefore the restriction map

$$H^0(A, \mathcal{O}_A(\Theta_a)) \to H^0(X, \mathcal{O}_X(\Theta_a))$$

is surjective. This implies that the length of X is 1. □

This provides in particular a different approach to [MP, Theorem 29.2] in *loc. cit.*, which is a proof of the multiplicity Conjecture C when Θ has isolated singularities. A careful reading of the proof in *loc. cit.* shows that it relies precisely on the statement of Corollary 5.4, proved there using Theorem 4.1, plus the fact that the linear series $|2\Theta|$ separates neither points nor tangent vectors on A.

6. **Geometric nondegeneracy.** The second ingredient for the proof of Theorem A is a result that combined with the work of Schreieder [Sch] ensures the geometric nondegeneracy of a closed subscheme X for which $\mathcal{I}_X(2\Theta)$ satisfies the IT_0 property.

Proposition 6.1. *Let (A, Θ) be an indecomposable principally polarized abelian variety of dimension g, and $X \subset A$ a proper closed subscheme of dimension d. Assume that $\mathcal{I}_X(2\Theta)$ satisfies IT_0. Then for every integral closed subscheme Y of X_{red} of dimension d, there exists an integral closed subscheme $Z \subseteq V(X_{\mathrm{red}})$ of dimension $g - d - 1$ such that*

$$Y - Z = \Theta.$$

In particular Y, hence also X, is geometrically nondegenerate.

Proof. The last assertion follows for Y from the first. If $1 \leq d \leq g-2$, it follows by [Sch, Theorem 1.1(1)]. If $d = g-1$, we get that Y is a translate of Θ, in which case it is clear. If $d = 0$, geometric nondegeneracy is automatic. Now since Y is a geometrically nondegenerate irreducible component of X_{red}, we deduce that the latter is also geometrically nondegenerate by [Sch, Lemma 4]. Hence so is X, because this property only depends on the reduced scheme structure.

We now focus on the first assertion. Let $\mathcal{P}$ be the normalized Poincaré bundle on $A \times \widehat{A}$. Under the isomorphism $A \simeq \widehat{A}$ induced by the principal polarization, we can regard $\mathcal{P}$ as a line bundle on $A \times A$.

Set

$$\theta = c_1(\mathcal{O}_A(\Theta)) \in H^2(A, \mathbb{Q}) \quad \text{and} \quad \ell = c_1(\mathcal{P}) \in H^2(A \times A, \mathbb{Q}).$$

Let

$$\delta : A \times A \to A, \quad (a, b) \mapsto a - b$$

be the difference map. By the construction of $\mathcal{P}$ in [Mu, p.78], our choice of the isomorphism $A \simeq \widehat{A}$, and the fact that $\mathcal{P}$ is trivial when restricted to $A \times \{0\}$ or $\{0\} \times A$, we have that

$$\ell = \delta^*\theta - p_1^*\theta - p_2^*\theta \in H^1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} H^1(A, \mathbb{Q}) \subseteq H^2(A \times A, \mathbb{Q}).$$

Denote the two natural projections by $p_1, p_2 \colon A \times A \to A$, and consider the Fourier-Mukai transform of $\mathcal{I}_X(\Theta)$, namely

$$C := \Phi\big(\mathcal{I}_X(\Theta)\big) = Rp_{2*}\big(p_1^*\mathcal{I}_X(\Theta) \otimes \mathcal{P}\big).$$

By Proposition 5.1, we have $C|_{A \smallsetminus V(X_{\mathrm{red}})} = 0$.

We now proceed to some Riemann-Roch calculations. Recall that there is an induced Fourier-Mukai transform at the level of cohomology with rational coefficients, defined on $H^\bullet(A, \mathbb{Q})$ by

$$\mathscr{F}(\alpha) = p_{2*}\big(p_1^*\alpha \cup e^{\ell}\big).$$

Using the fact that $\theta^g = g!$, we have

$$\mathscr{F}(e^{\theta}) = p_{2*}\big(e^{\delta^*\theta - p_2^*\theta}\big) = e^{-\theta}.$$

Using the triviality of the tangent bundle of A, as well as the standard exact sequence involving $\mathcal{I}_X$ and $\mathcal{O}_X$, Grothendieck-Riemann-Roch implies

$$\mathrm{ch}(C) = e^{-\theta} - \mathscr{F}\big(\mathrm{ch}(\mathcal{O}_X(\Theta))\big).$$

