Linear Series on Moduli Spaces of Vector Bundles on Curves

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2001

ACKNOWLEDGEMENTS

It has been an extraordinary privilege, both mathematically and personally, to be a student of Rob Lazarsfeld. Anyone who might ever appreciate something I do should correctly imagine that it is based on what I learned from him. I have also had an opportunity that I believe cannot be equaled, that is to meet and learn from Bill Fulton and Igor Dolgachev, who have always given me great advice and help. To say anything more would be useless, since each one of them knows how much I respect and owe them.

For the material described in this thesis I have benefitted from discussions with I. Dolgachev, W. Fulton, R. Lazarsfeld, M. S. Narasimhan, G. Pareschi, M. Roth and M. Teixidor I Bigas.

I would also like to thank very much my teachers in Bucharest: Lucian Bădescu, Iustin Coandă and Paltin Ionescu. Without them I would have never become an algebraic geometer.

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CHAPTER I

Introduction

The present work developes a geometric study of linear series and generalized theta functions on moduli spaces of vector bundles on curves, with the aim of understanding effective numerical statements in the spirit of higher dimensional geometry. It is essentially a combination of results contained in the papers [36], [37], [38], with references to further work in [39]. We give effective base point freeness and projective normality bounds for pluritheta linear series, as well as dimension bounds for the base loci of determinant linear series. This study has an "abelian" and a "nonabelian" part. For the "nonabelian" part the main technique is focused on giving upper bounds on the dimension of Quot schemes. On the other hand, from the "abelian" point of view, we introduce the notion of Verlinde bundle on the Jacobian of a curve, and study this type of bundle with methods specific to vector bundles on abelian varieties. As a by-product of these techniques we obtain a new global picture on duality for generalized theta functions and we formulate some further conjectures.

Let X be a smooth projective curve of genus $g \geq 2$ over the field of complex numbers. Our main objects of interest will be the moduli space $SU_X(r)$ of semistable rank r vector bundles with trivial determinant, and the moduli space $U_X(r,0)$ of semistable bundles of rank r and degree 0 on C. It is well understood what linear series live on these spaces, by work on Drezet-Narasimhan [9]. Concretely, the Picard group of $SU_X(r)$ is isomorphic to $\mathbf{Z}$, generated by an ample line bundle $\mathcal{L}$ called the *determinant bundle.* Also, the Picard group of $U_X(r,0)$ is isomorphic to $\mathbf{Z}\cdot\mathcal{O}(\Theta)\oplus\pi^*\mathrm{Pic}(J(X))$, where $\pi : U_X(r,0) \to J(X)$ is the determinant map, while Θ is a *generalized theta divisor*, which depends on the choice of a line bundle of degree $g-1$ on X (and such that $\mathcal{O}(\Theta)$ restricts to $\mathcal{L}$ on $SU_X(r)$). Results analogous to what we will describe below hold in fact on the more general spaces $SU_X(r,L)$ or $U_X(r,d)$, where the degree d is arbitrary (and L is a fixed line bundle of degree d), and we will also consider these.

In order to understand the geometry of these moduli spaces, a first step is to understand facts like the effective base point freeness, very ampleness or projective normality of the linear series involved. It is well understood that the best conceivable statements, like the global generation of $\mathcal{L}$ for any rank r, do not hold (cf. [41], [2]), and in fact there is strong evidence suggesting that $|\mathcal{L}^k|$ should be badly behaved when k is small with respect to r (cf. [36]). The particular shape of the Picard group implies that the base point freeness of $|\mathcal{L}^k|$ is in fact a Fujita type problem for adjoint linear series. Previous work in this direction appears in papers of Le Potier [26] and Hein [17], where the authors obtain bounds for the effective base point freeness of $|\mathcal{L}^k|$ with order of magnitude in the range suggested by Fujita's conjecture. On the other hand, the problem of very ampleness or projective normality of $\mathcal{L}$ has been treated in the literature only in the case of $SU_X(2)$ (or $SU_X(2,1)$) in a series of papers, e.g. [6], [24], [14], via methods specific to this particular case. On the "negative" side, it is certainly interesting to understand what causes bad behavior (like the existence of base points for $|\mathcal{L}|$) when this occurs. This is intimately related to the geometric form of a central conjecture in the subject, called the Strange Duality conjecture (cf. e.g. [8]). This circle of ideas, and the specific questions involved, is described in Beauville's survey [2], which provided in fact the main inspiration for us at the beginning of this work.

We turn to a brief description of the content of the specific chapters of the thesis. The beginning of each chapter contains more detailed background and motivation for the explicit problems involved there, and so this should be treated only as a preliminary introduction.

After a brief description of the main objects involved, given in the second chapter, we begin in Chapter 3 with a construction of new families of base points for the linear series $|\mathcal{L}|$ on $SU_X(r)$ for sufficiently large r. These show in particular that the dimension of the base locus has to grow as a function of the rank r. We also explain why the Strange Duality conjecture implies that the linear series $|\mathcal{L}^k|$ should still have base points for k small with respect to r.

In order to approach the problem of effective base point freeness for multiples of $\mathcal{L}$, we first treat in Chapter 4 a problem of significant interest in its own right, namely that of bounding the dimension of Quot schemes associated to arbitrary vector bundles on curves. This is achieved here via elementary methods, but we mention that in recent work [39] with M. Roth we give an essentialy optimal solution to this problem by considering compactifications via moduli spaces of stable maps (we will mention this briefly in the text). This dimension bound implies base point freeness bounds for multiples of the determinant line bundle by making effective a method originating in [11], namely generation via generalized theta divisors. This is the subject of Chapter 5, where we also give an application to Donaldson determinant line bundles on moduli spaces of sheaves on surfaces, following a technique of Le Potier [26].

Turning to the study of the generalized theta line bundles $\mathcal{O}(k\Theta)$ on the bigger moduli spaces $U_X(r,0)$, we note that this time there is also an "abelian" component of the problem, given by the natural determinant map $\det : U_X(r,0) \to J(X)$. In Chapter 6 we introduce a class of vector bundles on the Jacobian $J(X)$ naturally associated as the push-forwards of the line bundles above. We call these Verlinde bundles, as their fibers are isomorphic to the Verlinde vector spaces of nonabelian theta functions. We begin with a study of general properties of these bundles, as they turn out to be interesting for a number of different reasons. We continue in this chapter by showing how the Verlinde bundles can be used to globalize and understand certain dualities between nonabelian theta functions from a new perspective. In addition, we explain how they can be used to produce new examples for the determinant linear series, generalizing previous work of Raynaud.

In Chapter 7 we use the Verlinde bundles and some new techniques in theory of vector bundles on abelian varities, develped by Pareschi, to give a solution to the problem of effective global generation and projective normality for the linear series $|k\Theta|$ on $U_X(r,0)$. Inspired by some numerical results that appear naturally in this study, we formulate some further conjectures generalizing classical results on Jacobians. We also explain analogous results in the case of arbitrary degree vector bundles.

CHAPTER II

Generalized theta divisors on moduli spaces of vector bundles

2.1 Moduli spaces of vector bundles

In this section we review a few facts about moduli spaces of vector bundles on curves. General references for this are the texts [44], [27] or [19]. We will always work over the field of complex numbers.

Let X be a smooth projective complex curve of genus $g \geq 2$. Recall that for a vector bundle E of rank r and degree $d = c_1(E)$, the *slope* of E is the number

$$\mu(E) := d/r.$$

The vector bundle E is called *semistable* (respectively *stable*) if for any subbundle $F \subset E$ one has

$$\mu(F) \leq \mu(E) \text{ (respectively } \mu(F) < \mu(E)).$$

Every semistable bundle has a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E,$$

where the factors E_i/E_{i-1} are all stable bundles of slope equal to $\mu(E)$. Such a filtration is called a *Jordan-Hölder filtration*. Given one such, the vector bundle

$$\mathrm{gr}(E) := \bigoplus E_i/E_{i-1}$$

is called the *associated graded bundle* of E. Note that although there may exist different Jordan-Hölder filtrations of E, the associated graded bundle does not depend on such a choice. Two semistable bundles E and F are called S-*equivalent* if $\mathrm{gr}(E) \cong \mathrm{gr}(F)$.

Semistability is precisely the notion that allows one to organize vector bundles into moduli spaces. More precisely, there exists a moduli space $U_X(r,d)$ parametrizing S-equivalence classes of semistable bundles of rank r and degree d. One also considers the smaller moduli spaces $SU_X(r,L)$ of S-equivalence classes of semistable bundles of rank r and fixed determinant $L = \wedge^r E \in \mathrm{Pic}^d(X)$. The most familiar case is that of trivial determinant, corresponding to the existence of a reduction of structure group to SL_r, when the corresponding moduli space is denoted simply $SU_X(r)$. Also, when the choice of determinant is of no special importance, we will use the notation $SU_X(r,d)$.

Remark 2.1.1. It is clear that every stable bundle coincides with its associated graded bundle, thus in this case S-equivalence is the same as isomorphism. On the other hand all semistable bundles are automatically stable if and only if the rank and the degree are coprime. Thus the moduli spaces $U_X(r,d)$ with $(r,d) = 1$ parametrize precisely isomorphism classes of stable bundles. It is a fact (see e.g. [44]) that these are exactly those moduli spaces which are fine, i.e. have a universal family (with the exception of the special case of rank 2 bundles in genus 2).

We next list some of the most important properties of these moduli spaces, leaving the study of their linear series for the next section. We will refer only to $U_X(r,d)$, but the exact same properties hold for $SU_X(r,d)$. The proofs of these facts can be found in the references mentioned above.

- $U_X(r,d)$ is a normal projective variety with rational singularities.
- The smooth locus of $U_X(r,d)$ is the open subset $U_X^s(r,d)$ consisting of isomorphism classes of stable bundles (except in the case $g=2$ and $r=2$, when it is the whole moduli space).
- The singular (i.e. semistable) locus of $U_X(r,d)$ has codimension at least 2.
- $U_X(r,d)$ has a universal bundle if and only if $(r,d)=1$.

2.2 Generalized theta divisors

For the simplicity of the notations involved, we consider here only $U_X(r,0)$, i.e. the moduli space of semistable vector bundles of rank r and degree 0 on X, and $SU_X(r,L)$, the moduli space of vector bundles with fixed determinant $L \in \mathrm{Pic}^0(X)$. Let also $U_X^s(r,0)$ and $SU_X^s(r,L)$ be the open subsets corresponding to stable bundles. We recall the construction and some basic facts about generalized theta divisors on these moduli spaces, drawing especially on [9]. Analogous constructions work for any degree d, as we will recall in Section 7.2.

Let N be a line bundle of degree $g-1$ on X. Then $\chi(E \otimes N) = 0$ for all $E \in U_X(r,0)$. Consider the subset of $U_X^s(r,0)$

$$\Theta_N^s := \{E \in U_X^s(r,0) |\ h^0(E \otimes N) \neq 0\}$$

and the analogous set in $SU_X^s(r,L)$. One can prove (see [9] (7.4.2)) that Θ_N^s is a hypersurface in $U_X^s(r,0)$ (resp. $SU_X^s(r,L)$). Denote by Θ_N the closure of Θ_N^s in $U_X(r,0)$ and $SU_X(r,L)$. As we vary N, these hypersurfaces are called *generalized theta divisors*. It is proved in [9], Theorem A, that $U_X(r,0)$ and $SU_X(r,L)$ are locally factorial and so the generalized theta divisors determine line bundles $\mathcal{O}(\Theta_N)$ on these moduli spaces. We have the following important facts:

Theorem 2.2.1. *([9], Theorem B) The line bundle $\mathcal{O}(\Theta_N)$ on $SU_X(r,L)$ does not depend on the choice of N. The Picard group of $SU_X(r,L)$ is isomorphic to $\mathbf{Z}$, generated by $\mathcal{O}(\Theta_N)$.*

The line bundle in the theorem above, independent of the choice of N, is denoted by $\mathcal{L}$ and is called the *determinant bundle*.

Theorem 2.2.2. *([9], Theorem C) The inclusions $\mathrm{Pic}(J(X)) \subset \mathrm{Pic}(U_X(r,0))$ (given by the determinant morphism) and $\mathbf{Z}\cdot\mathcal{O}(\Theta_N) \subset \mathrm{Pic}(U_X(r,0))$ induce an isomorphism*

$$\mathrm{Pic}(U_X(r,0)) \cong \mathrm{Pic}(J(X)) \oplus \mathbf{Z}.$$

More generally, for any vector bundle F of rank k and degree $k(g-1)$, we can define

$$\Theta_F^s := \{E \in U_X^s(r,0) |\ h^0(E \otimes F) \neq 0\}$$

and denote by Θ_F the closure of Θ_F^s in $U_X(r,0)$. It is clear that, for generic F at least, Θ_F is strictly contained in $U_X(r,0)$ (in which case it is again a divisor). It is useful to know what is the dependence of $\mathcal{O}(\Theta_F)$ on F and in this direction we have:

Proposition 2.2.3. *([9] (7.4.3) and [8] Prop.3) Let F and G be two vector bundles of slope $g-1$ on X. If $\mathrm{rk}(F) = m \cdot \mathrm{rk}(G)$, then*

$$\mathcal{O}(\Theta_F) \cong \mathcal{O}(\Theta_G)^{\otimes m} \otimes \det{}^*(\det F \otimes (\mathrm{detG})^{\otimes -m})$$

where we use the natural identification of $\mathrm{Pic}^0(X)$ with $\mathrm{Pic}^0(J(X))$. In particular, if F has rank k and N is a line bundle of degree $g-1$, we get

$$\mathcal{O}(\Theta_F) \cong \mathcal{O}(\Theta_N)^{\otimes k} \otimes \det{}^*(\det F \otimes N^{\otimes -k}).$$

2.3 A convention on theta divisors

When looking at generalized theta divisors Θ_N and the corresponding linear series, it will be convenient to consider the line bundle N to be a *theta characteristic*, i.e. satisfying $N^{\otimes 2} \cong \omega_X$. The assumption brings some simplifications to most of the arguments, but on the other hand this case implies all the results for arbitrary N. This is true since for any N and M in $\mathrm{Pic}^{g-1}(X)$, if $\xi := N \otimes M^{-1}$, twisting by ξ gives an automorphism:

$$U_X(r,0) \xrightarrow{\otimes \xi} U_X(r,0)$$

by which $\mathcal{O}(\Theta_M)$ corresponds to $\mathcal{O}(\Theta_N)$. As an example, we will use freely isomorphisms of the type $r_J^*\mathcal{O}_J(\Theta_N) \cong \mathcal{O}_J(r^2\Theta_N)$, where r_J is multiplication by r on $J(X)$, which in general would work only up to numerical equivalence. One can easily see that in each particular proof the arguments could be worked out in the general situation with little extra effort (the main point is that the cohomological arguments work even if we use numerical equivalence instead of linear equivalence).

It is worth mentioning another convention about the notation that we will be using. The divisors Θ_N make sense of course on both $U_X(r,0)$ for $r \geq 2$ and $J(X) = U_X(1,0)$ and in some proofs both versions will be used. We will denote the associated line bundle simply by $\mathcal{O}(\Theta_N)$ if Θ_N lives on $U_X(r,0)$ and by $\mathcal{O}_J(\Theta_N)$ if it lives on the Jacobian.

CHAPTER III

Base points for generalized theta linear series

3.1 Background and statement of main results

Let X be a smooth projective complex curve of genus $g \geq 2$. In his survey [2], A. Beauville raises a few questions about the base locus of the linear system $|\mathcal{L}|$, where $\mathcal{L}$ is the theta (or determinant) bundle on the moduli space $SU_X(r)$ of semistable rank r vector bundles on X of trivial determinant. It is known (see [2], §3) that $E \in SU_X(r)$ is a base point for $|\mathcal{L}|$ if and only if $H^0(E \otimes L) \neq 0$ for every $L \in \mathrm{Pic}^{g-1}(X)$[1]. When $r = 2$, or $r = 3$ and X is of genus 2 or generic of any genus, it is known that $|\mathcal{L}|$ is base point free. However, M. Raynaud constructs in [41] examples of bundles which lead to the existence of base points of $|\mathcal{L}|$ for $r = n^g$, where n is an integer ≥ 2 dividing g. His construction gives finitely many base points in each genus. Among other things, Beauville asks in [2] if one could find new examples of such base points and if the base locus is actually of strictly positive dimension.

The purpose of this chapter is to give a positive answer to Beauville's question. From a qualitative point of view, our results can be summarized in the following:

[1]When E is semistable, this should be interpreted as a statement about its equivalence class: by a common argument using the Jordan-Hölder filtration of E and the fact that $\chi(E \otimes L) = 0$, it is easy to see that $H^0(E \otimes L) \neq 0, \forall L \in \mathrm{Pic}^{g-1}(X)$ iff $H^0(gr(E) \otimes L) \neq 0, \forall L \in \mathrm{Pic}^{g-1}(X)$. So this property does not depend on the choice of a bundle in the class of E.

Theorem 3.1.1. *(a) For every $g \geq 2$, there exists a rank $\rho(g)$ such that for all $r \geq \rho(g)$ the linear system $|\mathcal{L}|$ on $SU_X(r)$ has base points. Also, for every $g \geq 2$ there exist ranks where some base points are stable.*

(b) Moreover, for every $g \geq 2$ and every $k \geq 2$, there exists an integer $\rho(k,g)$ such that for all $r \geq \rho(k,g)$, the base locus of $|\mathcal{L}|$ on $SU_X(r)$ has dimension at least $(k-1)g$.

For the examples and for more precise numerical statements see Section 3.2.

Another question raised by Beauville adresses the freeness of $|\mathcal{L}^2|$, or more generally of low multiples of $\mathcal{L}$. In the spirit of [8], one could also look at other moduli spaces, not necessarily of trivial determinant. We remark, by using a theorem of Lange and Mukai-Sakai (see [22],[28]), that the global generation of low multiples of the theta divisor on such moduli spaces cannot go hand in hand with the strange duality conjecture (see [2], §8).

3.2 Base points for the determinant linear series

Consider a line bundle L on X of degree $d \geq 2g+1$. We will restrict to the case $g \geq 2$, since for $g \leq 1$ the space $SU_X(r)$ is well understood. Denote by M_L the kernel of the evaluation map:

$$0 \longrightarrow M_L \longrightarrow H^0(L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

and let $Q_L = M_L^*$. These vector bundles are well known for their importance in the study of the minimal resolution of X in the embedding defined by L (see [25], §1 for a survey).

Among the properties of Q_L, we quote from [25], §1.4, the following: if $x_1, \dots, x_d$ are the points of a generic hyperplane section of $X \subset \mathbb{P}(H^0(L))$, then Q_L sits in an extension:

$$0 \longrightarrow \bigoplus_{i=1}^{d-g-1} \mathcal{O}_X(x_i) \longrightarrow Q_L \longrightarrow \mathcal{O}_X(x_{d-g} + \ldots + x_d) \longrightarrow 0$$

This induces for every integer p an inclusion:

$$0 \longrightarrow \bigoplus_{1 \leq i_1 < \ldots < i_p \leq d-g-1} \mathcal{O}_X(x_{i_1} + \ldots + x_{i_p}) \longrightarrow \bigwedge^p Q_L \tag{3.1}$$

Recall also from [10], §3 that Q_L is stable, and so $\bigwedge^p Q_L$ is poly-stable (i.e. a direct sum of stable bundles of the same slope).

Definition. Similarly to a definition in [41], we say that a vector bundle E satisfies property $(*)$ if and only if:

$$H^0(E \otimes \xi) \neq 0, \forall \xi \in \mathrm{Pic}^0(X).$$

In all that follows we will denote $\gamma := [\frac{g+1}{2}]$.

Proof of Theorem 3.1.1(a). Notice first that to find a base point for $|\Theta|$ it is enough to exhibit a semistable bundle E of integral slope $0 \leq \mu(E) \leq g-1$ satisfying $(*)$, since we could then twist by a suitable line bundle.

Claim: *For every line bundle L on X of degree $d \geq 2g+1$, the bundle $\bigwedge^\gamma Q_L$ satisfies property $(*)$.*

Proof of claim. From (3.1) it is clear that for $x_1, \ldots, x_p$ general points on X we have $H^0(\bigwedge^p Q_L(-x_1 - \ldots - x_p)) \neq 0$. So for any p, a generic line bundle $\xi \in \mathrm{Pic}^0(X)$ of the form $\xi = \mathcal{O}_X(A_p - B_p)$, with A_p, B_p generic effective divisors of degree p, satisfies $H^0(\bigwedge^p Q_L \otimes \xi) \neq 0$.

On the other hand it is well known (see [1]) that every $\xi \in \mathrm{Pic}^0(X)$ can be written in the form $\xi = \mathcal{O}_X(A_\gamma - B_\gamma)$ with A_γ, B_γ effective divisors of degree γ.

Hence $H^0\left(\bigwedge^\gamma Q_L \otimes \xi\right) \neq 0$ for a general ξ and by semicontinuity the same must hold for every $\xi \in \mathrm{Pic}^0(X)$, which proves the claim.

So, as noted above, it is enough to get integral slopes for $\bigwedge^\gamma Q_L$ for suitable choices of d. Since the computations differ from case to case and tend to get messy, we will restrict to giving examples that work uniformly rather then trying to find the smallest possible rank for each genus.

We can obtain an uniform answer by choosing $d = g(\gamma+1)$, when we get $\mu\left(\bigwedge^\gamma Q_L\right) = \gamma + 1$. The corresponding rank will be $rk\left(\bigwedge^\gamma Q_L\right) = \binom{g\cdot\gamma}{\gamma}$ (actually for most g's this is by no means the best answer).

It is easy to see that since the bundles $\bigwedge^\gamma Q_L$ are poly-stable and satisfy $(*)$, at least one of their stable summands (which have the same slope) must also satisfy $(*)$. Thus the constructions above also give us examples of stable base points in each genus. On the other hand, the existence of a base point $E \in SU_X(r)$ induces the existence of decomposable base points for every rank $r' \geq r$: simply take $E \oplus \mathcal{O}_X^{\oplus(r'-r)}$. □

Remark: There are many versions of this construction that give additional examples. Let us just mention them without getting into numerology. One could look at $\bigwedge^p Q_L$ for $p > \gamma$ such that $\mu\left(\bigwedge^p Q_L\right) \leq g-1$ or work with $S^p Q_L$ instead of $\bigwedge^p Q_L$. It is probably most interesting though to replace Q_L by Q_E, where E is a semistable bundle of slope $\mu(E) > 2g$ (so automatically very ample) and Q_E is defined exactly as Q_L. By [7] Q_E is known to be semistable and a closer analysis shows that a result analogous to the claim above holds. Using this construction one can check that by good numerical choices we can make $\bigwedge^\gamma Q_E$ have any integral slope $[\frac{g+1}{2}] < \mu \leq g-1$.

The additional feature that makes these examples interesting is that they come in positive dimensional families (roughly speaking by varying L), so in the range covered by them the base locus is indeed positive dimensional.

Proof of Theorem 3.1.1(b). The following is the more precise statement referred to in the introduction:

Claim: *Fix $g \geq 2$, $d \geq 2g+1$, $k \geq 2$ and let L be any line bundles of degree kd. Then there exists a $(k-1)g$ dimensional family of (equivalence classes of) semistable bundles of rank $\binom{k(d-g)}{\gamma}$ and fixed determinant $L^{\otimes \binom{k(d-g)-1}{\gamma-1}}$ satisfying property $(*)$.*

Proof of claim. Fix L of degree kd. To every $k-1$ line bundles $L_1, \dots, L_{k-1} \in \mathrm{Pic}^d(X)$ associate $L_k := L \otimes L_1^* \otimes \dots \otimes L_{k-1}^* \in \mathrm{Pic}^d(X)$, so that $L_1 \otimes \dots \otimes L_k = L$. Set $F_{L_1,\dots,L_{k-1}} := L_1 \oplus \dots \oplus L_k$. Thus $\det(F_{L_1,\dots,L_{k-1}}) = L$ and clearly $Q_{F_{L_1,\dots,L_{k-1}}} = Q_{L_1} \oplus \dots \oplus Q_{L_k}$ (cf. the remark above for the definition).

It is enough to prove that the morphism:

$$\psi: \ \mathrm{Pic}^d(X) \times \dots \times \mathrm{Pic}^d(X) \ \longrightarrow \ SU_X\left(\binom{k(d-g)}{\gamma}, L^{\otimes \binom{k(d-g)-1}{\gamma-1}}\right)$$

$$(L_1, \dots, L_{k-1}) \quad \rightsquigarrow \quad \bigwedge^{\gamma} Q_{F_{L_1,\dots,L_{k-1}}}$$

is finite. We have:

$$\bigwedge^{\gamma} Q_{F_{L_1,\dots,L_{k-1}}} \cong \bigoplus_{i_1+\dots+i_k=\gamma} \left(\bigwedge^{i_1} Q_{L_1} \otimes \dots \otimes \bigwedge^{i_k} Q_{L_k} \right)$$

In particular $\bigwedge^{\gamma} Q_{F_{L_1,\dots,L_{k-1}}}$ is poly-stable of slope $\gamma \cdot \frac{d}{d-g}$.

Assume now that:

$$\bigwedge^{\gamma} Q_{F_{L_1,\dots,L_{k-1}}} \cong \bigwedge^{\gamma} Q_{F_{L'_1,\dots,L'_{k-1}}}$$

for some other $L'_1, \dots, L'_k$ as before. By the previous formula one has inclusions:

$$\bigwedge^{\gamma} Q_{L'_i} \hookrightarrow \bigwedge^{\gamma} Q_{F_{L'_1,\dots,L'_{k-1}}}.$$

As noted before, all the bundles above are poly-stable (of the same slope), so $\bigwedge^\gamma Q_{L'_i}$ is a direct sum of some collection of the stable summands of $\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$. There are finitely many ways in which this can occur, so it is enough then to notice that the morphism:

$$\begin{array}{cccc} \phi: & \mathrm{Pic}^d(X) & \longrightarrow & U_X\left(\binom{d-g}{\gamma}, \gamma\cdot\frac{d}{d-g}\cdot\binom{d-g}{\gamma}\right) \\ & M & \rightsquigarrow & \bigwedge^\gamma Q_M \end{array}$$

is finite. This is clear since $\det(Q_M)=M$ and the claim is proved.

Taking in particular $d = g(\gamma+1)$ as in part (a), we get that the bundles $\bigwedge^\gamma Q_{F_{L_1,\ldots,L_{k-1}}}$ have integral slope $\gamma+1$ and so they lead to base points as before. Hence we can take $\rho(k,g) = \binom{k\cdot g\cdot\gamma}{\gamma}$.

This argument actually gives a statement about equivalence classes, since the bundles in the family that we have constructed are all poly-stable. Again, by adding trivial bundles we get the same statement in all ranks $r \geq \rho(k,g)$. □

One could conjecture that the base locus is at least g-dimensional whenever it is non empty. Perhaps in view of the remark above an even more optimistic guess could be made.

3.3 Strange duality versus freenes of low multiples of theta

In connection with Beauville's question about low multiples of $\mathcal{L}$, we show that the strange duality conjecture implies the existence of base points on $|\mathcal{L}^k|$ for small k, on suitable moduli spaces.

Consider first, in general, the moduli space $SU_X(r,A)$ for some $A \in \mathrm{Pic}^m(X)$, $m \in \mathbb{N}$, $m \leq g-1$. Consider also $F \in U_X(k, k(g-1-m))$ and define Θ_F on $SU_X(r,A)$ to be $\Theta_F = \tau_F^*\Theta$, where τ_F is the map :

$$\begin{aligned} \tau_F : SU_X(r,A) &\longrightarrow U_X(kr, kr(g-1)) \\ E &\rightsquigarrow E \otimes F \end{aligned}$$

and Θ is the canonical theta divisor on $U_X(kr, kr(g-1))$. Set theoretically of course $\Theta_F = \{E \mid H^0(E \otimes F) \neq 0\}$. The famous strange duality conjecture, discussed at length in [8], or more precisely its geometric formulation (see [2], §8), asserts that the linear system $|\mathcal{L}^k|$ on $SU_X(r,A)$ is spanned by the divisors Θ_F as F varies in $U_X(k, k(g-1-m))$.

Let us consider in particular E to be one of Raynaud's examples (see [41], §3) i.e. a bundle E with $rk(E) = n^g$, $\mu(E) = \frac{g}{n}$, $n|g$ which satisfies $(*)$. A theorem of Lange and Mukai-Sakai (see [22], [28]) implies that every $F \in U_X(k, k(g-1-\frac{g}{n}))$ has a subbundle $M \hookrightarrow F$ of degree $\deg(M) \geq \frac{k(g-1-\frac{g}{n})-(k-1)g}{k}$. Assume further that $\frac{g}{k} \geq 1+\frac{g}{n}$, which can be achieved for good choices of g and n. Then $m :=\deg(M) \geq 0$ and so, by property $(*)$:

$$H^0(E \otimes M) = H^0(E \otimes M(-m\mathrm{p}) \otimes O_X(m\mathrm{p})) \neq 0,$$

where p is a point on X. Since $M \hookrightarrow F$, we obtain:

$$H^0(E \otimes F) \neq 0 \text{ for all } F \in U_X\left(k, k\left(g-1-\frac{g}{n}\right)\right).$$

By the discussion above, the strange duality conjecture then implies that $|\mathcal{L}^k|$ on $SU_X(n^g, \det(E))$ has a base point at E.

Remarks: 1. The conclusion above suggests (assuming the strange duality conjecture is true!) that one should expect $|\mathcal{L}^k|$ to have base points, say for example for k small enough with respect to g or r, even extrapolating to $SU_X(r)$.

2. In the discussion above we cannot use the examples from the previous section instead of Raynaud's examples, since the condition $\deg(M) \geq 0$ is not necessarily

satisfied any more.

Let us conclude with another analogous application of the strange duality conjecture:

Consider $F_L = \bigwedge^{\gamma} Q_L$ from the previous section (of integral slope) and denote $A_L = \det(F_L) = L^{\otimes \binom{d-g-1}{\gamma-1}}$, $r' = rk(F_L) = \binom{d-g}{\gamma}$. Then exactly by the same argument as above, one can check that F_L is a base point for $|\mathcal{L}|$ on $SU_X(r', A_L)$ under mild assumptions on d. One can also apply the same argument for at least the Raynaud examples such that $g \geq n$ and $(g,n) \neq 1$ (in particular for those of integral slope).

CHAPTER IV

Dimension estimates for Quot schemes

4.1 Background and statement

It is a well established fact that the solutions of many problems involving families of vector bundles should essentially depend on good estimates for the dimension of the Quot schemes of coherent quotients of a given bundle. Deformation theory provides basic cohomological dimension bounds, but most of the time the cohomology groups involved are hard to estimate accurately and moreover do not provide bounds that work uniformly. On smooth algebraic curves, an optimal answer to this problem has been previously given only in the case of quotients of minimal degree by Mukai and Sakai. If E is a vector bundle of rank r and

$$f_k = f_k(E) := \min_{\mathrm{rk}Q=k} \{\deg Q |\ E \to Q \to 0\},$$

is the minimal degree of a quotient of E of rank k, they show in [28] that dim $\mathrm{Quot}_{k,f_k}(E) \leq k(r-k)$, where in general $\mathrm{Quot}_{k,d}(E)$ stands for Grothendieck's Quot scheme of coherent quotients of E of rank k and degree d.

The goal of this chapter is to give an upper bound for the dimension of $\mathrm{Quot}_{k,d}(E)$ for any degree d. The bound involves in an essential (and somewhat unexpected) way the invariant f_k. Examples provided in 4.2.13 show that it is optimal at least in the case corresponding to line subbundles.

Theorem 4.1.1. *Let E be an arbitrary vector bundle of rank r on a smooth projective curve X over an algebraically closed field. Then:*

$$\dim \text{Quot}_{k,d}(E) \leq k(r-k) + (d - f_k)(k+1)(r-k).$$

This generalizes (and uses) the result from [28] mentioned above, which is exactly the case $d = f_k$. The proof is based on induction on the difference $d - f_k$ and the key ingredient is a technique that allows one to "eliminate" all the minimal quotient bundles of E while preserving a fixed nonminimal one. This is achieved via elementary transformations along zero-dimensional subschemes of arbitrary length. The problem has also been previously given some precise answers in the particular case of generic stable bundles in [42] and [5]. In this case the dimension (and f_k) can be computed exactly (cf. Example 4.2.13 below).

In the next chapter we will apply the estimate above to the basic problem of giving effective bounds for the global generation of multiples of generalized theta line bundles on moduli spaces of vector bundles.

4.2 Dimension estimates via elementary transformations

The aim of this section is to prove Theorem 4.1.1 above giving an upper bound on the dimension of the Quot schemes of coherent quotients of fixed rank and degree of a given vector bundle. For the general theory of Quot schemes the reader can consult for example [26] §4.

Concretely, fix a vector bundle E of rank r and degree e on X and denote by $\text{Quot}_{r-k,e-d}(E)$ the Quot scheme of coherent quotients of E of rank $r-k$ and degree $e-d$. We can (and will) identify $\text{Quot}_{r-k,e-d}(E)$ to the set of subsheaves of E of rank k and degree d. Consider also:

$$d_k := \max\{\deg(F)|\ F \subset E, \mathrm{rk}(F) = k\}$$

and

$$M_k(E) = \{F|\ F \subset E, \mathrm{rk}(F) = k, \deg(F) = d_k\}$$

the set of maximal subbundles of rank k. Clearly any $F \in M_k(E)$ has to be a vector subbundle of E. Note that the number $e - d_k$ is exactly the minimal degree of a quotient bundle of E of rank $r - k$. By [28] §2 we have the following basic result:

Proposition 4.2.1. *The following hold and are equivalent:*

(i) dim $M_k(E) \leq k(r-k)$

(ii) for any $x \in X$ and any $W \subset E(x)$ k-dimensional subspace of the fiber of E at x, there are at most finitely many $F \in M_k(E)$ such that $F(x) = W$.

Part (i) above thus gives an upper bound for the dimension of the Quot scheme in the case $d = d_k$. The next result is a generalization in the case of arbitrary degree d, which turns out to give an optimal result (see Example 4.2.13 below). It is a restatement of Theorem 4.1.1 above in a form convenient for the proof.

Theorem 4.2.2. *With the notation above, we have:*

$$\dim \mathrm{Quot}_{r-k,e-d}(E) \leq k(r-k) + (d_k - d)k(r-k+1).$$

Remark 4.2.3. To avoid any confusion, we emphasize here that the notation is slightly different from that used in the background section, in the sense that we are replacing k by $r-k$, d by $e-d$ and f_k by $e-d_k$. This is done for consistency in rewriting everything in terms of subbundles, but note that the statement is exactly the same.

The proof will proceed by induction on the difference $d_k - d$. In order to perform this induction we have to use a special case of the notion of *elementary transformation* along a torsion sheaf suported on a zero-dimensional subscheme of arbitrary length. We call this construction simply elementary transformation since there is no danger of confusion.

Definition 4.2.4. Let τ be a torsion sheaf supported on a zero-dimensional subscheme of X whose reduced structure is given by the points $P_1, \ldots, P_s$. An *elementary transformation* of E along τ is a vector bundle E' defined by a sequence of the form:

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\phi} \tau \longrightarrow 0.$$

where the morphism ϕ is determined by giving surjective maps $E_{P_i} \overset{\phi_i}{\to} \mathbf{C}_{P_i}^{a_i}$ induced by specifying a_i distinct hyperplanes in $E(P_i)$ (whose intersection is the kernel of ϕ_i) , $\forall i \in \{1, \ldots, s\}$. We call $m = a_1 + \ldots + a_s$ the *length* of τ and a_i the *weight* of P_i.

Let us briefly remark that this is not the most general definition, since we are imposing a condition on the choice of hyperplanes. We prefer to work with this notion because it is sufficient for our purposes and allows us to avoid some technicalities. Note though that the space parametrizing these transformations is not compact. One could equally well work with the general definition, when the hyperplanes could come together, and obtain a compact parameter space, which can be shown to be irreducible (it is basically a Hilbert scheme of rank zero quotients of fixed length).

In fact it is an immediate observation that the elementary transformations of E of length m, in the sense of the definition above, are parametrized by $Y := (\mathbf{P}E)_m - \Delta$, where $(\mathbf{P}E)_m$ is the m-th symmetric product of the projective bundle $\mathbf{P}E$ and Δ is the union of all its diagonals. There is an obvious forgetful map

$$\pi : Y \longrightarrow X_m,$$

where X_m is the m-th symmetric product of the curve X. We will denote by $Y_m \subset (\mathbf{P}E)_m - \Delta$ the open subset $(\mathbf{P}E)_m - \pi^{-1}(\delta)$, where δ is the union of the diagonals in X_m. Its points correspond to the elementary transformations of length m supported at m distinct points of X.

Definition 4.2.5. Let V be a subbundle of E. An elementary transformation

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\phi} \tau \longrightarrow 0$$

is said to *preserve* V if the inclusion $V \subset E$ factors through the inclusion $E' \subset E$.

Lemma 4.2.6. *If E' is determined by the hyperplanes $H_i^1, \dots, H_i^{a_i} \subset E(P_i)$ for $i \in \{1, \dots, s\}$ and $V_i := \bigcap_{j=1}^{a_i} H_i^j$, then V is preserved by E' if and only if $V(P) \subset V_i$, $\forall i \in \{1, \dots, s\}$.*

Proof. We have an induced diagram

$$\begin{array}{ccccccccc} & & & & V & & & & \\ & & & & \downarrow & \searrow^{\alpha} & & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \xrightarrow{\phi} & \tau & \longrightarrow & 0 \end{array}$$

where α is the composition of ϕ with the inclusion of V in E. It is clear that E' preserves V if and only if α is identically zero. The lemma follows then easily from the definitions. □

In general it is important to know the dimension of the set of elementary transformations of a certain type preserving a given subbundle. This is given by the following simple lemma:

Lemma 4.2.7. *Let $V \subset E$ be a subbundle of rank k. Consider the set of elementary transformations of E along a zero dimensional subscheme of length m belonging to an irreducible subvariety W of $X_m - \delta$ that preserve V:*

$$\mathcal{Z}_V := \{E' |\ V \subset E'\} \subset \pi^{-1}(W),$$

where $\pi : Y_m \to X_m$ is the natural projection. Then $\mathcal{Z}_V$ is irreducible of dimension $m(r-k-1) + \dim W$.

Proof. An elementary transformation at m points $x_1, \ldots, x_m$ is given by a choice of hyperplanes $H_i \subset E(x_i)$ for each i. By the previous lemma, such a transformation preserves V if and only if $V(x_i) \subset H_i$ for all i. We have a natural diagram

$$\begin{array}{ccc} \mathcal{Z}_V & \overset{i}{\hookrightarrow} & \pi^{-1}(W) \\ & \searrow^{p} & \downarrow \pi \\ & & W \end{array}$$

where π is the restriction to $\pi^{-1}(W)$ and p is the composition with the inclusion of $\mathcal{Z}_V$ in $\pi^{-1}(W)$. For $D = x_1 + \ldots + x_m \in W$, we have:

$$p^{-1}(D) \cong \{(H_1, \ldots, H_m) |\ V(x_i) \subset H_i \subset E(x_i), \forall i = 1, \ldots, m\}$$

$$\cong \mathbf{P}^{r-k-1} \times \ldots \times \mathbf{P}^{r-k-1}$$

where the product is taken m times. So $p^{-1}(D)$ is irreducible of dimension $m(r-k-1)$ and this gives that $\mathcal{Z}_V$ is irreducible of dimension $m(r-k-1) + \dim W$. □

The following proposition will be the key step in running the inductive argument. It computes "how fast" we can eliminate all the maximal subbundles of E while preserving a fixed nonmaximal subbundle.

Proposition 4.2.8. *Let $V \subset E$ be a subbundle of rank k and degree d, not maximal. Then if $m \geq r - k + 1$, there exists an elementary transformation of length m*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \tau \longrightarrow 0$$

such that $V \subset E'$, but $F \not\subset E'$ for any $F \in M_k(E)$. In other words E' preserves V, but does not preserve any maximal F. If we fix a point $P \in X$, then τ can be chosen to have weight $m - 1$ at P and weight 1 at a generic point $Q \in X$.

Proof. Fix a point $P \in X$. We can consider an elementary transformation of E of length $r - k$, supported only at P:

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \tau \longrightarrow 0,$$

such that $\mathrm{Im}(E'(P) \to E(P)) = V(P)$. Then as in Lemma 4.2.6, the only maximal subbundles F that are preserved by this transformation are exactly those such that $F(P) = V(P)$. By Proposition 4.2.1 this implies that only at most a finite number of F's can be preserved.