Now since the support of $\mathcal{O}_X(\Theta)$ has dimension d, we have $\mathrm{ch}_r(\mathcal{O}_X(\Theta)) = 0$ for $r < g-d$. Moreover, since the bidegree of ℓ is $(1,1)$ and the reverse Fourier-Mukai transform is the inverse of $\mathscr{F}$ up to sign and the standard involution, for all j we have

$$\mathscr{F}\big(H^{2j}(A, \mathbb{Q})\big) = H^{2(g-j)}(A, \mathbb{Q}).$$

Thus for every $j > d$ we deduce that

$$\mathrm{ch}_j(C) = (-1)^j \frac{\theta^j}{j!},$$

and in particular $\mathrm{ch}_{d+1}(C) = (-1)^{d+1}\frac{\theta^{d+1}}{(d+1)!}$.

Since $C|_{A \smallsetminus V(X_{\mathrm{red}})} = 0$, the Chern character $\mathrm{ch}(C)$ in the Chow ring of A restricts to 0 on this open set, and in particular $\mathrm{ch}_{d+1}(C)$ is represented by a rational cycle class with support in $V(X_{\mathrm{red}})$.

Passing to cohomology, we obtain an expression

$$\frac{\theta^{d+1}}{(d+1)!} = \sum_j r_j [Z_j],$$

where $r_j \in \mathbb{Q}$ and each $Z_j \subseteq V(X_{\mathrm{red}})$ is a closed subvariety of dimension $g-d-1$; here $[W]$ denotes the cohomology class of a closed subvariety W.

Now fix a d-dimensional component $Y \subseteq X_{\mathrm{red}}$ as in the hypothesis, and set

$$\gamma = \delta_* \left(p_1^*[Y] \cup p_2^* \frac{\theta^{d+1}}{(d+1)!} \right) = \sum_j r_j \, \delta_*[Y \times Z_j] \in H^2(A, \mathbb{Q}).$$

For the Pontrjagin product $u * v = m_*(p_1^* u \cup p_2^* v)$ of cohomology classes, we have the standard identity

$$\frac{\theta^{d+1}}{(d+1)!} * \frac{\theta^{g-1}}{(g-1)!} = (g-d)\frac{\theta^d}{d!}.$$

See for instance [BL, Corollary 16.5.8]. The automorphism $f\colon A \times A \to A \times A$ defined by $(a,b) \mapsto (b, a-b)$ satisfies $m \circ f = p_1$, $p_1 \circ f = p_2$, and $p_2 \circ f = \delta$. Thus we deduce that

$$\begin{aligned}
\int_A \gamma \cup \frac{\theta^{g-1}}{(g-1)!} &= \int_{A\times A} p_1^*[Y] \cup p_2^* \frac{\theta^{d+1}}{(d+1)!} \cup \delta^* \frac{\theta^{g-1}}{(g-1)!} \\
&= \int_{A\times A} m^*[Y] \cup p_1^* \frac{\theta^{d+1}}{(d+1)!} \cup p_2^* \frac{\theta^{g-1}}{(g-1)!} \\
&= \int_A [Y] \cup \left(\frac{\theta^{d+1}}{(d+1)!} * \frac{\theta^{g-1}}{(g-1)!} \right) = \frac{g-d}{d!} \int_Y \theta^d > 0.
\end{aligned}$$

Thus we deduce that $\delta_*[Y \times Z_j] \neq 0$ for some j and therefore

$$\dim(Y - Z_j) = \dim(Y \times Z_j) = g-1.$$

On the other hand,

$$Y - Z_j \subseteq X_{\mathrm{red}} - V(X_{\mathrm{red}}) \subseteq \Theta.$$

Since the theta divisor of an indecomposable ppav is integral, we deduce that $Y - Z_j = \Theta$. Taking $Z = Z_j$ concludes the proof. □

7. **Proof of Theorem A.** We assume that X is a proper closed subscheme of an indecomposable ppav (A, Θ) such that $\mathcal{I}_X(2\Theta)$ satisfies IT_0. By Proposition 6.1 we know that X is geometrically nondegenerate. Then by Corollary 5.3 we deduce that $\mathcal{I}_X(\Theta)$ is a GV-sheaf. As X is a geometrically nondegenerate closed GV-subscheme of A, [Sch, Theorem 2(1)] implies that X is integral.