If none of the maximal subbundles actually survive in E', then any further transformation at one point would do. Otherwise clearly for a generic $Q \in X$ we have $F(Q) \neq V(Q)$ for all the F's that are preserved and so we can choose a hyperplane $V(Q) \subset H \subset E(Q)$ such that $F(Q) \not\subset H$ for any such F. The elementary transformation of E' at Q corresponding to this hyperplane satisfies then the required property. □

Remark 4.2.9. (1) It can definitely happen that all the maximal subbundles are killed by elementary transformations of length less than $r - k + 1$ which preserve V. In any case, as it was already suggested in the proof above, by further elementary transforming we obviously don't change the property that we are interested in, so

$r - k + 1$ is a bound that works in all situations.

(2) By Lemma 4.2.7, the set $\mathcal{Z}_V$ of all elementary transformations of length m preserving V is irreducible of dimension $m(r-k)$. On the other hand the condition of preserving at least one maximal subbundle is closed, so once the lemma above is true for one elementary transformation, it applies for an open subset of $\mathcal{Z}_V$.

Finally we have all the ingredients necessary to prove the theorem. To simplify the formulations, it is convenient to introduce the following ad-hoc definition:

Definition 4.2.10. An irreducible component $\mathcal{Q} \subset \mathrm{Quot}_{r-k,e-d}(E)$ is called *non-special* if its generic point corresponds to a locally free quotient of E and *special* if all its points correspond to non-locally free quotients. For any $\mathcal{Q}$, denote by $\mathcal{Q}_0$ the open subset parametrizing locally free quotients and consider $\mathrm{Quot}^0_{r-k,e-d}(E) := \bigcup\limits_{\mathcal{Q}} \mathcal{Q}_0$.

Proof. (of Theorem 4.1.1) Denote by $\mathcal{Q}$ an irreducible component of $\mathrm{Quot}_{r-k,e-d}(E)$ (recall that we are thinking now of this Quot scheme as parametrizing subsheaves of rank k and degree d). The first step is to observe that it is enough to prove the statement when $\mathcal{Q}$ is non-special. To see this, note that every nonsaturated subsheaf $F \subset E$ determines a diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & \tau & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & G' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & \xrightarrow{\cong} & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where F' is the saturation of F, G is a quotient vector bundle and τ, the torsion subsheaf of G', is a nontrivial zero-dimensional subscheme, say of length a. We can stratify the set of all such F's according to the value of the parameter a, which obviously runs over a finite set. If we denote by $\{F\}_a$ the subset corresponding to a fixed a, this gives then:

$$\dim \{F\}_a \leq \dim \mathrm{Quot}^0_{r-k,e-d-a}(E) + ka.$$

The right hand side is clearly less than $k(r-k) + (d_k - d)k(r-k+1)$ if we assume that the statement of the theorem holds for $\mathrm{Quot}^0_{r-k,e-d-a}(E)$.

Let us then restrict to the case when $\mathcal{Q}$ is a non-special component. The proof goes by induction on $d_k - d$. If $d_k = d$, the statement is exactly the content of 4.2.1. Assume now that $d_k > d$ and that the statement holds for all the pairs where this difference is smaller. Recall that $\mathcal{Q}_0 \subset \mathcal{Q}$ denotes the open subset corresponding to vector subbundles and fix $V \in \mathcal{Q}_0$. Then by Proposition 4.2.8, there exists an elementary transformation

$$0 \longrightarrow E' \longrightarrow E \longrightarrow \tau \longrightarrow 0$$

of length $r-k+1$, such that $V \subset E'$, but $F \not\subset E'$ for any $F \in M_k(E)$. Then necessarily $d_k(E') < d_k(E) = d_k$ (consider the saturation in E of a maximal subbundle of E') and so $d_k(E') - d < d_k - d$. This means that we can apply the inductive hypothesis for any non-special component of the set of subsheaves of rank k and degree d of E'. To this end, consider the correspondence:

$$\begin{array}{ccccc} & \mathcal{W} & & = & \{(V,E')|\ V \subset E', F \not\subset E', \forall F \in M_k(E)\} \subset \mathcal{Q}_0 \times Y_{r-k+1} \\ {}^{p_1}\swarrow & & \searrow^{p_2} & & \\ \mathcal{Q}_0 & & Y_{r-k+1}. & & \end{array}$$

By Lemma 4.2.7 and Remark 4.2.9(b), for any $V \in \mathcal{Q}_0$, the fiber $p_1^{-1}(V)$ is a (quasi-projective irreducible) variety of dimension $(r-k+1)(r-k)$ and so:

$$\dim \mathcal{W} = \dim \mathcal{Q}_0 + (r-k+1)(r-k). \tag{4.1}$$

On the other hand, for $E' \in \mathrm{Im}(p_2)$, the inductive hypothesis implies that

$$\dim p_2^{-1}(E') \leq k(r-k) + (d_k(E') - d)k(r-k+1)$$

$$\leq k(r-k) + (d_k - d - 1)k(r-k+1)$$

and since $\dim Y_{r-k+1} = r(r-k+1)$ we have:

$$\dim \mathcal{W} \leq r(r-k+1) + k(r-k) + (d_k - d - 1)k(r-k+1). \tag{4.2}$$

Combining (4.1) and (4.2) we get:

$$\dim \mathcal{Q}_0 \leq k(r-k) + (d_k - d)k(r-k+1)$$

and of course the same holds for $\mathcal{Q} = \overline{\mathcal{Q}_0}$. This completes the proof. □

The formulation and the proof of the theorem give rise to a few natural questions and we address them in the following examples.

Example 4.2.11. It is easy to construct special components of Hilbert schemes. For example consider for any X the Hilbert scheme of quotients of $\mathcal{O}_X^{\oplus 2}$ of rank 1 and degree 1. There certainly exist such quotients which have torsion, like

$$\mathcal{O}_X^{\oplus 2} \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_P \longrightarrow 0,$$

where P is any point of X, but for obvious cohomological reasons there can be no sequence of the form

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow L \longrightarrow 0$$

with $\deg(L) = 1$. So in this case there are actually no non-special components.

Example 4.2.12. Going one step further, there may exist special components whose dimension is greater than that of any of the non-special ones. Note though that the proof shows that in this case that the bound cannot be optimal. To see an example, consider quotients of $\mathcal{O}_X^{\oplus 2}$ of rank 1 and degree $1 \leq d \leq g-2$ on a nonhyperelliptic curve X. Any such locally free quotient L gives a sequence:

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow L \longrightarrow 0$$

and so the dimension of any component of the Hilbert scheme containing it is bounded above by $h^0(L^{\otimes 2})$. Now Clifford's theorem says that $h^0(L^{\otimes 2}) \leq d+1$, but our choices make the equality case impossible, so in fact $h^0(L^{\otimes 2}) \leq d$.

On the other hand consider an effective divisor D of degree d. Then a point in the same Hilbert scheme is determined by a natural sequence:

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X^{\oplus 2} \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_D \longrightarrow 0$$

and it is not hard to see that the dimension of the Hilbert scheme at this point is equal to $d+1$ (essentially d parameters come from D and one from the sections of $\mathcal{O}_X^{\oplus 2}$). This gives then a special component whose dimension is greater than that of any non-special one.

Example 4.2.13. More significantly, the bound given in the theorem is optimal, at least in the case of line subbundles. Consider for this a line bundle L of degree 4 on a curve X of genus 2 and a generic extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow L \longrightarrow 0.$$

By standard arguments one can see that such an extension must be stable. Since $\mu(E) = 2$, by the classical theorem of Nagata [33] we get that $d_1(E) := \max_{M \subset E} \deg(M) =$

1 and so for the sequence above $d_1 - d = 1$. The theorem then tells us that the dimension of any component of the Hilbert scheme containing the given quotient is bounded above by 3. But on the other hand $h^1(L) = 0$, so this gives a smooth point and the dimension of the component is $h^0(L)$, which by Riemann-Roch is exactly 3.

This example turns out to be a special case of a general pattern, as suggested by M. Teixidor. In fact in [42] it is shown that whenever E is a generic stable bundle, the invariant d_k is the largest integer d that makes the expression $ke - rd - k(r-k)(g-1)$ nonnegative (cf. also [5]). Also the dimension of the Hilbert scheme can be computed exactly in this case (see [42] 0.2), and for example under the numerical assumptions above it is precisely equal to 3. Thus in fact for every generic stable bundle of rank 2 and degree 4 on a curve of genus 2, we have equality in the theorem. Much more generally, it can be seen analogously that for any r and g equality is satisfied for a generic stable bundle as long as d_1 satisfies a certain numerical condition.

The proof of the theorem given above can be slightly modified towards a more natural and compact form. We chose to follow the longer approach because it emphasizes very clearly what is the phenomenon involved, but below we would also like to briefly sketch this parallel argument, which grew out of conversations with I. Coandă.

We will use the same notations as above. There exists a natural specialization map:

$$X \times \mathrm{Quot}^0_{r-k,e-d}(E) \longrightarrow \mathbf{G}_{r-k}(E),$$

where $\mathbf{G}_{r-k}(E)$ is the Grassmann bundle of $r-k$ dimensional quotients of the fibers of E. Of course in the case $d = d_k$, $\mathrm{Quot}^0_{r-k,e-d}(E)$ is compact and the morphism above is finite. Fix now $P \in X$ and $w \in \mathbf{G}_{r-k}(E(P))$ a point corresponding to a quotient $E(P) \to W \to 0$. The choice of P determines a map:

$$\phi : \mathrm{Quot}^0_{r-k,e-d}(E) \longrightarrow \mathbf{G}_{r-k}(E(P))$$

and we would like to bound the dimension of $\phi^{-1}(w)$. There is a natural induced sequence:

$$0 \longrightarrow F \longrightarrow E \longrightarrow W \otimes \mathbf{C}_P \longrightarrow 0$$

and it is not hard to see that $\phi^{-1}(w)$ embeds as an open subset in $\mathrm{Quot}^0_{r-k,e-d-k}(F)$. Every locally free quotient of F has degree $\geq e - d_k - k$, and there are at most a finite number of quotients having precisely this degree (they come exactly from the minimal degree quotients of E having fixed fiber W at P). Let $G_1, \ldots, G_m$ be these quotients, sitting in exact sequences:

$$0 \longrightarrow F_i \longrightarrow F \longrightarrow G_i \longrightarrow 0.$$

The variety $Y := \mathbf{P}F - \bigcup_{i=1}^{m} \mathbf{P}G_i$ parametrizes then the one-point elementary transformations of F that do not preserve any of the F_i's. Consider the natural incidence

$$\mathcal{Z} \subset \mathrm{Quot}^0_{r-k,e-d-k}(F) \times Y$$

parametrizing the pairs $(F \to Q \to 0, F')$, where F' is an elementary transformation in Y and Q is not preserved as a quotient of F' (in other words the corresponding kernel is preserved). The fiber of $\mathcal{Z}$ over $F \to Q \to 0$ is isomorphic to $\mathbf{P}Q \cap Y$ and so

$$\dim\, \mathcal{Z} = \dim\, \mathrm{Quot}^0_{r-k,e-d-k}(F) + r - k.$$

On the other hand the fiber of $\mathcal{Z}$ over $F' \in Y$ is $\mathrm{Quot}^0_{r-k,e-d-k-1}(F')$. Now for F' the minimal degree of a quotient of rank $r-k$ is smaller, hence inductively as before:

$$\dim\, \mathrm{Quot}^0_{r-k,e-d-k-1}(F') \leq k(r-k) + (d_k - d - 1)k(r-k+1).$$

This immediately implies that

$$\dim\ \mathrm{Quot}^0_{r-k,e-d-k}(F) \leq k(r-k) + (d_k - d - 1)k(r-k+1) + k.$$

As this consequently holds for every fiber of the map ϕ, we conclude that

$$\dim\ \mathrm{Quot}^0_{r-k,e-d}(E) \leq k(r-k) + (d_k - d)k(r-k+1),$$

which finishes the proof.

4.3 Update on dimension estimates via stable maps and further work on Quot schemes

In recent work with M. Roth [39] we have improved the dimension bound given in the previous section, and also studied in some detail the structure of Quot schemes. This is achieved by considering a second compactification of the space of locally free quotients of rank k and degree d of a fixed vector bundle E, namely a Kontsevich type moduli space $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ via stable maps into the relative Grassmannian $\mathbb{G}(E,k)$ of k-dimensional quotient spaces of E. We mention here very briefly only the results concerning Quot schemes proved in [39].

Theorem 4.3.1. *([39] Theorem 4.1)*

$$\dim \mathrm{Quot}_{k,d}(E) \leq k(r-k) + (d - d_k)r, \ \textit{for all } d \geq d_k.$$

As mentioned above, this is an improvement of Theorem 4.1.1, but we preferred to keep the exposition here at a more elementary level.

Theorem 4.3.2. *([39] Theorem 6.4)*

For any vector bundle E on C, there is an integer $d_Q = d_Q(E,k)$ such that for all $d \geq d_Q$, $\mathrm{Quot}_{k,d}(E)$ is irreducible. For any such d, $\mathrm{Quot}_{k,d}(E)$ is generically smooth, *has dimension* $rd - ke - k(r-k)(g-1)$*, and its generic point corresponds to a vector bundle quotient.*

Theorem 4.3.3. *([39] Theorems 7.1 and 7.4)*

(a) If $k = (r-1)$ *then there is a surjective morphism from* $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ *to* $\mathrm{Quot}_{k,d}(E)$ *which extends the map on the locus where the domain curve is smooth. Such a morphism also exists for any* k *if* $d = d_k$ *or* $d = d_k + 1$*.*

(b) If $k \neq (r-1)$ *then in general there is no morphism from* $\overline{\mathcal{M}}_g(\mathbb{G}(E,k), \beta_d)$ *to* $\mathrm{Quot}_{k,d}(E)$ *extending the map on smooth curves.*

CHAPTER V

Effective base point freeness on $SU_X(r)$

5.1 Background

The underlying idea for studying linear series on the moduli spaces $SU_X(r,e)$ has its roots in the paper of Faltings [11], where a construction of the moduli space based on theta divisors is given. A very nice introduction to the subject is provided in [2].

Fix r and e and denote $h = \gcd(r,e)$, $r_1 = \frac{r}{h}$ and $e_1 = \frac{e}{h}$. Consider a vector bundle F of rank pr_1 and degree $p(r_1(g-1)-e_1)$. Generically such a choice determines (cf. [9] 0.2) a *theta divisor* Θ_F on $SU_X(r,e)$, supported on the set

$$\Theta_F = \{E|\ h^0(E \otimes F) \neq 0\}.$$

All the divisors Θ_F for $F \in U_X(pr_1, p(r_1(g-1)-e_1))$ belong to the linear system $|\mathcal{L}^p|$, where $\mathcal{L}$ is the *determinant* line bundle $\mathcal{L}$. We have the following well-known:

Lemma 5.1.1. *$E \in SU_X(r,e)$ is not a base point for $|\mathcal{L}^p|$ if there exists a vector bundle F of rank pr_1 and degree $p(r_1(g-1)-e_1)$ such that $h^0(E \otimes F) = 0$.*

It is easy to see that such an F must necessarily be semistable (cf. [26] (2.5)). It is also a simple consequence of the existence of Jordan-Hölder filtrations that one has to check the condition in the above lemma only for E stable. We sketch the proof for convenience:

Lemma 5.1.2. *If for any stable bundle V of rank $r' \leq r$ and slope e/r there exists $F \in U_X(pr_1, p(r_1(g-1)-e_1))$ such that $h^0(V \otimes F) = 0$, then the same is true for every $E \in SU_X(r,e)$.*

Proof. Assume that E is strictly semistable. Then it has a Jordan-Hölder filtration:

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that E_i/E_{i-1} are stable for $i \in \{1, \ldots, n\}$ and $\mu(E_i/E_{i-1}) = \frac{e}{r}$. By assumption there exist $F_i \in U_X(pr_1, p(r_1(g-1)-e_1))$ such that $h^0(E_i/E_{i-1} \otimes F_i) = 0$ and so if we denote

$$\Theta_{E_i/E_{i-1}} := \{F | h^0(E_i/E_{i-1} \otimes F) \neq 0\} \subset U_X(pr_1, p(r_1(g-1)-e_1)),$$

this is a proper subset for every i. It is clear that any

$$F \in U_X(pr_1, p(r_1(g-1)-e_1)) - \bigcup_{i=1}^{n} \Theta_{E_i/E_{i-1}}$$

satisfies $h^0(E \otimes F) = 0$. □

We also record a simple lemma which will be useful in Section 5.3. It is most certainly well known, but we sketch the proof for convenience (cf. also [42] 1.1).

Lemma 5.1.3. *Consider a sheaf extension:*

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0.$$

If E is stable, then $h^0(G^* \otimes F) = 0$.

Proof. Assuming the contrary, there is a nonzero morphism $G \to F$. Composing this with the maps $E \to G$ to the left and $F \to E$ to the right, we obtain a nontrivial endomorphism of E, which contradicts the stability assumption. □

As a final remark, note that we are always slightly abusing the notation by using vector bundles instead of S-equivalence classes. This is harmless, since it is easily seen that it is enough to check the assertions for any representative of the equivalence class.

5.2 Warm up for effective base point freeness: the case of $SU_X(2)$

In this section we give a very simple proof of a theorem which first appeared in [41] (see also [17]). It completely takes care of the case of $SU_X(2)$. Although the specific technique (based on the Clifford theorem for line bundles) is different from the methods that will be used in Section 5.3 to prove the main result, the general computational idea already appears here, in a particularly transparent form. This is the reason for including the proof.

Theorem 5.2.1. *The linear system $|\mathcal{L}|$ on $SU_X(2)$ has no base points.*

Proof. Recall from 5.1.1 and 5.1.2 that the statement of the theorem is equivalent to the following fact: for any stable bundle $E \in SU_X(2)$, there exists a line bundle $L \in \mathrm{Pic}^{g-1}(X)$ such that $H^0(E \otimes L) = 0$. This is certainly an open condition and it is sufficient to prove that the algebraic set

$$\{L \in \mathrm{Pic}^{g-1}(X)|\ H^0(E \otimes L) \neq 0\} \subset \mathrm{Pic}^{g-1}(X)$$

has dimension strictly less than g.

A nonzero map $E^* \to L$ comes together with a diagram of the form:

$$\begin{array}{ccccc} E^* & \longrightarrow & M & \longrightarrow & 0 \\ & \searrow & \downarrow & & \\ & & L & & \end{array}$$

where M is just the image in L. Then we have $M = L(-D)$ for some effective divisor D. Since E is stable, the degree of M can vary from 1 to $g-1$ and we want to count all these cases separately. So for $m = 1, \ldots, g-1$, consider the following algebraic subsets of $\mathrm{Pic}^{g-1}(X)$:

$$A_m := \{L \in \mathrm{Pic}^{g-1}(X) |\ \exists\ 0 \neq \phi : E^* \to L \text{ with } M = \mathrm{Im}(\phi), \deg(M) = m\}.$$

The claim is that dim $A_m \leq g-1$ for all such m. Then of course

$$A_1 \cup \ldots \cup A_{g-1} \subsetneq \mathrm{Pic}^{g-1}(X)$$

and any L outside this union satisfies our requirement. To prove the claim, denote by $\mathrm{Quot}_{1,m}(E)$ the Quot scheme of coherent quotients of E of rank 1 and degree m. The set of line bundle quotients $E^* \to M \to 0$ of degree m is a subset of $\mathrm{Quot}_{1,m}(E)$. On the other hand every $L \in A_m$ can be written as $L = M(D)$, with M as above and D effective of degree $g-1-m$. This gives the obvious bound:

$$\dim\ A_m \leq \dim\ \mathrm{Quot}_{1,m}(E) + g - 1 - m.$$

To bound the dimension of the Quot scheme in question, fix an M as before and consider the exact sequence that it determines:

$$0 \longrightarrow M^* \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Note that the kernel is isomorphic to M^* since E has trivial determinant. Now we use the well known fact from deformation theory that dim $\mathrm{Quot}_{1,m}(E) \leq h^0(M^{\otimes 2})$. To estimate $h^0(M^{\otimes 2})$, one uses all the information provided by Clifford's theorem. The initial bound that it gives is $h^0(M^{\otimes 2}) \leq m+1$ (note that $\deg(M) \leq g-1$). If actually $h^0(M^{\otimes 2}) \leq m$, then we immediately get dim $A_m \leq g-1$ as required. On the other hand if $h^0(M^{\otimes 2}) = m+1$, by the equality case in Clifford's theorem (see e.g. [1] III, §1) one of the following must hold: $M^{\otimes 2} \cong \mathcal{O}_X$ or $M^{\otimes 2} \cong \omega_X$ or X is hyperelliptic and $M^{\otimes 2} \cong m \cdot g^1_2$. The first case is impossible since $\deg(M) > 0$. In the second case M is a theta characteristic and we are done either by the fact that these are a finite number or by other overlapping cases. The third case can also happen only for a finite number of M's and if we're not in any of the other cases then of course $\dim A_m \leq g - 1 - m < g - 1$. This concludes the proof of the theorem. □

Remark 5.2.2. Note that the key point in the proof above is the ability to give a convenient upper bound on the dimension of certain Quot schemes. This will essentially be the main ingredient in the general result we will prove in the next section, and the needed estimate was provided in the previous chapter on Quot schemes.

5.3 Base point freeness for pluritheta linear series on $SU_X(r, e)$

Using the dimension bound for Quot schemes given in Chapter 4, we are now able to prove the main result of this chapter, namely an effective base point freeness bound for pluritheta linear series on $SU_X(r, e)$. The proof is computational in nature and the roots of the main technique involved have already appeared in in the proof of Theorem 5.2.1. Let $r \geq 2$ and e be arbitrary integers and let $h = \gcd(r, e)$, $r = r_1 h$ and $e = e_1 h$. For the statement it is convenient to introduce another invariant of the moduli space. If $E \in SU_X(r, e)$ and $1 \leq k \leq r - 1$, define $s_k(E) := ke - rd_k$, where d_k is the maximum degree of a subbundle of E of rank k (cf.[23]). Note that if E is stable one has $s_k(E) \geq h$ and we can further define $s_k = s_k(r, e) := \min_{E \text{ stable}} s_k(E)$ and $s = s(r, e) := \min_{1 \leq k \leq r-1} s_k$. Clearly $s \geq h$ and it is also an immediate observation that $s(r, e) = s(r, -e)$.

Theorem 5.3.1. *The linear series $|\mathcal{L}^p|$ on $SU_X(r,e)$ is base point free for*

$$p \geq \max\{\frac{(r+1)^2}{4r}h, \frac{r^2}{4s}h\}.$$

Remark 5.3.2. Note that the bound given in the theorem is always either a quadratic or a linear function in the rank r. It should also be said right away that although this bound works uniformly, in almost any particular situation one can do a little better. Unfortunately there doesn't seem to be a better uniform way to express it, but we will comment more on this at the end of the section (cf. Remark 5.3.5).

Proof. (of Theorem 5.3.1) Let us denote for simplicity $M := \max\{\frac{(r+1)^2}{4r}h, \frac{r^2}{4s}h\}$. Since the problem depends only on the residues of e modulo r, there is no loss of generality in looking only at $SU_X(r,-e)$ with $0 \leq e \leq r-1$. The statement of the theorem is implied by the following assertion, as described in 5.1.1 and 5.1.2:

For any stable $E \in SU_X(r,-e)$ and any $p \geq M$, there is an

$F \in U_X(pr_1, p(r_1(g-1)+e_1))$ such that $h^0(E \otimes F) = 0$.

Fix a stable bundle $E \in SU_X(r,-e)$. Note that only in this proof, as opposed to the rest of the chapter, e in fact denotes the degree of E^*, and not that of E. If for some $F \in U_X(pr_1, p(r_1(g-1)+e_1))$ there is a nonzero map $E^* \xrightarrow{\phi} F$, then this comes together with a diagram of the form:

$$\begin{array}{ccccc} E^* & \longrightarrow & V & \longrightarrow & 0 \\ & \searrow^{\phi} & \downarrow & & \\ & & F & & \end{array}$$

where the vector bundle V is the image of ϕ. The idea is essentially to count all such diagrams assuming that the rank and degree of V are fixed and see that the F's involved in at least one of them cover only a proper subset of the whole moduli space. Denote as before by $\text{Quot}_{k,d}(E^*)$ the Quot scheme of quotients of E^* of rank k and degree d and for any $1 \leq k \leq r$ and any d in the suitable range (given by the stability of E and F) consider its subset:

$$A_{k,d} := \{V \in \text{Quot}_{k,d}(E^*) | \, \exists F \in U_X(pr_1, p(r_1(g-1)+e_1)), \, \exists 0 \neq \phi : E^* \to F \text{ with } V = \text{Im}(\phi)\}.$$

The theorem on Quot schemes stated in Section 4.2 then gives us the dimension estimate:

$$\dim A_{k,d} \leq k(r-k) + (d - f_k)(k+1)(r-k), \tag{5.1}$$

where $f_k = f_k(E^*)$ is the minimum possible degree of a quotient bundle of E^* of rank k (which is the same as $-d_k$). Define now the following subsets of $U_X(pr_1, p(r_1(g-1)+e_1))$:

$$U_{k,d} := \{F | \, \exists V \in A_{k,d} \text{ with } V \subset F\} \subset U_X(pr_1, p(r_1(g-1)+e_1)).$$

The elements of $U_{k,d}$ are all the F's that appear in diagrams as above for fixed k and d. The claim is that

$$\dim U_{k,d} < (pr_1)^2(g-1) + 1,$$

which would imply that $U_{k,d} \subsetneq U_X(pr_1, p(r_1(g-1)+e_1))$. Assuming that this is true, and since k and d run over a finite set, any $F \in U_X(pr_1, p(r_1(g-1)+e_1)) - \bigcup_{k,d} U_{k,d}$ satisfies the desired property that $h^0(E \otimes F) = 0$, which gives the statement of the theorem. It is easy to see, and in fact a particular case of the computation below, that in the case $k = r$ (i.e. $V = E^*$) $U_{k,d}$ has dimension exactly $(pr_1)^2(g-1)$.

Let us concentrate then on proving the claim above for $1 \leq k \leq r-1$. Note that the inclusions $V \subset F$ appearing in the definition of $U_{k,d}$ are valid in general only at the sheaf level. Any such inclusion determines an exact sequence:

$$0 \longrightarrow V \longrightarrow F \longrightarrow G' \longrightarrow 0, \tag{5.2}$$

where $G' = G \oplus \tau_a$, with G locally free and τ_a a zero dimensional subscheme of length a. We stratify $U_{k,d}$ by the subsets

$$U^a_{k,d} := \{F|\ F \text{ given by an extension of type (4)}\} \subset U_{k,d},$$

where a runs over the obvious allowable finite set of integers. A simple computation shows that G has rank $pr_1 - k$ and degree $p(r_1(g-1)+e_1)-d-a$. Denote by $T^a_{k,d}$ the set of all vector bundles G that are quotients of some $F \in U_X(pr_1, p(r_1(g-1)+e_1))$. These can be parametrized by a relative Hilbert scheme (see e.g. [27] §8.6) over (an étale cover of) $U_X(pr_1, p(r_1(g-1)+e_1))$ and so they form a bounded family. We invoke a general result, proved in [4] 4.1 and 4.2, saying that the dimension of such a family is always at most what we get if we assume that the generic member is stable. Thus we get the bound:

$$\dim T^a_{k,d} \leq (pr_1 - k)^2(g-1) + 1.$$

Now we only have to compute the dimension of the family of all possible extensions of the form (4) when V and G are allowed to vary over $A_{k,d}$ and $T^a_{k,d}$ respectively and τ_a varies over the symmetric product X_a. Any such extension induces a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & V' & \longrightarrow & \tau_a & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \longrightarrow & F & \longrightarrow & G' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & \xrightarrow{\cong} & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

If we denote by $A^a_{k,d}$ the set of isomorphism classes of vector bundles V' that are (inverse) elementary transformations of length a of vector bundles in $A_{k,d}$, then we have the obvious:

$$\dim A^a_{k,d} \leq \dim A_{k,d} + ka.$$

On the other hand any F is obtained as an extension of a bundle in $T^a_{k,d}$ by a bundle in $A^a_{k,d}$. Denote by $\mathcal{U} \subset A^a_{k,d} \times T^a_{k,d}$ the open subset consisting of pairs (V', G) such that there exists an extension

$$0 \longrightarrow V' \longrightarrow F \longrightarrow G \longrightarrow 0$$

with F stable. Note that by Lemma 5.1.3 for any such pair we have $h^0(G^* \otimes V') = 0$ and so by Riemann-Roch $h^1(G^* \otimes V')$ is constant, given by:

$$h^1(G^* \otimes V') = 2kpr_1(g-1) - k^2(g-1) + kpe_1 - pr_1 d - pr_1 a. \tag{5.3}$$

In this situation it is a well known result (see e.g. [40] (2.4) or [22] §4) that there exists a universal space of extension classes $\mathbf{P}(\mathcal{U}) \to \mathcal{U}$ whose dimension is computed by the formula:

$$\dim \mathbf{P}(\mathcal{U}) = \dim A^a_{k,d} + \dim T^a_{k,d} + h^1(G^* \otimes V') - 1.$$

There is an obvious forgetful map:

$$\mathbf{P}(\mathcal{U}) \longrightarrow U_X(pr_1, p(r_1(g-1) + e_1))$$

whose image is exactly $U^a_{k,d}$. Thus by putting together all the inequalities above we obtain:

$$\dim U^a_{k,d} \leq \dim A^a_{k,d} + \dim T^a_{k,d} + h^1(G^* \otimes V') - 1$$

$$\leq k(r-k) + (d - f_k)(k+1)(r-k) + ka + (pr_1 - k)^2(g-1) + 1$$

$$+2kpr_1(g-1) - k^2(g-1) + kpe_1 - pr_1 d - pr_1 a - 1$$

$$\leq k(r-k) + (d - f_k)(k+1)(r-k) + (pr_1)^2(g-1) + kpe_1 - pr_1 d + ka - pr_1 a$$

$$\leq k(r-k)+(d-f_k)(k+1)(r-k)+(pr_1)^2(g-1)+kpe_1-pr_1d,$$

where the last inequality is due to the obvious fact that $k \leq r-1 < pr_1$ if $p \geq M$. Since a runs over a finite set, to conclude the proof of the claim it is enough to see that $\dim U^a_{k,d} \leq (pr_1)^2(g-1)$. By the inequality above this is true if

$$p(r_1d-ke_1) \geq k(r-k)+(d-f_k)(k+1)(r-k),$$

or equivalently if

$$p(rd-ke) \geq k(r-k)h+(d-f_k)(k+1)(r-k)h$$

for any k and d. This can be rewritten in the following more manageable form:

$$p(r(d-f_k)+rf_k-ke) \geq k(r-k)h+(d-f_k)(k+1)(r-k)h.$$

The first case to look at is $d=f_k$, when we should have $p(rf_k-ke) \geq k(r-k)h$ and this should hold for every k. But clearly $rf_k-ke = s_{r-k}(E^*) = s_k(E)$, defined above in terms of maximal subbundles, and $h|s_k(E)$. Since E is stable we then have $s_k(E) \geq h$ for all k and so $s \geq h$ as mentioned before. Note though that in general one cannot do better (cf. Remark 5.3.6). In any case, this says that the inequality $p \geq \frac{r^2}{4s}h$ must be satisfied (which would certainly hold if $p \geq \frac{r^2}{4}$).

When $d > f_k$, it is convenient to collect together all the terms containing $d-f_k$. The last inequality above then reads:

$$(d-f_k)(pr-(k+1)(r-k)h)+ps_k(E) \geq k(r-k)h.$$

For p as before it is then sufficient to have $pr \geq (k+1)(r-k)h$, which again by simple optimization is satisfied for $p \geq \frac{(r+1)^2}{4r}h$. Concluding, the desired inequality holds as long as $p \geq M$. □

The most important instances of this theorem are the cases of vector bundles of degree 0 (more generally $d \equiv 0(\text{mod } r)$) and degree 1 or -1 (more generally $d \equiv \pm 1(\text{mod} r)$). In the second situation the moduli space in question is smooth. It is somewhat surprising that the results obtained in these cases have different orders of magnitude.

Corollary 5.3.3. *$|\mathcal{L}^p|$ is base point free on $SU_X(r)$ for $p \geq \frac{(r+1)^2}{4}$.*

Proof. This is clear since $h = r$. □

Corollary 5.3.4. *$|\mathcal{L}^p|$ is base point free on $SU_X(r,1)$ and $SU_X(r,-1)$ for $p \geq r-1$.*

Proof. Note that by duality it suffices to prove the claim for one of the moduli spaces, say $SU_X(r,-1)$. In this case $h = 1$ and for any $E \in SU_X(r,-1)$, $s_{r-k}(E^*) = rf_k - k \geq r - k$. Following the proof of the theorem we see thus that it suffices to have

$$p \geq \max\{r-1, \frac{(r+1)^2}{4r}\},$$

which is equal to $r-1$ for $r \geq 3$. For $r = 2$ one can slightly improve the last inequality in the proof of the theorem (actually this is true whenever r is even) to see that $p = 1$ already works. □

Remark 5.3.5. Corollary 5.3.4 can be seen as a generalization of the well-known fact that $|\mathcal{L}|$ is base point free on $SU_X(2,1)$. Also, as already noted in its proof, the general bound obtained in the theorem can be slightly improved in each particular case, due to the fact that the two optimization problems do not simultaneously have integral solutions. Thus for example if r is even, the proof of the theorem actually gives that $|\mathcal{L}^p|$ is base point free on $SU_X(r)$ for $p \geq r(r+2)/4$.

Remark 5.3.6. As already noted, in any given numerical situation the bound given by the theorem is either linear or quadratic in the rank r. One may thus hope that at least in the case $h = 1$ (i.e. r and e coprime), a closer study of the number $s(r, e)$ might always produce by this method a linear bound. Examples show though that this is not the case: one can take $r = 4l$, $e = 2l - 1$, $k = 2l + 1$ and $f_k = l$ (this works for special vector bundles) for a positive integer l, which implies $s = s_k = 1$.

5.4 An application to surfaces à la Le Potier

Another, in some sense algorithmic, application of the effective bound 5.3.1 can be given following the paper of Le Potier [26]. By a simple use of a restriction theorem due to Flenner [13] (cf. also [27] §11), Le Potier shows that effective results for the determinant bundle $\mathcal{L}$ induce effective results for the Donaldson determinant line bundles on moduli spaces of semistable sheaves on surfaces. For the appropriate definitions and basic results, the reader can consult [19] §8. Using the uniform bound $k \geq (r+1)^2/4$ that works on every moduli space $SU_X(r, d)$, the result can be formulated as follows:

Theorem 5.4.1. *Let $(X, \mathcal{O}_X(1))$ be a polarized smooth projective surface and L a line bundle on X. Let $M = M_X(r, L, c_2)$ be the moduli space of semistable sheaves of rank r, fixed determinant L and second Chern class c_2 on X and denote $n = \deg(X) = \mathcal{O}_X(1)^2$ and $d = n[\frac{r^2}{2}]$. If $\mathcal{D}$ is the Donaldson determinant line bundle on M, then $\mathcal{D}^{\otimes p}$ is globally generated for $p \geq d \cdot \frac{(r+1)^2}{4}$ divisible by d.*

Note that it is not true that $\mathcal{D}$ is ample, which accounts for the formulation of the theorem. The significance of the map to projective space given by some multiple of $\mathcal{D}$ is well known. Its image is the moduli space of μ-semistable sheaves and in the rank 2 and degree 0 case this is homeomorphic to the Donaldson-Uhlenbeck compactification of the moduli space of ASD-connections in gauge theory, the map realizing the transition between the Gieseker and Uhlenbeck points of view (see e.g. [19] §8.2 for a survey). Better bounds for the global generation of the Donaldson line bundle thus limit the dimension of an ambient projective space for this moduli space. The main improvement brought by the results in the present paper comes from the fact that our result is not influenced by the genus of the curve given by Flenner's theorem. Effectively that reduces the bound given in [26] §3.2 by an order of four, namely from a polynomial of degree 8 in the rank r to a polynomial of degree 4.

Sketch of proof of Theorem 5.4.1.(cf. [26] 3.6) In analogy with the curve situation, given $E \in M$, the problem is to find a complementary 1-dimensional sheaf F on X such that $h^1(E \otimes F) = 0$. Flenner's theorem says that there exists a smooth curve C belonging to the linear series $|\mathcal{O}_X(d)|$ such that $E_{|C}$ is semistable. By theorem 5.3.1, on C one can find for any $k \geq (r+1)^2/4$ a vector bundle V of rank kr_1 such that $h^1(E \otimes V) = 0$. The F that we are looking for is obtained by considering V as a 1-dimensional sheaf on X and a simple computation shows that if $p = dk$, this gives the global generation of $\mathcal{D}^{\otimes p}$.

Remark 5.4.2. Depending on the values of the invariants involved, this bound may sometimes be improved to a polynomial of degree 3 in r, according to the precise statement of Theorem 5.3.1.

CHAPTER VI

Verlinde bundles and duality for generalized theta functions

6.1 The Verlinde bundles $E_{r,k}$

The purpose of this section is to introduce and study some very interesting vector bundles on the Jacobian of a curve which are associated to generalized theta linear series on moduli spaces of vector bundles on that curve. By way of application, we establish in the present and the next chapter, effective bounds for the global and normal generation of such linear series and we give new proofs of the known results and a new perspective on the familiar conjectures concerning duality for generalized theta functions. We also show that these bundles lead to new examples (in the spirit of [41]) of base points for the determinant linear series.

Definition 6.1.1. For every positive integers r and k and every $N \in \mathrm{Pic}^{g-1}(X)$, the (r,k) - *Verlinde* bundle on $J(X)$ is the vector bundle

$$E_{r,k}(= E_{r,k}^N) := \det_* \mathcal{O}(k\Theta_N),$$

where $\det : U_X(r,0) \to J(X)$ is the determinant map. The dependence of the definition on the choice of N is implicitly assumed, but not emphasized by the notation. Recall that by the convention in Section 2.3 we will (and it is enough to) assume that N is a theta characteristic. Also, most of the time the rank r is fixed and we refer to $E_{r,k}$ as the *level* k Verlinde bundle.

Remark 6.1.2. The fibers of the determinant map are the moduli spaces of vector bundles of fixed determinant $SU_X(r,L)$ with $L \in \mathrm{Pic}^0(X)$. The restriction of $\mathcal{O}(k\Theta_N)$ to such a fiber is exactly $\mathcal{L}^k$, where $\mathcal{L}$ is the determinant bundle. Since on $SU_X(r)$ the dualizing sheaf is isomorphic to $\mathcal{L}^{-2r}$, by the rational singularities version of the Kodaira vanishing theorem these have no higher cohomology and it is clear that $E_{r,k}$ is a vector bundle of rank $s_{r,k} := h^0(SU_X(r), \mathcal{L}^k)$. The fiber of $E_{r,k}$ at a point L is naturally isomorphic to the Verlinde vector space $H^0(SU_X(r,L), \mathcal{L}^k)$ and this justifies the terminology *Verlinde bundles* that we are using. Our viewpoint is that they allow a translation of statements about spaces of generalized theta functions into global statements about vector bundles on Jacobians. This in turn opens the way towards a more geometric study of these spaces.

Much of the study of the vector bundles $E_{r,k}$ is governed by the fact that they decompose very nicely when pulled back via multiplication by r. To see this, first recall from [8], §2 and §4, that there is a cartesian diagram:

$$\begin{array}{ccc} SU_X(r) \times J(X) & \xrightarrow{\tau} & U_X(r,0) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \det} \\ J(X) & \xrightarrow{r_J} & J(X) \end{array}$$

where τ is the tensor product of vector bundles, p_2 is the projection on the second factor and r_J is multiplication by r. The top and bottom maps are étale covers of degree r^{2g} and one finds in [8] the formula:

$$\tau^*\mathcal{O}(\Theta_N) \cong \mathcal{L} \boxtimes \mathcal{O}_J(r\Theta_N). \tag{6.1}$$

Using the notation $V_{r,k} := H^0(SU_X(r), \mathcal{L}^k)$, we have the following simple but very important

Lemma 6.1.3. $r_J^* E_{r,k} \cong V_{r,k} \otimes \mathcal{O}_J(kr\Theta_N)$.