Remark 7.1. Note that once it is shown that X is a GV-scheme, and therefore automatically geometrically nondegenerate by Theorem A, then various other nice properties of X follow. We have already seen that X is integral by [Sch]. It is also known that X is normal, and has even rational singularities when $\dim X = 2$, by [JP]. Further generation properties relying only on the GV-condition were studied in [LT].

D. Singularities of theta divisors

8. **Multiplicities of points, and multiplicity loci.** Let (A, Θ) be a complex ppav. We will make use of the Hodge ideal $I_1(\Theta)$, see §4. From here on, write

$$I_1(\Theta) = \mathcal{I}_Z,$$

where Z is the closed subscheme of A determined by this ideal.

We start with the following result towards Conjecture C, which is independent of the generic vanishing conjecture or Theorem A. Note first that when Θ is irreducible, it has rational singularities by [EL, Theorem 1], and in particular it is normal. Therefore we have

$$\dim Z \leq \dim \mathrm{Sing}(\Theta) \leq \dim \Theta - 2 = g - 3.$$

For this reason, Conjecture C is immediate for $m = 2$.

Theorem 8.1. *If (A, Θ) is an indecomposable ppav of dimension g, then for all $m \geq 2$ we have*

$$\dim \Sigma_m(\Theta) \leq \max\, \{g - 2m + 1, \dim \mathrm{Sing}(Z)\},$$

with the convention that $\dim \emptyset = -\infty$.[5]

Proof. For $m = 2$ the result is clear by the discussion above, so we may assume $m \geq 3$. Recall that we consider $\Sigma_m(\Theta)$ with its reduced scheme structure, and let W be an irreducible component. Denote

$$r = \mathrm{codim}_A W \quad \text{and} \quad e = \mathrm{mult}_W(\Theta).$$

In particular we have $e \geq m \geq 3$. If $r \geq 2m - 1$, or equivalently $\dim W \leq g - 2m + 1$, we are in good shape already. So we may assume $r \leq 2m - 2$.

With this data, we apply Theorem 4.1(1) to obtain

$$\mathcal{I}_Z \subseteq \mathcal{I}_W^{(q)}, \quad \text{for} \quad q = \min\, \{e - 1, 2e - r\},$$

where $\mathcal{I}_W^{(q)}$ is the q-th symbolic power of $\mathcal{I}_W$. Given our numerical assumptions, we have $q \geq 2$, so that at the very least

$$\mathcal{I}_Z \subseteq \mathcal{I}_W^{(2)}.$$

Let $x \in W$ be a general point, at which W is smooth. Since the symbolic and usual powers agree at smooth points, we have

$$\mathcal{I}_{Z,x} \subseteq \mathcal{I}_{W,x}^2 \subseteq \mathfrak{m}_x^2,$$

where the latter is the maximal ideal of x in A. This implies that x is a singular point of Z, and therefore $W \subseteq \mathrm{Sing}(Z)$. □

Studying the case $\dim Z = 0$ relies on Corollary 5.4, which says that Z is a reduced point. Next we begin to make use of Theorem A, if $\dim Z \geq 1$. The sheaf $\mathcal{I}_Z(2\Theta)$ satisfies IT_0 by Theorem 4.1(2), and consequently Z is a GV-scheme by Theorem A. It follows that Z is itself normal by [JP, Lemma 7.8], and so overall

$$\dim \mathrm{Sing}(Z) \leq \dim Z - 2 \leq g - 5.$$

Putting everything together, we get a conclusion that includes Theorem F.

[5] Note that this result only uses statements from [MP], and nothing about GV-schemes or minimal classes.

Corollary 8.2. *Conjecture C(2) holds in arbitrary dimension for*

$$m \leq \frac{g+1-\dim \mathrm{Sing}(Z)}{2}.$$

Hence it always holds for $m \leq 3$. It holds for all m if Z is smooth or empty.

The condition $Z = \emptyset$ is equivalent to having $I_1(\Theta) = \mathcal{O}_A$, which is in turn equivalent to the condition $\widetilde{\alpha}(\Theta) \geq 2$ for the minimal exponent of Θ by [Sa]. This is known as the 1-Du Bois condition; equivalently, the pair (A, Θ) is 1-log canonical in the terminology of [MP]. This is stronger than rationality; for instance, in the case of Jacobians it holds for Brill-Noether general curves, by [SY, Theorem 5.1].