Proof. By the push-pull formula (see [16], III.9.3) and (2) we obtain:

$$r_J^* E_{r,k} \cong r_J^* \mathrm{det}_* \mathcal{O}(k\Theta_N) \cong p_{2*}\tau^* \mathcal{O}(k\Theta_N)$$

$$\cong p_{2*}(\mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N)) \cong V_{r,k} \otimes \mathcal{O}_J(kr\Theta_N).$$

□

An immediate consequence of this property is the following (recall that a vector bundle is called *polystable* if it decomposes as a direct sum of stable bundles of the same slope):

Corollary 6.1.4. *$E_{r,k}$ is an ample vector bundle, polystable with respect to any polarization on $J(X)$.*

Proof. Both properties can be checked up to finite covers (see e.g. [19] §3.2) and they are obvious for $r_J^* E_{r,k}$. □

For future reference it is also necessary to study how the bundles $E_{r,k}$ behave under the Fourier-Mukai transform (for the definitions see Appendix A).

Lemma 6.1.5. *$E_{r,k}$ satisfies I.T. with index 0 and so $\widehat{E_{r,k}}$ is a vector bundle of rank $h^0(U_X(r,0), \mathcal{O}(k\Theta_N))$ satisfying I.T. with index g.*

Proof. Using the usual identification between $\mathrm{Pic}^0(J(X))$ and $\mathrm{Pic}^0(X)$, we have to show that $H^i(E_{r,k} \otimes P) = 0$ for all $P \in \mathrm{Pic}^0(X)$ and all $i > 0$. But $H^i(E_{r,k} \otimes P)$ is a direct summand in $H^i(r_{J*} r_J^*(E_{r,k} \otimes P))$ and so it is enough to have the vanishing of $H^i(r_J^*(E_{r,k} \otimes P))$. This is obvious by the formula in 6.1.3. By A.3(5) we have:

$$rk(\widehat{E_{r,k}}) = h^0(E_{r,k}) = h^0(U_X(r,0), \mathcal{O}(k\Theta_N)).$$

The last statement follows also from A.3(5). □

By Serre duality, one obtains in a similar way the statement:

Lemma 6.1.6. *$E^*_{r,k}$ satisfies I.T. with index g and so $\widehat{E^*_{r,k}}$ is a vector bundle of rank $h^0(U_X(r,0), \mathcal{O}(k\Theta_N))$ satisfying I.T. with index 0.*

The Verlinde bundles are particularly easy to compute when k is a multiple of the rank r, but note that for general k the decomposition of $E_{r,k}$ into stable factors is not so easy to describe.

Proposition 6.1.7. *For all $m \geq 1$*

$$E_{r,mr} \cong \bigoplus_{s_{r,mr}} \mathcal{O}_J(m\Theta_N) (\cong V_{r,mr} \otimes \mathcal{O}_J(m\Theta_N)).$$

Proof. By 6.1.3 we have

$$r_J^* E_{r,mr} \cong V_{r,mr} \otimes \mathcal{O}_J(mr^2\Theta_N).$$

Since N is a theta characteristic, Θ_N is symmetric and so $r_J^*\mathcal{O}(m\Theta_N) \cong \mathcal{O}_J(mr^2\Theta_N)$. We get

$$r_J^* E_{r,mr} \cong r_J^*(V_{r,mr} \otimes \mathcal{O}(m\Theta_N)).$$

Note now that the diagram giving 6.1.3 is equivariant with respect to the X_r, the group of r-torsion points of $J(X)$, if we let X_r act on $SU_X(r) \times J(X)$ by $(\xi, (F, L)) \to (F, L \otimes \xi)$ and naturally on all the other spaces. Also the action of X_r on $\tau^*\mathcal{O}(mr\Theta_N) = \mathcal{L}^{mr} \boxtimes \mathcal{O}_J(mr^2\Theta_N)$ is on $\mathcal{O}_J(mr^2\Theta_N)$ the same as the natural (pullback) action. By chasing the diagram we see then that the vector bundle isomorphism above is equivariant with respect to the natural X_r action on both sides. Since we have an an induced isomorphism:

$$r_{J*} r_J^* E_{r,mr} \cong r_{J*} r_J^*(V_{r,mr} \otimes \mathcal{O}(m\Theta_N))$$

and $E_{r,mr}$ and $V_{r,mr} \otimes \mathcal{O}_J(m\Theta_N)$ are both eigenbundles with respect to the trivial character, the lemma follows. □

The rest of the section will be devoted to a further study of these bundles in the case $k = 1$, where more tools are available. The main result is that $E_{r,1}$ is a simple vector bundle and this fact will be exploited in the next section. We show this after proving a very simple lemma, to the effect that twisting by r-torsion line bundles does not change $E_{r,1}$.

Lemma 6.1.8. *$E_{r,1} \otimes P_\xi \cong E_{r,1}$ for any r-torsion line bundle P_ξ on $J(X)$ corresponding to an r-torsion $\xi \in \mathrm{Pic}^0(X)$ by the usual identification.*

Proof. By definition and the projection formula one has

$$E_{r,1} \otimes P_\xi \cong \det{}_*(\mathcal{O}(\Theta_N) \otimes \det{}^* P_\xi) \cong \det{}_*\mathcal{O}(\Theta_{N\otimes\xi}),$$

where the last isomorphism is an application of 2.2.3. Now the following commutative diagram:

$$\begin{array}{ccc} U_X(r,0) & \xrightarrow{\otimes\xi} & U_X(r,0) \\ {\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \det} \\ J(X) & \xrightarrow{id} & J(X) \end{array}$$

shows that $\det_*\mathcal{O}(\Theta_N) \cong \det_*\mathcal{O}(\Theta_{N\otimes\xi})$, which is exactly the statement of the lemma.

∎

Proposition 6.1.9. *$E_{r,1}$ is a simple vector bundle.*

Proof. This follows from a direct computation of the number of endomorphisms of $E_{r,1}$. By lemma 6.1.3 and the Verlinde formula at level 1, $r_J^* E_{r,1} \cong \bigoplus_{r^g} \mathcal{O}_J(r\Theta_N)$. Then:

$$h^0(r_{J*}r_J^*(E_{r,1}^* \otimes E_{r,1})) = h^0(r_J^*(E_{r,1}^* \otimes E_{r,1}))$$

$$= h^0((\bigoplus_{r^g}\mathcal{O}_J(-r\Theta_N)) \otimes (\bigoplus_{r^g}\mathcal{O}_J(r\Theta_N))) = r^{2g}.$$

On the other hand, since r_J is a Galois cover with Galois group X_r, we have the formula $r_{J*}\mathcal{O}_J \cong \bigoplus_{\xi \in X_r} P_\xi$. Combined with lemma 6.1.8, this gives

$$r_{J*}r_J^*(E_{r,1}^* \otimes E_{r,1}) \cong \bigoplus_{\xi \in X_r} E_{r,1}^* \otimes E_{r,1} \otimes P_\xi \cong \bigoplus_{r^{2g}} E_{r,1}^* \otimes E_{r,1}.$$

The two relations imply that $h^0(E_{r,1}^* \otimes E_{r,1}) = 1$, so $E_{r,1}$ is simple. □

Remark 6.1.10. Using an argument similar to the one given above, plus the Verlinde formula, it is not hard to see that the bundles $E_{r,k}$ are not simple for $k \geq 2$. In the case when k is a multiple of r they even decompose as direct sums of line bundles, as we have already seen in 6.1.7. This shows why some special results that we will obtain in the case $k = 1$ do not admit straightforward extensions to higher k's.

6.2 Stability of Fourier transforms and duality for generalized theta functions

One of the main features of the vector bundles $E_{r,1}$, already observed in the previous section, is that they are simple. We will see below that in fact they are even stable (with respect to any polarization on $J(X)$). This fact, combined with the fact that $\widehat{\mathcal{O}_J(r\Theta_N)}$ is also stable (see Proposition 6.2.1 below), gives very quick proofs of some results of Beauville-Narasimhan-Ramanan [3] and Donagi-Tu [8]. Note that for the proofs of these applications it is enough to use the simpleness of the vector bundles mentioned above.

Proposition 6.2.1. $\widehat{E_{1,r}} = \widehat{\mathcal{O}_J(r\Theta_N)}$ *is stable with respect to any polarization on* $J(X)$.

This is a consequence of a more general fact of independent interest, saying that this is indeed true for an arbitrary nondegenerate line bundle on an abelian variety. Recall that a line bundle A on the abelian variety X is called *nondegenerate* if

$\chi(A) \neq 0$. By [32] §16 this implies that there is a unique i (the *index* of A) such that $H^i(A) \neq 0$.

Proposition 6.2.2. *Let A be a nondegenerate line bundle on an abelian variety X. The Fourier-Mukai transform $\widehat{A}$ is stable with respect to any polarization on X.*

Proof. Let's begin by fixing a polarization on $\widehat{X}$, so that stability will be understood with respect to this polarization. Consider the isogeny defined by A

$$\begin{array}{cccc} \phi_A : & X & \longrightarrow & \mathrm{Pic}^0(X) \cong \widehat{X} \\ & x & \rightsquigarrow & t_x^* A \otimes A^{-1} \end{array}$$

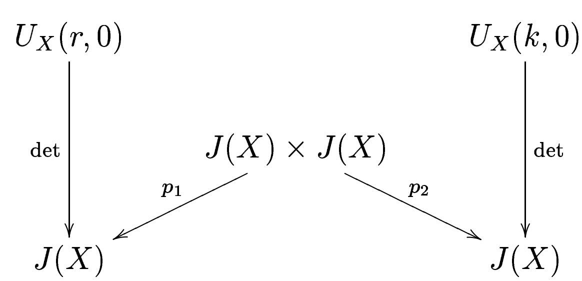
If i is the index of A, it follows from A.3(3) that $\phi_A^* \widehat{A} \cong V \otimes A^{-1}$, where $V := H^i(A)$. As we already mentioned in the proof of 6.1.4, by [19] §3.2 this already implies that $\widehat{A}$ is polystable. On the other hand, by A.3(5) the Fourier transform of any line bundle is simple, so $\widehat{A}$ must be stable. □

Remark 6.2.3. It is worth noting that one can avoid the use of [19] §3.2 quoted above and use only the easier fact that semistability is preserved by finite covers. More precisely, $\widehat{A}$ has to be semistable, but assume that it is not stable. Then we can choose a maximal destabilizing subbundle $F \subsetneq \widehat{A}$, which must obviously be semistable and satisfy $\mu(F) = \mu(\widehat{A})$. Again $\phi_A^* F$ must be semistable, with respect to the pull-back polarization. But $\phi_A^* F \subsetneq \phi_A^* \widehat{A} \cong V \otimes A^{-1}$ and by semistability this implies that $\phi_A^* F \cong V' \otimes A^{-1}$ with $V' \subsetneq V$. This situation is overruled by the presence of the action of Mumford's theta-group $\mathcal{G}(A)$. Recall from [31] that A is endowed with a natural $\mathcal{G}(A)$-linearization of weight 1. On the other hand, $\phi_A^* F$ has a natural $K(A)$-linearization which can be seen as a $\mathcal{G}(A)$-linearization of weight 0, where $K(A)$ is the kernel of the isogeny ϕ_A above. By tensoring we obtain a weight 1 $\mathcal{G}(A)$-linearization on $V' \otimes \mathcal{O}_X \cong \phi_A^* F \otimes A$ and thus an induced weight 1 representation on V'. It is known though from [31] that V is the unique irreducible representation of $\mathcal{G}(A)$ up to isomorphism, so we get a contradiction.

We now return to the study of the relationship between the bundles $E_{r,k}$ and their Fourier transforms. First note that throughout the rest of the paper we will use the fact that $J(X)$ is canonically isomorphic to its dual $\mathrm{Pic}^0(J(X))$ and consequently we will use the same notation for both. Thus all the Fourier transforms should be thought of as coming from the dual via this identification, although since there is no danger of confusion this will not be visible in the notation. The fiber of $E_{r,k}$ over a point $\xi \in J(X)$ is $H^0(SU_X(r,\xi),\mathcal{L}^k)$, while the fiber of $\widehat{E_{k,r}}$ over the same point is canonically isomorphic to $H^0(J(X), E_{r,k} \otimes P_\xi)$. By 2.2.3 the latter is isomorphic to $H^0(U_X(k,0),\mathcal{O}(r\Theta_{N\otimes\eta}))$, where $\eta^{\otimes r} \cong \xi$ (it is easy to see that this does not depend on the choice of η). The Strange Duality conjecture (see [2] §8 and [8] §5) says that there is a canonical isomorphism:

$$H^0(SU_X(r,\xi),\mathcal{L}^k)^* \cong H^0(U_X(k,0),\mathcal{O}(r\Theta_{N\otimes\eta}))$$

which will be described more precisely later in this section. This suggests then that one should relate somehow the vector bundles $E_{r,k}$ and $\widehat{E_{k,r}}$ via the diagram:



The first proposition treats the case $k = 1$ and establishes the fact that the dual of $E_{r,1}$ is nothing else but the Fourier transform of $\mathcal{O}_J(r\Theta_N)$.

Proposition 6.2.4. $E^*_{r,1} \cong \widehat{E_{1,r}}$.

Proof. By Mukai's duality theorem A.1, it is enough to show that $\widehat{E^*_{r,1}} \cong (-1_J)^* E_{1,r}$. But $E_{1,r}$ is just $\mathcal{O}_J(r\Theta_N)$, which is symmetric since N is a theta characteristic. So what we have to prove is

$$\widehat{E^*_{r,1}} \cong \mathcal{O}_J(r\Theta_N).$$

Since r_J is Galois with Galois group X_r, we have (see e.g [34] (2.1)):

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong \bigoplus_{\xi \in X_r} t_\xi^* \widehat{E^*_{r,1}}.$$

But translates commute with tensor products via the Fourier transform (cf. A.3(4)) and so

$$t_\xi^* \widehat{E^*_{r,1}} \cong \widehat{E^*_{r,1} \otimes P_\xi} \cong E^*_{r,1},$$

where as usual P_ξ is the line bundle in $\mathrm{Pic}^0(J(X))$ that corresponds to ξ and the last isomorphism follows from 6.1.8. Thus we get

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong \bigoplus_{r^{2g}} \widehat{E^*_{r,1}}$$

and the idea is to compute this bundle in a different way, by using the behavior of the Fourier transform under isogenies. More precisely, by applying A.3(1) we get the isomorphisms:

$$r_J^* r_{J*} \widehat{E^*_{r,1}} \cong r_J^* \widehat{r_J^* E^*_{r,1}} \cong r_J^* (V_{r,1} \otimes \mathcal{O}_J(-r\Theta_N))\widehat{}$$

$$\cong \bigoplus_{r^g} r_J^* \mathcal{O}_J\widehat{(-r\Theta_N)} \cong \bigoplus_{r^{2g}} \mathcal{O}_J(r\Theta_N).$$

The second isomorphism follows from 6.1.3, while the fourth follows from A.3(3) and the Verlinde formula at level 1. The outcome is the isomorphism

$$\bigoplus_{r^{2g}} \widehat{E^*_{r,1}} \cong \bigoplus_{r^{2g}} \mathcal{O}_J(r\Theta_N).$$

Now $\widehat{E^*_{r,1}}$ is simple since $E_{r,1}$ is simple, so the previous isomorphism implies the stronger fact that

$$\widehat{E^*_{r,1}} \cong \mathcal{O}_J(r\Theta_N).$$

□

Example 6.2.5. The relationship between the Chern character of a sheaf and that of its Fourier transform established in [29] (1.18) allows us to easily compute the first Chern class of $E_{r,1}$ as a consequence of the previous proposition. More precisely, if θ is the class of a theta divisor on $J(X)$, we have

$$c_1(\widehat{\mathcal{O}_J(r\Theta_N)}) = \mathrm{ch}_1(\widehat{\mathcal{O}_J(r\Theta_N)}) = (-1)\cdot PD_{2g-2}(\mathrm{ch}_{g-1}(\mathcal{O}_J(r\Theta_N))) =$$

$$= (-1)\cdot PD_{2g-2}(r^{g-1}\cdot\theta^{g-1}/(g-1)!) = -r^{g-1}\cdot\theta.$$

where $PD_{2g-2} : H^{2g-2}(J(X),\mathbf{Z}) \to H^2(J(X),\mathbf{Z})$ is the Poincaré duality. We get:

$$c_1(E_{r,1}) = r^{g-1}\cdot\theta.$$

We are able to prove the analogous fact for higher k's only modulo multiplication by r:

Proposition 6.2.6. $r_J^* E_{r,k}^* \cong r_J^* \widehat{E_{k,r}}$.

Proof. We know that $k_J^* E_{k,r} \cong \bigoplus_{s_{k,r}} \mathcal{O}_J(kr\Theta_N)$, so by A.3(1) we obtain

$$k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{s_{k,r}} \widehat{\mathcal{O}_J(kr\Theta_N)}.$$

As in the previous proposition, since the Galois group of k_J is X_k, we have

$$k_J^* k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{\xi\in X_k} t_\xi^*\widehat{E_{k,r}} \text{ and so } (kr)_J^* k_{J*}\widehat{E_{k,r}} \cong \bigoplus_{\xi\in X_k} r_J^* t_\xi^*\widehat{E_{k,r}}.$$

Moreover the isomorphisms above show as before that $\widehat{E_{k,r}}$ has to be semistable with respect to an arbitrary polarization. On the other by A.3(3) we have

$$(kr)_J^* k_{J*}\widehat{E_{r,k}} \cong \bigoplus_{s_{k,r}} (kr)_J^*\widehat{\mathcal{O}_J(kr\Theta_N)} \cong \bigoplus_{k^g r^g s_{k,r}} \mathcal{O}_J(-kr\Theta_N).$$

This gives us the isomorphism

$$\bigoplus_{\xi \in X_k} r_J^* t_\xi^* \widehat{E_{k,r}} \cong \bigoplus_{k^g r^g s_{k,r}} \mathcal{O}_J(-kr\Theta_N)$$

which in particular implies (recall semistability) that

$$r_J^* \widehat{E_{k,r}} \cong \bigoplus_{s_{r,k}} \mathcal{O}_J(-kr\Theta_N) \cong r_J^* E_{r,k}^*.$$

The important fact that the last index of summation is $s_{r,k}$ follows from the well-known "symmetry" of the Verlinde formula, which is:

$$r^g s_{k,r} = k^g s_{r,k}.$$

□

The propositions above allow us to give quick proofs of some results in [3] and [8] concerning duality between spaces of generalized theta functions. We first show how one can recapture a theorem of Donagi-Tu in the present context. The full version of the theorem (i.e. for arbitrary degree) can be obtained by the same method (cf. Section 7.2).

Theorem 6.2.7. *([8], Theorem 1) For any $L \in \mathrm{Pic}^0(X)$ and any $N \in \mathrm{Pic}^{g-1}(X)$, we have:*

$$h^0(SU_X(k,L), \mathcal{L}^r) \cdot r^g = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)) \cdot k^g.$$

Proof. We will actually prove the following equality:

$$h^0(SU_X(r,L), \mathcal{L}^k) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)).$$

The statement will then follow from the same symmetry of the Verlinde formula $r^g s_{k,r} = k^g s_{r,k}$ mentioned in the proof of 6.2.6. To this end we can use Proposition 6.2.6 to obtain $\mathrm{rk}(E_{r,k}^*) = \mathrm{rk}(\widehat{E_{k,r}})$. But on one hand

$$\mathrm{rk}(E_{r,k}^*) = h^0(SU_X(r,L), \mathcal{L}^k)$$

while on the other hand by A.3(5)

$$\mathrm{rk}(\widehat{E_{k,r}}) = h^0(J(X), E_{k,r}) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N))$$

as required. □

It is worth mentioning that since we are assuming the Verlinde formula all throughout, an important particular case of the theorem above is:

Corollary 6.2.8. *([3], Theorem 2)* $h^0(U_X(r,0), \mathcal{O}(\Theta_N)) = 1$.

As it is very well known, this fact is essential in setting up the strange duality. Turning to an application in this direction, the results above also lead to a simple proof of the strange duality at level 1, which has been first given a proof in [3], Theorem 3. It is important though to emphasize again that here, unlike in the quoted paper, the Verlinde formula is granted. So the purpose of the next application is to show how, with the knowledge of the Verlinde numbers, the strange duality at level 1 can simply be seen as the solution of a stability problem for vector bundles on $J(X)$. This may also provide a global method for understanding the conjecture for higher levels. A few facts in this direction will be mentioned at the end of this section (cf. Remark 6.2.13).

Before turning to the proof, we need the following general result, which is a globalization of §3 in [8].

Proposition 6.2.9. *Consider the tensor product map*

$$\tau : U_X(r,0) \times U_X(k,0) \longrightarrow U_X(kr,0)$$

and the map

$$\phi := \det \times \det : U_X(r,0) \times U_X(k,0) \longrightarrow J(X) \times J(X).$$

Then

$$\tau^*\mathcal{O}(\Theta_N) \cong p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P},$$

where $\mathcal{P}$ *is a Poincaré line bundle on* $J(X)\times J(X)$, *normalized such that* $\mathcal{P}_{|\{0\}\times J(X)} \cong \mathcal{O}_J$.

Proof. We will compare the restrictions of the two line bundles to fibers of the projections. First fix $F \in U_X(k,0)$. We have

$$\tau^*\mathcal{O}(\Theta_N)|_{U_X(r,0)\times\{F\}} \cong \tau_F^*\mathcal{O}(\Theta_N) \cong \mathcal{O}(\Theta_{F\otimes N}),$$

where τ_F is the map given by twisting with F. But by 2.2.3 one has

$$\mathcal{O}(\Theta_{F\otimes N}) \cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\mathrm{detF})(\cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\mathcal{P}_{|J(X)\times\{\det F\}})).$$

On the other hand obviously

$$p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}|_{U_X(r,0)\times\{F\}} \cong \mathcal{O}(k\Theta_N) \otimes \det{}^*(\det F).$$

Let's now fix $E \in U_X(r,0)$ such that $\det E \cong \mathcal{O}_X$. Using the same 2.2.3 we get

$$\tau^*\mathcal{O}(\Theta_N)|_{\{E\}\times U_X(k,0)} \cong \mathcal{O}(\Theta_{E\otimes N}) \cong \mathcal{O}(r\Theta_N)$$

and we also have

$$p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}|_{\{E\}\times U_X(k,0)}$$

$$\cong \mathcal{O}(r\Theta_N) \otimes \det{}^*(\mathcal{P}_{|\{\mathcal{O}_X\}\times J(X)}) \cong \mathcal{O}(r\Theta_N).$$

The desired isomorphism follows now from the see-saw principle(see e.g. [32], I.5.6). □

Theorem 6.2.7 tells us that there is essentially a unique nonzero section

$$s \in H^0(U_X(kr,0), \mathcal{O}(\Theta_N)),$$

which induces via τ a nonzero section

$$t \in H^0(U_X(r,0) \times U_X(k,0), \tau^*\mathcal{O}(\Theta_N)$$

$$\cong H^0(U_X(r,0) \times U_X(k,0), p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P}).$$

But notice that from the projection formula we get

$$\phi_*(p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N) \otimes \phi^*\mathcal{P})$$

$$\cong \phi_*(p_1^*\mathcal{O}_J(k\Theta_N) \otimes p_2^*\mathcal{O}_J(r\Theta_N)) \otimes \mathcal{P} \cong p_1^*E_{r,k} \otimes p_2^*E_{k,r} \otimes \mathcal{P},$$

so t induces a section (denoted also by t):

$$0 \neq t \in H^0(J(X) \times J(X), p_1^*E_{r,k} \otimes p_2^*E_{k,r} \otimes \mathcal{P}) \cong H^0(E_{r,k} \otimes \widehat{E_{k,r}}).$$

This is nothing else but a globalization of the section defining the strange duality morphism, as explained in [8] §5 (simply because for any $\xi \in$Pic$^0(X)$ the restriction of τ in 6.2.9 to $SU_X(r,\xi) \times U_X(k,0)$ is again the tensor product map

$$\tau : SU_X(r,\xi) \times U_X(k,0) \longrightarrow U_X(kr,0)$$

and $\tau^*\mathcal{O}(\Theta_N) \cong \mathcal{L}^k \boxtimes \mathcal{O}(r\Theta_N))$. In other words, t corresponds to a morphism of vector bundles

$$SD : E_{r,k}^* \longrightarrow \widehat{E_{k,r}}$$

which fiberwise is exactly the strange duality morphism

$$H^0(SU_X(r,\xi), \mathcal{L}^k)^* \xrightarrow{SD} H^0(U_X(k,0), \mathcal{O}(r\Theta_{N\otimes\eta})),$$

where $\eta^{\otimes r} \cong \xi$. Since this global morphism collects together all the strange duality morphisms as we vary ξ, the strange duality conjecture is equivalent to SD being an isomorphism.

Conjecture 6.2.10. $SD : E^*_{r,k} \longrightarrow \widehat{E_{k,r}}$ is an isomorphism of vector bundles.

In this context Proposition 6.2.6 can be seen as a weak form of "global" evidence for the conjecture in the case $k \geq 2$. For $k = 1$ it can now be easily proved.

Theorem 6.2.11. $SD : E^*_{r,1} \longrightarrow \widehat{E_{1,r}}$ *is an isomorphism.*

Corollary 6.2.12. *(cf. [3], Theorem 3) The level* 1 *strange duality morphism*

$$H^0(SU_X(r), \mathcal{L})^* \xrightarrow{SD} H^0(J(X), \mathcal{O}(r\Theta_N))$$

is an isomorphism.

Proof. (of 6.2.11) All the ingredients necessary for proving this have been discussed above: by 6.2.1 and 6.2.4, the bundles $E^*_{r,1}$ and $\widehat{E_{1,r}}$ are isomorphic and stable. This means that SD is essentially the unique nonzero morphism between them, and it must be an isomorphism. □

Remark 6.2.13. One step towards 6.2.10 is a better understanding of the properties of the kernel F of SD. A couple of interesting remarks in this direction can already be made. Since $E_{r,k}$ and $\widehat{E_{k,r}}$ are polystable of the same slope, the same will be true about F. On the other hand some simple calculus involving 2.2.3 and A.3(4) shows that F gets multiplied by a line bundle in $\mathrm{Pic}^0(J(X))$ when we translate it, so in the language of Mukai (e.g. [29] §3) it is a semi-homogeneous vector bundle (although clearly not homogeneous, i.e. not fixed by all translations).

6.3 A generalization of Raynaud's examples

In this section we would like to discuss a generalization of the examples of base points of the theta linear system $|\mathcal{L}|$ on $SU_X(m)$ constructed by Raynaud in [41]. For an introduction to this problem, the reader can consult Chapter 3. Let us recall here (see [36] §2) only that for a semistable vector bundle E of rank m to induce a base point of $|\mathcal{L}|$ it is sufficient that it satisfies the property

$$0 \leq \mu(E) \leq g-1 \text{ and } h^0(E \otimes L) \neq 0 \text{ for } L \in \mathrm{Pic}^0(X) \text{ generic.}$$

The examples of Raynaud are essentially the restrictions of $\widehat{E^*_{1,r}} = \widehat{\mathcal{O}_J(-r\Theta_N)}$ to some embedding of the curve X in the Jacobian. We will generalize this by considering the Fourier transform of higher level Verlinde bundles $\widehat{E^*_{k,r}}$ with $k \geq 2$.

For simplicity, let's fix k and r and denote $F := \widehat{E^*_{k,r}}$. This is a vector bundle by Lemma 6.1.6. Consider also an arbitrary embedding

$$j : X \to J(X)$$

and denote by E the restriction $F_{|X}$.

Proposition 6.3.1. *E is a semistable vector bundle.*

Proof. This follows basically from the proof of Proposition 6.2.6. One can see in a completely analogous way that:

$$r_J^* F \cong r_J^* E_{r,k} \cong \bigoplus_{s_{r,k}} \mathcal{O}_J(kr\Theta_N).$$

Now if we consider Y to be the preimage of X by r_J, this shows that $r_J^* F_{|Y}$ is semistable and so $F_{|X}$ is semistable. □

Proposition 6.3.2. *There is an embedding of X in $J(X)$ such that $E = F_{|X}$ satisfies $H^0(E \otimes L) \neq 0$ for $L \in \mathrm{Pic}^0(X)$ generic.*

Proof. The proof goes like in [41] (3.1) and we repeat it here for convenience: choose U a nonempty open subset of $J(X)$ on which $(-1_J)^* E_{k,r}$ is trivial. By Mukai's duality theorem A.1 we know that $\widehat{F} \cong (-1_J)^* E^*_{k,r}$, so there exists a nonzero section $s \in$

$\Gamma({p_2}^{-1}(U), p_1^*F \otimes \mathcal{P})$. Choose now an $x_0 \in J(X)$ such that $s_{|\{x_0\}\times U} \neq 0$ and consider an embedding of X in $J(X)$ passing through x_0. The image of s in $\Gamma(X \times U, p_1^*F \otimes \mathcal{P})$ is nonzero and this implies that $H^0(E \otimes L) \neq 0$ for L generic. □

We are only left with computing the invariants of E.

Proposition 6.3.3. *The rank and the slope of E are given by:*

$$\mathrm{rk}(E) = s_{r,k} = h^0(SU_X(r), \mathcal{L}^k) \text{ and } \mu(E) = gk/r.$$

Proof. By A.3(5) we have

$$\mathrm{rk}(F) = \mathrm{rk}(\widehat{E_{k,r}^*}) = h^0(E_{k,r}) = h^0(U_X(k,0), \mathcal{O}(r\Theta_N)).$$

But from the proof of 6.2.7 we know that $h^0(U_X(k,0), \mathcal{O}(r\Theta_N)) = h^0(SU_X(r,0), \mathcal{L}^k)$ and so $\mathrm{rk}(F) = s_{r,k}$. To compute the slope of E, first notice that by the proof of Proposition 6.3.1 we know that

$$r^{2g} \cdot \mu(E) = \deg(\mathcal{O}_J(kr\Theta_N)_{|Y}).$$

But $r_J^*\mathcal{O}_J(kr\Theta_N) \equiv \mathcal{O}_J(kr^3\Theta_N)$, so

$$r^2 \cdot \deg(\mathcal{O}_J(kr\Theta_N)_{|Y}) = \deg(r_J^*\mathcal{O}_J(kr\Theta_N)_{|Y}) = r^{2g} \cdot \deg(\mathcal{O}_J(kr\Theta_N)_{|X}) = r^{2g+1}kg.$$

Combining the two equalities we get

$$\mu(E) = \deg(\mathcal{O}_J(kr\Theta_N)_{|Y})/r^{2g} = gk/r.$$

□

In conclusion, for each r and k we obtain a semistable vector bundle E on X of rank equal to the Verlinde number $s_{r,k}$ and of slope gk/r, satisfying the property that $H^0(E \otimes L) \neq 0$ for L generic in $\mathrm{Pic}^0(X)$. So as long as $k < r$ and r divides

gk we obtain new examples of base points for $|\mathcal{L}|$ on the moduli spaces $SU_X(s_{r,k})$, as explained above. Raynaud's examples correspond to the case $k = 1$. See also [36] for a study of a different kind of examples of such base points and for bounds on the dimension of the base locus of $|\mathcal{L}|$.

CHAPTER VII

Effective global and normal generation on $U_X(r,0)$

7.1 Linear series on $U_X(r,0)$

The main application of the Verlinde vector bundles studied in the previous chapter concerns the global generation and normal generation of line bundles on $U_X(r,0)$. The specific goal is to give effective bounds for multiples of the generalized theta line bundles that satisfy the properties mentioned above. In this direction analogous results for $SU_X(r)$ will be used. The starting point is the following general result:

Proposition 7.1.1. *Let $f : X \to Y$ be a flat morphism of projective schemes, with reduced fibers, L a line bundle on X, and $E := f_*L$. Assume that if X_y denotes the fiber of f over $y \in Y$ the following conditions hold:*
(i) $h^1(L) = 0$
(ii) $h^i(L_{|X_y}) = 0, \forall y \in Y, \forall i > 0$.
Then L is globally generated as long as $L_{|X_y}$ is globally generated for all $y \in Y$ and E is globally generated.

Proof. Start with $x \in X$ and consider $y = f(x)$. By (i) we have the exact sequence on X:

$$0 \longrightarrow H^0(L \otimes \mathcal{I}_{X_y}) \longrightarrow H^0(L) \longrightarrow H^0(L_{|X_y}) \longrightarrow H^1(L \otimes \mathcal{I}_{X_y}) \longrightarrow 0.$$

The global generation of $L_{|X_y}$ implies that there exists a section $s \in H^0(L_{|X_y})$ such that $s(x) \neq 0$. We would like to lift s to some $\bar{s}$, so it is enough to prove that $H^1(L \otimes \mathcal{I}_{X_y}) = 0$. The fibers of f are reduced, so $\mathcal{I}_{X_y} \cong f^*\mathcal{I}_{\{y\}}$. Condition (ii) implies, by the base change theorem, that $R^if_*L = 0$ for all $i > 0$ and so by the projection formula we also get $R^if_*(L \otimes \mathcal{I}_{X_y}) = 0$ for all $i > 0$. The Leray spectral sequence then gives $H^i(E) \cong H^i(L)$ and $H^i(E \otimes \mathcal{I}_{\{y\}}) \cong H^i(L \otimes \mathcal{I}_{Xy})$ for all $i > 0$. The first isomorphism implies that there is an exact sequence:

$$0 \longrightarrow H^0(E \otimes \mathcal{I}_{\{y\}}) \longrightarrow H^0(E) \xrightarrow{ev_y} H^0(E_y) \longrightarrow H^1(E \otimes \mathcal{I}_{\{y\}}) \longrightarrow 0.$$

But E is globally generated, which means that ev_y is surjective. This implies the vanishing of $H^1(E \otimes \mathcal{I}_{\{y\}})$, which by the second isomorphism is equivalent to $H^1(L \otimes \mathcal{I}_{X_y}) = 0$. $\square$

The idea is to apply this result to the situation when the map is $\det : U_X(r, 0) \to J(X)$, $L = \mathcal{O}(k\Theta_N)$ and $E = E_{r,k}$. Modulo detecting for what k global generation is attained, the conditions of the proposition are satisfied. The fiberwise global generation problem (i.e. the $SU_X(r)$ case) has been given some effective solutions in the literature. The most recent is the author's result in [37], described in Chapter V, improving earlier bounds of Le Potier [26] and Hein [17]. We show there that $\mathcal{L}^k$ on $SU_X(r)$ is globally generated if $k \geq \frac{(r+1)^2}{4}$. We now turn to the effective statement for $E_{r,k}$ and in this direction we make essential use of Pareschi's cohomological criterion described in Appendix B.

Proposition 7.1.2. *$E_{r,k}$ is globally generated if and only if $k \geq r + 1$.*

Proof. The trick is to write $E_{r,k}$ as $E_{r,k} \otimes \mathcal{O}_J(-\Theta_N) \otimes \mathcal{O}_J(\Theta_N)$. Denote $E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)$ by F. Pareschi's criterion B.1 says in our case that $E_{r,k}$ is globally generated as long as the condition

$$h^i(F \otimes \alpha) = 0, \forall \alpha \in \mathrm{Pic}^0(X)$$

is satisfied (in fact under this assumption $F \otimes A$ will be globally generated for every ample line bundle A). Arguing as usual, $h^i(F \otimes \alpha) = 0$ is implied by $h^i(r_J^*(F \otimes \alpha)) = 0$. We have

$$r_J^* F \cong r_J^* E_{r,k} \otimes r_J^* \mathcal{O}_J(-\Theta_N) \cong \mathcal{O}_J((kr - r^2)\Theta_N)$$

and this easily gives the desired vanishing for $k \geq r+1$. On the other hand from 6.1.7 we know that $E_{r,r} \cong \bigoplus_{s_{r,r}} \mathcal{O}_J(\Theta_N)$, which is clearly not globally generated. This shows that the bound is optimal. □

Combining all these we obtain effective bounds for global generation on $U_X(r,0)$ in terms of the analogous bounds on $SU_X(r)$. We prefer to state the general result in a non-effective form though, in order to emphasize that it applies algorithmically (but see the corollaries for effective statements):

Theorem 7.1.3. *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ as long as $k \geq r+1$ and $\mathcal{L}^k$ is globally generated on $SU_X(r)$. Moreover, $\mathcal{O}(r\Theta_N)$ is not globally generated.*

Proof. The first part follows by puting together 7.1.2 and 7.1.1 in our particular setting. To prove that $\mathcal{O}(r\Theta_N)$ is not globally generated, let us begin by assuming the contrary. Then the restriction $\mathcal{L}^r$ of $\mathcal{O}(r\Theta_N)$ to any of the fibers $SU_X(r,L)$ is also globally generated.

Choose in particular a line bundle L outside the support of Θ_N on $J(X)$. Restriction to the fiber gives the following long exact sequence on cohomology:

$$0 \longrightarrow H^0(\mathcal{O}(r\Theta_N) \otimes \mathcal{I}_{SU_X(r,L)}) \longrightarrow H^0(\mathcal{O}(r\Theta_N)) \xrightarrow{\alpha}$$

$$\longrightarrow H^0(\mathcal{L}^r) \longrightarrow H^1(\mathcal{O}(r\Theta_N) \otimes \mathcal{I}_{SU_X(r,L)}) \longrightarrow 0.$$

As in the proof of 7.1.1, this sequence can be written in terms of the cohomology of $E_{r,r}$:

$$0 \longrightarrow H^0(E_{r,r} \otimes \mathcal{I}_{\{L\}}) \longrightarrow H^0(\mathcal{O}(r\Theta_N)) \xrightarrow{\alpha} H^0(\mathcal{L}^r) \longrightarrow H^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) \longrightarrow 0.$$

The assumption on $\mathcal{O}(r\Theta_N)$ ensures the fact that the map α in the sequence above is nonzero, and as a result

$$h^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) < h^0(\mathcal{L}^r) = s_{r,r}.$$

On the other hand we use again the fact 6.1.7 that $E_{r,r}$ is isomorphic to $\bigoplus_{s_{r,r}} \mathcal{O}_J(\Theta_N)$. The additional hypothesis that $L \in \Theta_N$ says then that

$$h^1(E_{r,r} \otimes \mathcal{I}_{\{L\}}) = s_{r,r}$$

which is a contradiction. □

As suggested above, by combining this with the effective result on $SU_X(r)$ given in Theorem 5.3.1, we get:

Corollary 7.1.4. *$\mathcal{O}(k\Theta_N)$ is globally generated on $U_X(r,0)$ for*

$$k \geq \frac{(r+1)^2}{4}.$$

It is important, as noted in the introduction, to emphasize the fact that the $SU_X(r)$ bound may still allow for improvement. Thus the content and formulation of the theorem certainly go beyond this corollary. For moduli spaces of vector bundles of rank 2 and 3 though, in view of the second part of 7.1.3, we actually have optimal results (see also [37] (5.4)).

Corollary 7.1.5. *(i) $\mathcal{O}(3\Theta_N)$ is globally generated on $U_X(2,0)$.*

(ii) $\mathcal{O}(4\Theta_N)$ is globally generated on $U_X(3,0)$.

These are natural extensions of the fact that $\mathcal{O}(2\Theta_N)$ is globally generated on $J(X) \cong U_X(1,0)$ (see e.g. [15], Theorem 2, p.317). In Section 7.2 we will also state some questions and conjectures about optimal bounds in general.

Remark 7.1.6. A similar technique can be applied to study the base point freeness of more general linear series on $U_X(r,0)$. This is done in Section 7.2.

Remark 7.1.7. A result analogous to 7.1.1, combined with a more careful study of the cohomological properties of $E_{r,k}$, gives information about effective separation of points by the linear series $|k\Theta_N|$. We will not insist on this aspect in the present work.

In the same spirit of studying properties of linear series on $U_X(r,0)$ via vector bundle techniques, one can look at multiplication maps on spaces of sections and normal generation. The Verlinde bundles are again an essential tool. The underlying theme is the study of surjectivity of the multiplication map

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \xrightarrow{\mu_k} H^0(\mathcal{O}(2k\Theta_N)).$$

To this respect we have to start by assuming that k is already chosen such that $\mathcal{L}^k$ and $E_{r,k}$ are globaly generated (in particular $k \geq r+1$). As proved in Theorem 7.1.3, this also induces the global generation of $\mathcal{O}(k\Theta_N)$. The method will be to look at the kernels of various multiplication maps–in the spirit of [25] for example–and study their cohomology vanishing properties. Let M_k on $U_X(r,0)$ and $M_{r,k}$ on $J(X)$ be the vector bundles defined by the sequences:

$$0 \longrightarrow M_k \longrightarrow H^0(\mathcal{O}(k\Theta_N)) \otimes \mathcal{O} \longrightarrow \mathcal{O}(k\Theta_N) \longrightarrow 0 \qquad (7.1)$$

and

$$0 \longrightarrow M_{r,k} \longrightarrow H^0(E_{r,k}) \otimes \mathcal{O}_J \longrightarrow E_{r,k} \longrightarrow 0. \tag{7.2}$$

By twisting (7.1) with $\mathcal{O}(k\Theta_N)$ and taking cohomology, it is clear that the surjectivity of μ_k is equivalent to $H^1(M_k \otimes \mathcal{O}(k\Theta_N)) = 0$.