We next prove the main result in this direction.

Proof of Theorem D. We need to show that the generic vanishing Conjecture B implies the strong multiplicity Conjecture C.

If Z is smooth or empty, the conjecture holds by Corollary 8.2. In particular this applies when $\dim Z = 0$, as we may apply Corollary 5.4 to deduce that Z is smooth. Assume now that $\dim Z > 0$. Then by Theorem A we have that $I_Z(\Theta)$ is a GV-sheaf, and Z is integral and geometrically nondegenerate. Conjecture B then leads to two possibilities.

On one hand, (A, Θ) could be the intermediate Jacobian of a smooth cubic threefold; this however cannot be the case, since its theta divisor has only one singular point, contradicting the assumption on $\dim Z$. On the other hand, (A, Θ) could be the Jacobian of a smooth curve of genus g. Then the bound in Conjecture C(2) holds by Martens' theorem. □

The following is a slightly stronger version of Corollary E.

Corollary 8.3. *Conjecture C holds unconditionally when:*

(1) $g = \dim A \leq 6$
(2) *More generally, when* $\dim Z \leq 2$ *or* $\dim Z \geq g-3$, *in any dimension g; this holds in particular when* $\dim \mathrm{Sing}(\Theta) \leq 2$ *or* $\dim \mathrm{Sing}(\Theta) \geq g-3$.

Proof. It suffices to show (2). The case $\dim Z = 0$ is dealt with by Corollary 5.4. Next note that the case $\dim Z \leq 2$ is technically equivalent to the case $\dim Z \geq g-3$. This is because by theta-duality, see [PP3, Theorem 5.2], if Z is a GV-scheme of dimension d, its theta-dual W is a GV-scheme of dimension $g-d-1$, and the argument in the proof of Theorem D works for either.

Hence all we need to know is that the GV-conjecture holds for all positive dimensional closed subschemes X of dimension at most two. If $\dim X = 1$, this is known by [PP3, Theorem C]. If $\dim X = 2$, it is known if $g \leq 5$ by [CMPS, Theorem A], and if $g \geq 6$ by [Ch, Corollary 1.3].

Finally, if $\dim \mathrm{Sing}(\Theta) \leq 2$, then clearly $\dim Z \leq 2$. On the other hand, if $\dim \mathrm{Sing}(\Theta) \geq g-3$, then in fact $\dim \mathrm{Sing}(\Theta) = g-3$ since Θ is normal, in which case we also have $\dim Z = g-3$ by Lemma 9.1 below. □

Remark 8.4. The first case when we do not have the multiplicity conjecture unconditionally is when $g = 7$ and $\dim Z = \dim \mathrm{Sing}(\Theta) = 3$. This is the only case left to deal with in dimension 7, and the conjecture is that it should not be possible (as it is not hard to see that it cannot happen on Jacobians, with $Z = W_3$ up to sign and translation).

Remark 8.5 (Equality case)**.** This refers to the Remark right after Conjecture C in the Introduction, regarding the maximal multiplicity case

$$\mathrm{mult}_x(\Theta) = \frac{g+1}{2}.$$

On the Jacobian of a smooth curve it is well known that the multiplicity of each point on Θ is at most $(g+1)/2$, by the Riemann Singularity Theorem combined with Clifford's Theorem. Moreover, Clifford also tells us that the equality case happens only on hyperelliptic Jacobians, at a unique point (the multiple of the g^1_2) when g is odd, and not at all when g is even. It is also well known that on the intermediate Jacobian of a smooth cubic threefold the only singular point is $0 \in \Theta$, which also satisfies the equality case $\mathrm{mult}_0(\Theta) = 3$. It it natural to conjecture that these are the only situations when equality holds.

9. **Characterization of hyperelliptic Jacobians.** The methods discussed above also lead to a proof of Beauville's conjecture characterizing hyperelliptic Jacobians; see Theorem G and the preceding paragraph. We start with a technical result:

Lemma 9.1. *If* $\dim \mathrm{Sing}(\Theta) = g-3$*, then* $\dim Z = g-3$*. Moreover,* $\mathrm{Sing}(\Theta)$ *has a unique irreducible component of maximal dimension* $g-3$*, equal to* Z*.*

Proof. Clearly we have $Z \subseteq \mathrm{Sing}(\Theta)$. By Theorem A, we also know that Z is integral. It suffices therefore to show that if W is a $(g-3)$-dimensional component of $\mathrm{Sing}(\Theta)$, then $W \subseteq Z$.