On the other hand, the global generation of $\mathcal{O}(k\Theta_N)$ implies that the natural map $\det^* E_{r,k} \to \mathcal{O}(k\Theta_N)$ is surjective, so we can consider the vector bundle K defined by the following sequence:

$$0 \longrightarrow K \longrightarrow \det^* E_{r,k} \longrightarrow \mathcal{O}(k\Theta_N) \longrightarrow 0. \tag{7.3}$$

Remark 7.1.8. Fixing $L \in \mathrm{Pic}^0(X)$, we can also look at the evaluation sequence for $\mathcal{L}^k$ on $SU_X(r, L)$:

$$0 \longrightarrow M_{\mathcal{L}^k} \longrightarrow H^0(\mathcal{L}^k) \otimes \mathcal{O}_{SU_X} \longrightarrow \mathcal{L}^k \longrightarrow 0.$$

The sequence (7.3) should be interpreted as globalizing this picture. It induces the above sequence when restricted to the fiber of the determinant map over each L.

The study of vanishing for $M_{r,k}$ and K will be the key to obtaining the required vanishing for M_k. This is reflected in the top exact sequence in the following commutative diagram, obtained from (7.1),(7.2) and (7.3) as an application of the snake lemma:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \det^* M_{r,k} & \longrightarrow & M_k & \longrightarrow & K \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \det^* H^0(E_{r,k}) \otimes \mathcal{O} & \xrightarrow{\cong} & H^0(k\Theta_N) \otimes \mathcal{O} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & \det^* E_{r,k} & \longrightarrow & \mathcal{O}(k\Theta_N) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

We again state our result in part (b) of the following theorem in a form that allows algorithmic applications. The main ingredient is an effective normal generation bound for $E_{r,k}$, which is the content of part (a).

Theorem 7.1.9. *(a) The multiplication map*

$$H^0(E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(E_{r,k}^{\otimes 2})$$

is surjective for $k \geq 2r + 1$.

(b) Under the global generation assumptions formulated above, the multiplication map

$$\mu_k : H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \longrightarrow H^0(\mathcal{O}(2k\Theta_N))$$

is surjective as long as the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ on $SU_X(r)$ is surjective and $k \geq 2r + 1$.

Proof. (a) This is not hard to deal with when k is a multiple of r, since we know from 6.1.7 that $E_{r,k}$ decomposes in a particularly nice way. To tackle the general case though, we have to appeal to Proposition B.3. Concretely, we have to see precisely when the skew Pontrjagin product

$$E_{r,k} \widehat{*} E_{r,k} \cong E_{r,k} * (-1_J)^* E_{r,k}$$

is globally generated, and since by the initial choice of a theta characteristic $E_{r,k}$ is symmetric, this is the same as the global generation of the usual Pontrjagin product $E_{r,k} * E_{r,k}$. As an aside, recall from B.3 that this would imply the surjectivity of all the multiplication maps

$$H^0(t_x^* E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(t_x^* E_{r,k} \otimes E_{r,k})$$

for all $x \in J(X)$.

The global generation of this Pontrjagin product is in turn another application of the general cohomological criterion B.1 for vector bundles on abelian varieties. We first prove that $E_{r,k} * E_{r,k}$ also has a nice decomposition when pulled back by an isogeny, namely this time by multiplication by $2r$. Denote by F the Fourier transform $\widehat{E_{r,k}}$, so that $\widehat{F} \cong (-1_J)^* E_{r,k} \cong E_{r,k}$. Then we have the following isomorphisms:

$$E_{r,k} * E_{r,k} \cong \widehat{F} * \widehat{F} \cong \widehat{F \otimes F},$$

where the second one is obtained by the correspondence between the Potrjagin product and the tensor product via the Fourier-Mukai transform, as in A.3(2). Next, as in the previous sections, we look at the behaviour of our bundle when pulled back via certain isogenies (cf. A.3(1)):

$$k_{J*}(E_{r,k} * E_{r,k}) \cong \widehat{k_J^* F \otimes k_J^* F}. \tag{7.4}$$

In 6.2.6 we proved that

$$k_J^* F \cong k_J^* \widehat{E_{r,k}} \cong k_J^* E_{k,r}^* \cong \bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N).$$

and by plugging this into (5) we obtain

$$k_{J*}(E_{r,k} * E_{r,k}) \cong \Big(\bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N) \bigotimes \bigoplus_{s_{k,r}} \mathcal{O}_J(-kr\Theta_N)\Big)\widehat{}$$

$$\cong \bigoplus_{s^2_{k,r}} \widehat{\mathcal{O}_J(-2kr\Theta_N)}.$$

Finally we apply $(2r)_J^* \circ k_J^*$ to both sides of the isomorphism above and use the behavior of the Fourier transform of a line bundle when pulled back via the isogeny that it determines (see A.3(3)). Since $E_{r,k} * E_{r,k}$ is a direct summand in $k_J^* k_{J*}(E_{r,k} * E_{r,k})$, we obtain the desired decomposition:

$$(2r)_J^*(E_{r,k} * E_{r,k}) \cong \bigoplus \mathcal{O}_J(2kr\Theta_N). \tag{7.5}$$

This allows us to apply a trick analogous to the one used in the proof of 7.1.2. Namely (7) implies that

$$(2r)_J^*(E_{r,k} * E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)) \cong \bigoplus \mathcal{O}_J((2kr - 4r^2)\Theta_N).$$

Thus if we denote by $U_{r,k}$ the vector bundle $E_{r,k} * E_{r,k} \otimes \mathcal{O}_J(-\Theta_N)$ we clearly have:

$$h^i(U_{r,k} \otimes \alpha) = 0,\ \forall \alpha \in \mathrm{Pic}^0(X),\ \forall i > 0 \text{ and } \forall k \geq 2r+1.$$

Pareschi's criterion 7.3 immediately gives then that $E_{r,k} * E_{r,k}$ is globally generated for $k \geq 2r+1$, since

$$E_{r,k} * E_{r,k} \cong U_{r,k} \otimes \mathcal{O}_J(\Theta_N).$$

(b) We will show the vanishing of $H^1(M_k \otimes \mathcal{O}(k\Theta_N))$. By the top sequence in the diagram preceding the theorem, it is enough to prove that

$$H^1(K \otimes \mathcal{O}(k\Theta_N)) = 0 \quad\text{and}\quad H^1(\det{}^* M_{r,k} \otimes \mathcal{O}(k\Theta_N)) = 0.$$

First we prove the vanishing of $H^1(K \otimes \mathcal{O}(k\Theta_N))$. The key point is to identify the pull-back of K by the étale cover τ in the diagram

$$\begin{array}{ccc} SU_X(r) \times J(X) & \xrightarrow{\ \tau\ } & U_X(r,0) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \det} \\ J(X) & \xrightarrow{\ r_J\ } & J(X) \end{array}$$

described in §6.1. In the pull-back sequence

$$0 \longrightarrow \tau^*K \longrightarrow \tau^*\det{}^*E_{r,k} \longrightarrow \tau^*\mathcal{O}(k\Theta_N) \longrightarrow 0$$

we can identify $\tau^*\det{}^*E_{r,k}$ with $p_2^*r_J{}^*E_{r,k}$ and $\tau^*\mathcal{O}(k\Theta_N)$ with $\mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N)$. In other words we have the exact sequence

$$0 \longrightarrow \tau^*K \longrightarrow H^0(\mathcal{L}^k) \otimes \mathcal{O}_J(kr\Theta_N) \longrightarrow \mathcal{L}^k \boxtimes \mathcal{O}_J(kr\Theta_N) \longrightarrow 0,$$

which shows that the following isomorphism holds (cf. 7.1.8):

$$\tau^*K \cong M_{\mathcal{L}^k} \boxtimes \mathcal{O}_J(kr\Theta_N).$$

Finally we obtain the isomorphism

$$\tau^*(K \otimes \mathcal{O}(k\Theta_N)) \cong (M_{\mathcal{L}^k} \otimes \mathcal{L}^k) \boxtimes \mathcal{O}_J(2kr\Theta_N).$$

Certainly by the argument mentioned earlier the surjectivity of the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ is also equivalent to $H^1(M_{\mathcal{L}^k} \otimes \mathcal{L}^k) = 0$. The required vanishing is then an easy application of the Künneth formula.

The next step is to prove the vanishing of $H^1(\det{}^*M_{r,k} \otimes \mathcal{O}(k\Theta_N))$. From the projection formula we know that

$$R^i\det{}_*(\det{}^*M_{r,k} \otimes \mathcal{O}(k\Theta_N)) \cong M_{r,k} \otimes R^i\det{}_*\mathcal{O}(k\Theta_N) = 0 \text{ for all } i > 0$$

since obviously $R^i\det{}_*\mathcal{O}(k\Theta_N) = 0$ for all $i > 0$. The Leray spectral sequence reduces then our problem to proving the vanishing $H^1(M_{r,k} \otimes E_{r,k}) = 0$, which is basically equivalent to the surjectivity of the multiplication map

$$H^0(E_{r,k}) \otimes H^0(E_{r,k}) \longrightarrow H^0(E_{r,k}^{\otimes 2}).$$

This is the content of part (a). □

Corollary 7.1.10. *For k as in 7.1.9, $\mathcal{O}(k\Theta_N)$ is very ample.*

Proof. Since Θ_N is ample, by a standard argument the assertion is true if the multiplication maps

$$H^0(\mathcal{O}(k\Theta_N) \otimes H^0(\mathcal{O}(kl\Theta_N)) \longrightarrow H^0(\mathcal{O}(k(l+1)\Theta_N))$$

are surjective for $l \geq 1$. For $l = 1$ this is proved in the theorem and the case $l \geq 2$ is similar but easier. □

Remark 7.1.11. The case of line bundles in the theorem above (i.e. $r = 1$) is the statement for Jacobians of a well known theorem of Koizumi (see [21] and [43]). Applied to that particular case, the metod of proof in Step 2 above is of course implicit in Pareschi's paper [35].

For an effective bound implied by the previous theorem we have to restrict ourselves to the case of rank 2 vector bundles, since to the best of our knowledge nothing is known about multiplication maps on $SU_X(r)$ for $r \geq 3$.

Corollary 7.1.12. *For a generic curve X the multiplication map*

$$H^0(\mathcal{O}(k\Theta_N)) \otimes H^0(\mathcal{O}(k\Theta_N)) \xrightarrow{\mu_k} H^0(\mathcal{O}(2k\Theta_N))$$

on $U_X(2,0)$ is surjective for $k \geq \max\{5, g-2\}$ and so $\mathcal{O}(k\Theta_N)$ is very ample for such k.

Proof. This follows by a theorem of Laszlo [24], which says that on a generic curve the multiplication map

$$S^k H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^k)$$

on $SU_X(2)$ is surjective for $k \geq 2g - 4$. See also [2] §4 for a survey of results in this direction. □

Remark 7.1.13. A more refined study along these lines gives analogous results in the extended setting of higher syzygies and N_p properties. We hope to come back to this somewhere else.

We would like to end this section with another application to multiplication maps. Although probably not of the same significance as the previous results, it still brings some new insight through the use of methods characteristic to abelian varieties. Recall from 2.2.2 that the Picard group of $U_X(r,0)$ is generated by $\mathcal{O}(\Theta_N)$ and the preimages of line bundles on $J(X)$. We want to study "mixed" multiplication maps of the form:

$$H^0(\mathcal{O}(k\Theta_N) \otimes H^0(\det^*\mathcal{O}_J(m\Theta_N)) \xrightarrow{\alpha} H^0(\mathcal{O}(k\Theta_N) \otimes \det^*\mathcal{O}_J(m\Theta_N)). \qquad (7.6)$$

Proposition 7.1.14. *The multiplication map α in (7.6) is surjective if $m \geq 2$ and $k \geq 2r+1$.*

Proof. By repeated use of the projection formula, from the commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}(k\Theta_N) \otimes H^0(\det^*\mathcal{O}_J(m\Theta_N)) & \xrightarrow{\alpha} & H^0(\mathcal{O}(k\Theta_N) \otimes \det^*\mathcal{O}_J(m\Theta_N)) \\ \cong\downarrow & & \downarrow\cong \\ H^0(E_{r,k}) \otimes H^0(\mathcal{O}_J(m\Theta_N)) & \xrightarrow{\beta} & H^0(E_{r,k} \otimes \mathcal{O}_J(m\Theta_N)) \end{array}$$

we see that it is enough to prove the surjectivity of the multiplication map β on $J(X)$.

This is an application of the cohomological criterion B.4 going back to Kempf [20]. What we need to check is that

$$h^i(E_{r,k} \otimes \mathcal{O}_J(l\Theta_N) \otimes \alpha) = 0,\ \forall i > 0,\ \forall l \geq -2 \text{ and } \forall \alpha \in \mathrm{Pic}^0(J(X)).$$

It is again enough to prove this after pulling back by multiplication by r. Now:

$$r_J^*(E_{r,k} \otimes \mathcal{O}_J(l\Theta_N) \otimes \alpha) \cong \bigoplus_{s_{r,k}} \mathcal{O}_J((kr + lr^2)\Theta_N) \otimes r_J^*\alpha$$

hence the required vanishings are obvious as long as $l \geq -2$ and $k \geq 2r+1$. □

7.2 Variants for arbitrary degree

As it is natural to expect, some of the facts discussed in the previous sections for moduli spaces of vector bundles of degree 0 can be extended to arbitrary degree. On the other hand, as the reader might have already observed, there are results that do not admit (at least straightforward) such extensions. In this last paragraph we would like to emphasize what can and what cannot be generalized using the present techniques.

Fix r and d arbitrary positive integers. Then we can look at the moduli space $U_X(r,d)$ of semistable vector bundles of rank r and degree d. For $A \in \mathrm{Pic}^d(X)$, denote also by $SU_X(r,A)$ the moduli space of rank r bundles with fixed determinant A. We will write $SU_X(r,d)$ when it is not important what determinant is involved. On these moduli spaces one can construct generalized theta divisors as in the degree 0 case. More precisely, denote

$$h = \gcd(r,d),\ r_1 = r/h \text{ and } d_1 = d/h.$$

Then for any vector bundle F of rank r_1 and degree d_1, we can consider Θ_F to be the closure in $U_X(r,d)$ of the locus

$$\Theta_F^s = \{E|\ h^0(E \otimes F) \neq 0\} \subset U_X^s(r,d).$$

This does not always have to be a proper subset, but it is so for generic F (see [18]) and in that case Θ_F is a divisor. We can of course do the same thing with vector bundles of rank kr_1 and degree kd_1 for any $k \geq 1$ and a formula analogous to 2.2.3 holds. On $SU_X(r,d)$ there are similar divisors Θ_F and they all determine the same determinant line bundle. As before this generates $\mathrm{Pic}(SU_X(r,d))$ and is denoted by $\mathcal{L}$.

To study the linear series determined by these divisors we define Verlinde type bundles as in the previous chapter. They will now depend on two parameters (again not emphasized by the notation). Namely fix $F \in U_X(r_1, r_1(g-1) - d_1)$ and $L \in \mathrm{Pic}^d(X)$. Consider the composition

$$\pi_L : U_X(r,d) \xrightarrow{\det} \mathrm{Pic}^d(X) \xrightarrow{\otimes L^{-1}} J(X)$$

and define:

$$E_{r,d,k} (= E^{F,L}_{r,d,k}) := \pi_{L*}\mathcal{O}(k\Theta_F).$$

This is a vector bundle on $J(X)$ of rank $s_{r,d,k} := h^0(SU_X(r,d), \mathcal{L}^k)$. There is again a fiber diagram

$$\begin{array}{ccc} SU_X(r,A) \times J(X) & \xrightarrow{\tau} & U_X(r,d) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi_L} \\ J(X) & \xrightarrow{r_J} & J(X) \end{array}$$

where τ is given by tensor product and the top and bottom maps are Galois with Galois group X_r. By [8] §3 one has the formula

$$\tau^*\mathcal{O}(\Theta_F) \cong \mathcal{L} \boxtimes \mathcal{O}_J(krr_1\Theta_N),$$

where $N \in \mathrm{Pic}^{g-1}(X)$ is a line bundle such that $N^{\otimes r} \cong L \otimes (\det F)^{\otimes h}$. As in 6.1.3 we obtain the decomposition:

$$r_J^* E_{r,d,k} \cong \bigoplus_{s_{r,d,k}} \mathcal{O}_J(krr_1\Theta_N).$$

The basic duality setup via Fourier-Mukai transform presented in Section 6.2 can be extended with a little extra care to this general setting. The purpose is to relate linear series on the complementary moduli spaces $SU_X(r,d)$ and $U_X(kr_1, kr_1(g-1) - kd_1)$

(cf. [8] §5) and this is realized via a diagram of the form:

$$\begin{array}{ccccc} U_X(r,d) & & & & U_X(kr_1, kr_1(g-1)-kd_1) \\ \Big\downarrow \pi_L & & J(X)\times J(X) & & \Big\downarrow \pi_M \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ J(X) & & & & J(X) \end{array}$$

where $L \in \mathrm{Pic}^d(X)$, $M \in \mathrm{Pic}^{kr_1(g-1)-kd_1}(X)$ and π_L and π_M are defined as above. One can also choose a vector bundle $G \in U_X(r_1, d_1)$ and consider the Verlinde bundle $E_{kr_1,kr_1(g-1)-kd_1,h}$ associated to G and M. As before, there is an obvious tensor product map

$$U_X(r,d) \times U_X(kr_1, kr_1(g-1)-kd_1) \xrightarrow{\tau} U_X(krr_1, krr_1(g-1)).$$

With the extra (harmless) assumption on our choices that $L \cong (\det G)^{\otimes h}$ and $M \cong (\det F)^{\otimes k}$ we can show exactly as in 6.2.9 that

$$\tau^*\mathcal{O}(\Theta) \cong \mathcal{O}(k\Theta_F) \boxtimes \mathcal{O}(h\Theta_G) \otimes (\pi_L \times \pi_M)^*\mathcal{P},$$

where $\mathcal{P}$ is a normalized Poincaré line bundle on $J(X) \times J(X)$. Note that this is slightly different from 6.2.9 in the sense that we are twisting up to slope $g-1$ and on $U_X(krr_1, krr_1(g-1))$, Θ represents the canonical theta divisor. The two formulations are of course equivalent.

The unique nonzero section of $\mathcal{O}(\Theta)$ induces then a nonzero map:

$$SD : E^*_{r,d,k} \longrightarrow (E_{kr_1,kr_1(g-1)-kd_1,h})\widehat{}$$

and the global formulation of the full strange duality conjecture is:

Conjecture 7.2.1. (cf. [8] §5) SD is an isomorphism.

On the positive side, the properties of the kernel of this map described in 6.2.13 still hold. On the other hand, the method of proof of Theorem 6.2.11 cannot be used for arbitrary degree even in the level 1 situation. The point is that these new Verlinde type bundles may always fail to be simple. Probably the most suggestive example is the case of r and d coprime, when the other extreme is attained for any k:

Example 7.2.2. If $\gcd(r,d) = 1$, then $E_{r,d,k}$ decomposes as a direct sum of line bundles for any k. More precisely:

$$E_{r,d,k} = \bigoplus_{s_{r,d,k}} \mathcal{O}(k\Theta_N), \ \forall k \geq 1.$$

This can be seen by imitating the proof of 6.1.7.

On a more modest note, the main result of [8] can be naturally integrated into these global arguments on $J(X)$. It is obtained by calculus with Fourier transforms in the spirit of Section 6.1 and we do not repeat the argument here:

Proposition 7.2.3. *([8], Theorem 1)* $h^0(U_X(r,d), \mathcal{O}(k\Theta_F)) = \frac{k^g}{h^g} \cdot s_{r,d,k}$.

Turning to effective global generation and normal generation $U_X(r,d)$, the picture described in Section 7.1 completely extends, with the appropriate modifications, to the general case. All the effective bounds turn out to depend on the number $h = \gcd(r,d)$. The global generation result analogous to 7.1.3 is formulated as follows:

Theorem 7.2.4. *$\mathcal{O}(k\Theta_F)$ is globally generated on $U_X(r,d)$ as long as $k \geq h+1$ and $\mathcal{L}^k$ is globally generated on $SU_X(r,d)$. Moreover, $\mathcal{O}(k\Theta_F)$ is not globally generated for $k \leq h$.*

In Theorem 5.3.1 it is proved that $\mathcal{L}^k$ is globally generated on $SU_X(r,d)$ for $k \geq \max\{\frac{(r+1)^2}{4r}h, \frac{r^2}{4s}h\}$, where s is an invariant of the moduli space that we will not define here, but satisfying $s \geq h$ so that in particular $k \geq \frac{(r+1)^2}{4}$ always works. This implies then:

Corollary 7.2.5. *$\mathcal{O}(k\Theta_F)$ is globally generated on $U_X(r,d)$ for $k \geq \max\{\frac{(r+1)^2}{4r}h, \frac{r^2}{4s}h\}$.*

This again produces optimal results in the case of rank 2 and rank 3 vector bundles:

Corollary 7.2.6. *$\mathcal{O}(2\Theta_F)$ is globally generated on $U_X(2,1)$ and $U_X(3,\pm 1)$.*

In the same vein, the normal generation result 7.1.9 can be generalized to:

Theorem 7.2.7. *The multiplication map*

$$\mu_k : H^0(\mathcal{O}(k\Theta_F)) \otimes H^0(\mathcal{O}(k\Theta_F)) \longrightarrow H^0(\mathcal{O}(2k\Theta_F))$$

on $U_X(r,d)$ is surjective as long as the multiplication map $H^0(\mathcal{L}^k) \otimes H^0(\mathcal{L}^k) \to H^0(\mathcal{L}^{2k})$ on $SU_X(r,d)$ is surjective and $k \geq 2h+1$. For such k, $\mathcal{O}(k\Theta_F)$ is very ample.

Corollary 7.2.8. *For X generic μ_k is surjective on $U_X(2,1)$ if $k \geq \max\{3, \frac{g-2}{2}\}$.*

Proof. This is a consequence of Theorem 7.2.7 and [24], where it is proved that on $SU_X(2,1)$

$$S^k H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}^k)$$

is surjective for $k \geq g-2$. $\square$

It is certainly not hard to formulate further results corresponding to 7.1.14. We leave this to the interested reader.

7.3 Some further conjectures in both the GL_n and SL_n cases and a generalization

It is interesting to see how the bounds given in Chapter V and in this chapter relate to possible optimal bounds and consequently formulate some conjectures and questions in this direction. Given the shape of the results, we will carry out the discussion in the case of $SU_X(r)$ (and $U_X(r,0)$) and $SU_X(r,\pm 1)$, based on the results 5.3.3 and 5.3.4. A similar analysis can be applied to any other case, but we will not give any details here.

We begin by looking at degree 0 vector bundles, where global generation is attained for $k \geq \frac{(r+1)^2}{4}$, with the improvement 5.2.1 in the case of rank 2, when $k \geq 1$ suffices. In view of 7.1.3, the bound in 7.1.4 is optimal in the case of rank 2 and rank 3 vector bundles. Let us recall the result for convenience.

Corollary 7.3.1. *Let $N \in \mathrm{Pic}^{g-1}(X)$. Then:*

(i) $\mathcal{O}_U(3\Theta_N)$ is globally generated on $U_X(2,0)$.

(ii) $\mathcal{O}_U(4\Theta_N)$ is globally generated on $U_X(3,0)$.

This could be seen as a natural extension of the classical fact that $\mathcal{O}_J(2\Theta_N)$ is globally generated on $J(X) \cong U_X(1,0)$. In presence of this evidence it is natural to conjecture that this is indeed the case for any rank:

Conjecture 7.3.2. For any $r \geq 1$, $\mathcal{O}_U(k\Theta_N)$ is globally generated on $U_X(r,0)$ for $k \geq r+1$.

This is the best that one can hope for and there is a possibility that it might be a little too optimistic, or in other words that Corollary 7.3.1 might be an accident of low values of a quadratic function. On the other hand if that is the case, the theorem should be very close to being optimal. Turning to $SU_X(r)$, in Chapter III we showed that, granting the Strange Duality conjecture, the optimal bound for the global generation of $\mathcal{L}^k$ should also go up as we increase the rank r. The case of rank 2 vector bundles 5.2.1 suggests though that we could ask for a slightly better result than for $U_X(r,0)$, but unfortunately further evidence is still missing:

Conjecture/Question 7.3.3. Is $\mathcal{L}^k$ globally generated on $SU_X(r)$ for $k \geq r-1$?

Note also that in view of 7.1.3, any positive answer in the range $\{r-1, r, r+1\}$ would imply the optimal conjecture 7.3.2.

In the case of $SU_X(r, \pm 1)$ 5.3.4 and 7.1.3 give that $\mathcal{O}_U(k\Theta_F)$ is globally generated for $k \geq \max\{2, r-1\}$, while $\mathcal{O}_U(\Theta_F)$ cannot be. We obtain thus again optimal bounds for rank 2 and rank 3 vector bundles.

Corollary 7.3.4. *$\mathcal{O}_U(2\Theta_F)$ is globally generated on $U_X(2,1)$ and $U_X(3,\pm 1)$.*

Note also that for all the examples of special vector bundles constructed in [41], [36] and [38] we have $h \neq 1$, therefore theoretically an optimal bound that does not depend on the rank r is still possible. It is natural to ask if the best possible result always holds:

Question 7.3.5. Is $\mathcal{O}_U(2\Theta_F)$ on $U_X(r,\pm 1)$, and so also $\mathcal{L}^2$ on $SU_X(r,\pm 1)$, globally generated? More generally, is this true whenever r and d are coprime?

We conclude the section with a generalization of Theorem 7.1.3. For simplicity we present it only in the degree 0 case, but the extension to other degrees is immediate. Recall from Section 2.2 that for $N \in \mathrm{Pic}^{g-1}(X)$, $\mathrm{Pic}(U_X(r,0)) \cong \mathbf{Z} \cdot \mathcal{O}(\Theta_N) \oplus \det^*\mathrm{Pic}(J(X))$. The method provided by the Verlinde bundles allows one to study effective global generation for "mixed" line bundles of the form $\mathcal{O}(k\Theta_N) \otimes \det * L$ with $L \in \mathrm{Pic}(J(X))$. Concretely we have the following cohomological criterion (assume $r \geq 2$):

Theorem 7.3.6. *$\mathcal{O}(k\Theta_N) \otimes \det^* L$ is globally generated if $k \geq \frac{(r+1)^2}{4}$ and*

$$h^i(\mathcal{O}_J((kr - r^2)\Theta_N) \otimes L^{\otimes r^2} \otimes \alpha) = 0,\ \forall i > 0,\ \forall \alpha \in \mathrm{Pic}^0(J(X)).$$

Proof. By the projection formula, for every $i > 0$ we have:

$$R^i\det_*(\mathcal{O}_U(k\Theta_N) \otimes \det^* L) \cong R^i\det_*\mathcal{O}_U(k\Theta_N) \otimes L = 0.$$

Also the restriction of $\mathcal{O}_U(k\Theta_N) \otimes \det^* L$ to any fiber $SU_X(r,\xi)$ of the determinant map is isomorphic to $\mathcal{L}^k$ and so globally generated for $k \geq \frac{(r+1)^2}{4}$. It is a simple consequence of the general machinery described in Section 7.1 Proposition 7.1.1, that in these conditions the statement holds as soon as

$$\det_*(\mathcal{O}_U(k\Theta_N) \otimes \det^* L) \cong E_{r,k} \otimes L$$

is globally generated on $J(X)$, where $E_{r,k}$. To study this we make use, as in Section 7.1, of the cohomological criterion for global generation of vector bundles on abelian varieties B.1. In our particular setting it says that $E_{r,k} \otimes L$ is globally generated if there exists some ample line bundle A on $J(X)$ such that

$$h^i(E_{r,k} \otimes L \otimes A^{-1} \otimes \alpha) = 0,\ \forall i > 0,\ \forall \alpha \in \mathrm{Pic}^0(J(X)).$$

We chose A to be $\mathcal{O}_J(\Theta_N)$, where Θ_N is the theta divisor on $J(X)$ associated to N. The cohomology vanishing that we need is true if it holds for the pullback of $E_{r,k} \otimes L \otimes \mathcal{O}_J(-\Theta_N) \otimes \alpha$ by any finite cover of $J(X)$. But recall from Lemma 6.1.3 that $r_J^* E_{r,k} \cong \bigoplus \mathcal{O}_J(kr\Theta_N)$, where r_J is the multiplication by r. Since $r_J^* L \equiv L^{\otimes r^2}$, via pulling back by r_J the required vanishing certainly holds if

$$h^i(\mathcal{O}_J((kr - r^2)\Theta_N) \otimes L^{\otimes r^2} \otimes \alpha) = 0,\ \forall i > 0,\ \forall \alpha \in \mathrm{Pic}^0(J(X)).$$

□

Corollary 7.3.7. *If* $l \in \mathbf{Z}$, $\mathcal{O}(k\Theta_N) \otimes \det^*\mathcal{O}_J(l\Theta_N)$ *is globally generated for*

$$k \geq \max\{r + 1 - lr, \frac{(r+1)^2}{4}\}.$$

APPENDICES

Appendix A. The Fourier-Mukai transform on an abelian variety

Here we give a brief overview of some basic facts on the Fourier-Mukai transform on an abelian variety, following the original paper of Mukai [28]. Let X be an abelian variety of dimension g, $\widehat{X}$ its dual and $\mathcal{P}$ the Poincaré line bundle on $X \times \widehat{X}$, normalized such that $\mathcal{P}_{|X\times\{0\}}$ and $\mathcal{P}_{|\{0\}\times\widehat{X}}$ are trivial. To any coherent sheaf $\mathcal{F}$ on X we can associate the sheaf $p_{2*}(p_1{}^*\mathcal{F} \otimes \mathcal{P})$ on $\widehat{X}$ via the natural diagram:

$$\begin{array}{ccccc} & & X \times \widehat{X} & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & \widehat{X} \end{array}$$

This correspondence gives a functor

$$\mathcal{S} : \mathrm{Coh}(\mathrm{X}) \to \mathrm{Coh}(\widehat{X}).$$

If we denote by $D(X)$ and $D(\widehat{X})$ the derived categories of $\mathrm{Coh}(X)$ and $\mathrm{Coh}(\widehat{X})$, then the derived functor $\mathbf{R}\mathcal{S} : D(X) \to D(\widehat{X})$ is defined (and called the Fourier functor) and one can consider $\mathbf{R}\widehat{\mathcal{S}} : D(\widehat{X}) \to D(X)$ in a similar way. Mukai's main theorem is the following:

Theorem. A.1*([28] (2.2)) The Fourier functor establishes an equivalence of categories between $D(X)$ and $D(\widehat{X})$. More precisely there are isomorphisms of functors:*

$$\mathbf{R}\mathcal{S} \circ \mathbf{R}\widehat{\mathcal{S}} \cong (-1_{\widehat{X}})^*[-g]$$

$$\mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\mathcal{S} \cong (-1_X)^*[-g].$$

In this paper we will essentially have to deal with the simple situation when by applying the Fourier functor we get back another vector bundle, i.e. a complex with only one nonzero (and locally free) term. This is packaged in the following definition (see [28] (2.3):

Definition. A.2 *A coherent sheaf $\mathcal{F}$ on X satisfies I.T. (index theorem) with index j if $H^i(\mathcal{F}\otimes\alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0(X)$ and all $i \neq j$. In this situation we have $R^i\mathcal{S}(\mathcal{F}) = 0$ for all $i \neq j$ and by the base change theorem $R^j\mathcal{S}(\mathcal{F})$ is locally free. We denote $R^j\mathcal{S}(\mathcal{F})$ by $\widehat{\mathcal{F}}$ and call it the Fourier transform of $\mathcal{F}$. Note that then* $\mathbf{R}\mathcal{S}(\mathcal{F}) \cong \widehat{F}[-j]$.

For later reference we list some basic properties of the Fourier transform that will be used repeatedly throughout the paper:

Proposition. A.3 *Let X be an abelian variety and $\widehat{X}$ its dual and let $\mathbf{R}\mathcal{S}$ and $\mathbf{R}\widehat{\mathcal{S}}$ the corresponding Fourier functors. Then the following are true:*

(1)([28] (3.4)) Let Y be an abelian veriety, $f : Y \to X$ an isogeny and $\hat{f} : \widehat{X} \to \widehat{Y}$ the dual isogeny. Then there are isomorphisms of functors:

$$f^* \circ \mathbf{R}\widehat{\mathcal{S}}_X \cong \mathbf{R}\widehat{\mathcal{S}}_Y \circ \hat{f}_*$$

$$f_* \circ \mathbf{R}\widehat{\mathcal{S}}_Y \cong \mathbf{R}\widehat{\mathcal{S}}_X \circ \hat{f}^*.$$

(2)([28] (3.7)) Let $\mathcal{F}$ and $\mathcal{G}$ coherent sheaves on X and define their Pontrjagin product by $\mathcal{F} * \mathcal{G} := \mu_*({p_1}^*\mathcal{F} \otimes {p_2}^*\mathcal{G})$, where $\mu : X \times X \to X$ is the multiplication on X. Then we have the following isomorphisms:

$$\mathbf{R}\mathcal{S}(\mathcal{F} * \mathcal{G}) \cong \mathbf{R}\mathcal{S}(\mathcal{F}) \otimes \mathbf{R}\mathcal{S}(\mathcal{G})$$

$$\mathbf{R}\mathcal{S}(\mathcal{F} \otimes \mathcal{G}) \cong \mathbf{R}\mathcal{S}(\mathcal{F}) * \mathbf{R}\mathcal{S}(\mathcal{G})[g],$$

where the operations on the right hand side should be thought of in the derived category.

(3)([28] (3.11)) Let L be a nondegenerate line bundle on X of index i, i.e. $h^i(L) \neq 0$ and $h^j(L) = 0$ for all $j \neq i$. Then by [32] §16, I.T. holds for L and there is an

isomorphism

$$\phi_L^* \widehat{L} \cong H^i(L) \otimes L^{-1} \ (\cong \bigoplus_{|\chi(L)|} L^{-1}),$$

with ϕ_L *the isogeny canonically defined by* L.

(4)([28] (3.1)) Let $\mathcal{F}$ *be a coherent sheaf on* X, $x \in \widehat{X}$ *and* $P_x \in \mathrm{Pic}^0(X)$ *the corresponding line bundle. Then we have an isomorphism:*

$$\mathbf{R}\mathcal{S}(\mathcal{F} \otimes P_x) \cong t_x^* \mathbf{R}\mathcal{S}(\mathcal{F}),$$

where t_x *is translation by* x.

(5)([28] (2.4),(2.5),(2.8)) Assume that E *satisfies I.T. with index* i. *Then* $\widehat{E}$ *satisfies I.T. with index* $g - i$ *and in this case*

$$\chi(E) = (-1)^i \cdot \mathrm{rk}(\widehat{E}).$$

Moreover, there are isomorphisms $\mathrm{Ext}^k(E, E) \cong \mathrm{Ext}^k(\widehat{E}, \widehat{E})$ *for all* k, *and in particular* E *is simple if and only if* $\widehat{E}$ *is simple.*

Appendix B. Global generation and normal generation of vector bundles on abelian varieties

For the reader's convenience, in the present paragraph we give a brief account of some very recent results and techniques in the study of vector bundles on abelian varieties – following work of Pareschi [35] – in a form convenient for our purposes. The underlying theme is to give useful criteria for the global generation and surjectivity of multiplication maps of such vector bundles.

Let X be an abelian variety and E a vector bundle on X. Building on earlier work of Kempf [20], Pareschi proves the following cohomological criterion for global generation:

Theorem. B.1 *([35] (2.1)) Assume that E satisfies the following vanishing property:*

$$h^i(E \otimes \alpha) = 0, \ \forall \alpha \in \mathrm{Pic}^0(X) \text{ and } \forall i > 0.$$

Then for any ample line bundle L on X, $E \otimes L$ is globally generated.

In another direction, in order to attack questions about multiplication maps of the form

$$H^0(E) \otimes H^0(F) \longrightarrow H^0(E \otimes F) \tag{.7}$$

for E and F vector bundles on X, the right notion turns out to be that of skew Pontrjagin product:

Definition. B.2 *Let E and F be coherent sheaves on the abelian variety X. Then the skew Pontrjagin product of E and F is defined by*

$$E \hat{*} F := p_{1*}((p_1 + p_2)^* E \otimes p_2^* F)$$

where $p_1, p_2 : X \times X \to X$ are the two projections.

The following is a simple but essential result relating the skew Pontrjagin product to the surjectivity of the multiplication map (1). It is a restatement of [35] (1.1) in a form convenient to us and we reproduce Pareschi's argument for the sake of completeness.

Proposition. B.3 *Assume that $E\hat{*}F$ is globally generated and that $h^i(t_x^*E \otimes F) = 0$ for all $x \in X$ and all $i > 0$, where t_x is the translation by x. Then for all $x \in X$ the multiplication map*

$$H^0(t_x^*E) \otimes H^0(F) \longrightarrow H^0(t_x^*E \otimes F)$$

is surjective and in particular (1) is surjective.

Proof. The fact that $h^i(t_x^*E \otimes F) = 0$ for all $i > 0$ and all $x \in X$ implies by base change that $E\hat{*}F$ is locally free with fiber

$$E\hat{*}F(x) \cong H^0(t_x^*E \otimes F).$$

We also have a natural isomorphism

$$\varphi : H^0(E\hat{*}F) \xrightarrow{\sim} H^0(E) \otimes H^0(F)$$

obtained as follows. One one hand by Leray we naturally have

$$H^0(p_{1*}((p_1 + p_2)^*E \otimes p_2^*F)) \cong H^0((p_1 + p_2)^*E \otimes p_2^*F)$$

and on the other hand the automorphism $(p_1 + p_2, p_2)$ of $X \times X$ induces an isomorphism

$$H^0((p_1 + p_2)^*E \otimes p_2^*F) \cong H^0(E) \otimes H^0(F),$$

so φ is obtained by composition. If we identify $H^0(E) \otimes H^0(F)$ with both $H^0(E\hat{*}F)$ (via φ) and $H^0(t_x^*E) \otimes H^0(F)$ (via $t_x^* \times id$), then it is easily seen that the multiplication map

$$H^0(t_x^*E) \otimes H^0(F) \longrightarrow H^0(t_x^*E \otimes F)$$

coincides with the evaluation map

$$H^0(E\hat{*}F) \xrightarrow{ev_x} E\hat{*}F(x)$$

and this proves the assertion. □

Remark. *There is a clear relationship between the skew Pontrjagin product and the usual Pontrjagin product as defined in 7.3(2). In fact (see [35] (1.2)):*

$$E\hat{*}F \cong E * (-1_X)^*F.$$

If F is symmetric the two notions coincide and one can hope to apply results like 7.3(2). This whole circle of ideas will be used in Section 5 below.

Finally we extract another result on multiplication maps that will be useful in the sequel. It is a particular case of [35] (3.8), implicit in Kempf's work [20] on syzygies of abelian varieties.

Proposition. B.4 *Let E be a vector bundle on X, L an ample line bundle and $m \geq 2$ an integer. Assume that*

$$h^i(E \otimes L^{\otimes k} \otimes \alpha) = 0,\ \forall i > 0,\ \forall k \geq -2 \text{ and } \forall \alpha \in \mathrm{Pic}^0(X).$$

Then the multiplication map

$$H^0(L^{\otimes m}) \otimes H^0(E) \longrightarrow H^0(L^{\otimes m} \otimes E)$$

is surjective.

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ABSTRACT

Linear Series on Moduli Spaces of Vector Bundles on Curves

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The present work developes a geometric study of linear series and generalized theta functions on moduli spaces of vector bundles on curves, with the aim of understanding effective numerical statements in the spirit of higher dimensional geometry. We give effective base point freeness and projective normality bounds for pluritheta linear series, as well as dimension bounds for the base loci of determinant linear series. This study has an "abelian" and a "nonabelian" part. For the "nonabelian" part the main technique is focused on giving upper bounds on the dimension of Quot schemes, via constructions known as elementary transformations. On the other hand, from the "abelian" point of view, we introduce the notion of Verlinde bundle on the Jacobian of a curve, and study this type of bundle with methods specific to the theory of vector bundles on abelian varieties. As a byproduct of these techniques we obtain a new global picture on duality for generalized theta functions and we formulate some further conjectures on optimal effective statements.