Denote by m the generic multiplicity of Θ along W. Since the pair (A, Θ) is log canonical (see §4), it is standard that, for any $k \geq 2$, each irreducible component of $\Sigma_k(\Theta)$ has codimension at least k in A; moreover, by [EL, Corollary 2], the existence of a component of codimension precisely k implies that Θ is reducible. Therefore in our case we have the inequality

$$g-3 = \dim W \leq g-m-1,$$

and therefore $m = 2$. We now use the relationship between Hodge ideals and multiplicity loci in Theorem 4.1(1) ; in our case, with codimension $r = 3$ and multiplicity $m = 2$, it implies that

$$I_1(\Theta) \subseteq I_W,$$

which is precisely what we want. □

Proof of Theorem G. It is a standard fact that if Θ is reducible, then $\dim \mathrm{Sing}(\Theta) = g-2$. Therefore under our assumption Θ is irreducible.

As discussed in the Introduction, Beauville [Be] proved this result for $g \leq 5$, hence we may assume $g \geq 6$. By Lemma 9.1, we see that if Z is the zero-scheme of the Hodge ideal $I_1(\Theta)$, then Z itself has dimension $g-3$. Moreover, as before Theorem A implies that $\mathcal{I}_Z(\Theta)$ is a GV-sheaf. Using the main result of [Ch], the GV-conjecture holds for Z, and therefore (A, Θ) is a Jacobian. But on the Jacobian of a curve C it is well known that $\dim \mathrm{Sing}(\Theta) = g-3$ if and only if C is hyperelliptic. □

10. **Centers of minimal exponent.** Another closed subvariety that is amenable to the same techniques as above is the center of minimal exponent Y of Θ, defined by Schnell-Yang [SY, §4.2]. From the point of view of this paper, Y plays a role similar to that of the zero-scheme Z of the ideal $I_1(\Theta)$, but it is not hard to check that in general they are different. This is also the place to note that some of the arguments in this paper can be equally applied to the zero-schemes of the higher multiplier ideals of *loc. cit.*; they are indeed considered in [HJ].

It is shown in [SY, Theorem 1.12] that if $\dim Y = 1$, then Y must be a hyperelliptic curve, and if moreover there exists a component of the maximal allowed dimension $g - 2m + 1$ in some $\Sigma_m(\Theta)$, then A is the Jacobian of Y. Please see the interesting discussion in §1.6 of *loc. cit.* for the significance of such a statement. Using the discussion in the paragraph above, we have:

Corollary 10.1. *Assume that* $\dim Y \geq 1$*, and the minimal exponent of* Θ *satisfies* $1 < \widetilde{\alpha}(\Theta) < 2$*. If Conjecture B holds, then* (A, Θ) *is the Jacobian of a hyperelliptic curve.*[6]

In particular, this holds unconditionally if $1 \leq \dim Y \leq 2$*, or if* $\dim Y = g - 3$*.*

Proof. In the proof of [SY, Lemma 5.5] it is shown that under this minimal exponent hypothesis $\mathcal{I}_Y(2\Theta)$ satisfies IT_0, so the methods we developed here apply: by Theorem A, $\mathcal{I}_Y(\Theta)$ is GV, and Y is a geometrically nondegenerate (integral, normal) closed subvariety of A.

We obviously have $Y \subseteq \mathrm{Sing}(\Theta)$, and therefore $1 \leq \dim Y \leq g - 3$. Assuming Conjecture B, we deduce that since Y is positive dimensional, (A, Θ) must be the Jacobian of a curve C. The conjecture also says that Y must be $+$ or $-$ a translate of $W_d(C)$ where $d = \dim Y$. But we may also assume that $Y = -Y$, by choosing Θ to be symmetric, as the construction of the center of minimal exponent is canonical. It is however well known that if this property holds for some $W_d(C)$, then C must be hyperelliptic; see also [De1, Lemma 5.5] for a more general statement.

The unconditional statement refers to the cases of the GV-conjecture we know to hold, as everywhere else in the paper. □

References

[ACP] E. Arbarello, G. Codogni, G. Pareschi, *Characterizing Jacobians via the KP equation and via flexes and degenerate trisecants to the Kummer variety: an algebro-geometric approach* J. reine angew. Math. **777** (2021), 251–271.

[Be] A. Beauville, *Prym varieties and the Schottky problem*, Invent. Math. **41** (1977), 149–196.

[BL] Ch. Birkenhake and H. Lange, *Complex abelian varieties*, Second, Grundlehren der mathema- tischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302, Springer-Verlag, Berlin, 2004.

[Ca1] S. Casalaina-Martin, *Singularities of theta divisors in algebraic geometry*, In Curves and abelian varieties, Contemp. Math. **465**, 25–43. Amer. Math. Soc., Providence, RI, 2008.

[Ca2] S. Casalaina-Martin, *Singularities of the Prym theta divisor*, Ann. of Math. (2) **170** (2009), no.1, 162–204.

[CMPS] S. Casalaina-Martin, M. Popa, and S. Schreieder, *Generic vanishing and minimal cohomology classes on abelian fivefolds*, J. Algebraic Geom. **27** (2018), 553–581.

[Ch] Y. Chen, *Generic vanishing subschemes of codimension three*, preprint arXiv:2609.15391.

[De1] O. Debarre, *Minimal cohomology classes and Jacobians*, J. Algebraic Geom. **4** (1995), 321–335.

[De2] O. Debarre, *Update on: Singularities of divisors on abelian varieties*, lecture notes at https://perso.imj-prg.fr/olivier-debarre/exposes-talks/.

[EL] L. Ein and R. Lazarsfeld, *Singularities of theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), no.1, 243–258.

[Gr] S. Grushevsky, *The Schottky problem*, In *Current developments in algebraic geometry*, Math. Sci. Res. Inst. Publ. **59**, 129–164. Cambridge Univ. Press, Cambridge, 2012.

[HJ] S. Huang and Z. Jiang, *GV-subschemes and hyperelliptic Jacobians*, preprint.

[JP] Z. Jiang and G. Pareschi, *Cohomological rank functions on abelian varieties*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 4, 815–846.

[Ko] J. Kollár, *Shafarevich maps and automorphic forms*, M.B. Porter Lectures, Princeton Univ. Press, 1995.

[LT] L. Lombardi and S. Tirabassi, *GV-subschemes and their embeddings in principally polarized abelian varieties*, Math. Nachr. **288** (2015), no. 11-12, 1405–1412.

[Me] F. Meng, *Generic vanishing conjecture for divisors*, preprint arXiv:2609.11377.

[6] It is shown in [SY] that every Jacobian satisfies $1 < \widetilde{\alpha}(\Theta) \leq 2$, while for Brill-Noether general curves we have $\widetilde{\alpha}(\Theta) = 2$.

[Muk] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[Mu] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[MP] M. Mustaţă and M. Popa, *Hodge ideals*, Mem. Amer. Math. Soc. **262** (2019), no.1268.

[Pa] G. Pareschi, *Singularities of base loci on abelian varieties*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **36** (2025), no. 2, 365–375.

[PP1] G. Pareschi and M. Popa, *Regularity on abelian varieties I*, J. Amer. Math. Soc. **16** (2003), 285–302.

[PP2] G. Pareschi and M. Popa, *Regularity on abelian varieties II: Basic results on linear series and defining equations*, J. Algebraic Geom. **13** (2004), no. 1, 167–193.

[PP3] G. Pareschi and M. Popa, *Generic vanishing and minimal cohomology classes on abelian varieties*, Math. Ann. **340** (2008), 209–222.

[PP4] G. Pareschi and M. Popa, *Regularity on abelian varieties III: relationship with Generic Vanishing and applications*, in *Grassmannians, Moduli Spaces and Vector Bundles* 2008, Clay Mathematics Proceedings **14**, Amer. Math. Soc., Providence RI, 2011, 141–167.

[Po] M. Popa, *Connections between some conjectures on subvarieties of abelian varieties*, Oberwolfach Report 33/2018, 1993–1995.

[Ra] Z. Ran, *On subvarieties of abelian varieties*, Invent. Math. **62** (1981), 459–479.

[Sa] M. Saito, *Hodge ideals and microlocal V-filtration*, preprint arXiv:1612.08667.

[Sch] S. Schreieder, *Decomposable theta divisors and generic vanishing*, Int. Math. Res. Not. IMRN **16** (2017), 4984–5009.

[SY] C. Schnell and R. Yang, *Higher multiplier ideals*, J. reine angew. Math. **832** (2026), 153–197.

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