

REGULARITY ON ABELIAN VARIETIES III: RELATIONSHIP WITH GENERIC VANISHING AND APPLICATIONS

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1. INTRODUCTION

In previous work we have introduced the notion of M -regularity for coherent sheaves on abelian varieties ([PP1], [PP2]). This is useful because M -regular sheaves enjoy strong generation properties, in such a way that M -regularity on abelian varieties presents close analogies with the classical notion of Castelnuovo-Mumford regularity on projective spaces. Later we studied objects in the derived category of a smooth projective variety subject to Generic Vanishing conditions (GV -objects for short, [PP4]). The main ingredients are Fourier-Mukai transforms and the systematic use of homological and commutative algebra techniques. It turns out that, from the general perspective, M -regularity is a natural strenghtening of a Generic Vanishing condition. In this paper we describe in detail the relationship between the two notions in the case of abelian varieties, and deduce new basic properties of both M -regular and GV -sheaves. We also collect a few extra applications of the generation properties of M -regular sheaves, mostly announced but not contained in [PP1] and [PP2]. This second part of the paper is based on our earlier preprint [PP6].

We start in §2 by recalling some basic definitions and results from [PP4] on GV -conditions, restricted to the context of the present paper (coherent sheaves on abelian varieties). The rest of the section is devoted to the relationship between GV -sheaves and M -regular sheaves. More precisely, we prove a criterion, Proposition 2.8, characterizing the latter among the former: M -regular

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sheaves are those GV -sheaves \mathcal{F} for which the Fourier-Mukai transform of the Grothendieck-dual object $\mathbf{R}\Delta\mathcal{F}$ is a torsion-free sheaf. (This will be extended to higher regularity conditions, or strong Generic Vanishing conditions, in our upcoming work [PP5].)

We apply this relationship in §3 to the basic problem of the behavior of cohomological support loci under tensor products. We first prove that tensor products of GV -sheaves are again GV when one of the factors is locally free, and then use this and the torsion-freeness characterization to deduce a similar result for M -regular sheaves. The question of the behavior of M -regularity under tensor products had been posed to us by A. Beauville as well. It is worth mentioning that Theorem 3.2 does not seem to follow by any more standard methods.

In the other direction, in §4 we prove a result on GV -sheaves based on results on M -regularity. Specifically, we show that GV -sheaves on abelian varieties are nef. We deduce this from a theorem of Debarre [De2], stating that M -regular sheaves are ample, and the results in §2. This is especially interesting for the well-known problem of semipositivity: higher direct images of dualizing sheaves via maps to abelian varieties are known to be GV (cf. [Hac], [PP3]).

In §5 we survey generation properties of M -regular sheaves. This section is mostly expository, but the presentation of some known results, as Theorem 5.1(a) \Rightarrow (b) (which was proved in [PP1]), is new and more natural with respect to the Generic Vanishing perspective, providing also the new implication (b) \Rightarrow (a). In combination with well-known results of Green-Lazarsfeld and Ein-Lazarsfeld, we deduce some basic generation properties of the canonical bundle on a variety of maximal Albanese dimension, used in the following section.

The second part of the paper contains miscellaneous applications of the generation properties enjoyed by M -regular sheaves on abelian varieties, extracted or reworked from our older preprint [PP6]. In §6 we give effective results for pluricanonical maps on irregular varieties of general type and maximal Albanese dimension via M -regularity for direct images of canonical bundles, extending work in [PP1] §5. In particular we show, with a rather quick argument, that on a smooth projective variety Y of general type, maximal Albanese dimension, and whose Albanese image is not ruled by subtori, the pluricanonical series $|3K_Y|$ is very ample outside the exceptional locus of the Albanese map (Theorem 6.1). This is a slight strengthening, but also under a slightly stronger hypothesis, of a result of Chen and Hacon ([CH], Theorem 4.4), both statements being generalizations of the fact that the tricanonical bundle is very ample for curves of genus at least 2.

In §7.1 we look at bounding the Seshadri constant measuring the local positivity of an ample line bundle. There is already extensive literature on this in the case of abelian varieties (cf. [La1], [Nak], [Ba1], [Ba2], [De1] and also [La2] for further references). Here we explain how the Seshadri constant of a polarization L on an abelian variety is bounded below by an asymptotic version – and in particular by the usual – M -regularity index of the line bundle L , defined in [PP2] (cf. Theorem 7.4). Combining this with various bounds for Seshadri constants proved in [La1], we obtain bounds for M -regularity indices which are not apparent otherwise.

In §7.2 we shift our attention towards a cohomological study of Picard bundles, vector bundles on Jacobians of curves closely related to Brill-Noether theory (cf. [La2] 6.3.C and 7.2.C for a general introduction). We combine Fourier-Mukai techniques with the use of the Eagon-Northcott resolution for special determinantal varieties in order to compute their regularity, as well as that of their relatively small tensor powers (cf. Theorem 7.15). This vanishing theorem has practical applications. In particular we recover in a more direct fashion the main results of

[PP1] §4 on the equations of the W_d 's in Jacobians, and on vanishing for pull-backs of pluritheta line bundles to symmetric products.

By work of Mukai and others ([Muk3], [Muk4], [Muk1], [Um] and [Or]) it has emerged that on abelian varieties the class of vector bundles most closely resembling semistable vector bundles on curves and line bundles on abelian varieties is that of *semihomogeneous* vector bundles. In §7.3 we show that there exist numerical criteria for their geometric properties like global or normal generation, based on their Theta regularity. More generally, we give a result on the surjectivity of the multiplication map on global sections for two such vector bundles (cf. Theorem 7.29). Basic examples are the projective normality of ample line bundles on any abelian variety, and the normal generation of the Verlinde bundles on the Jacobian of a curve, coming from moduli spaces of vector bundles on that curve.

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2. GV -SHEAVES AND M -REGULAR SHEAVES ON ABELIAN VARIETIES

GV -sheaves. We recall definitions and results from [PP4] on *Generic Vanishing* conditions (GV for short). In relationship to the treatment of [PP4] we confine ourselves to a more limited setting, with respect to the following three aspects: (a) we consider only coherent sheaves (rather than complexes) subject to generic vanishing conditions; (b) we consider only the simplest such condition, i.e. GV_0 , henceforth denoted GV ; (c) we work only on abelian varieties, with the classical Fourier-Mukai functor associated to the Poincaré line bundle on $X \times \text{Pic}^0(X)$ (rather than arbitrary integral transforms).

Let X be an abelian variety of dimension g over an algebraically closed field, $\widehat{X} = \text{Pic}^0(X)$, P a normalized Poincaré bundle on $X \times \widehat{X}$, and $\mathbf{R}\widehat{\mathcal{S}} : \mathbf{D}(X) \rightarrow \mathbf{D}(\widehat{X})$ the standard Fourier-Mukai functor given by $\mathbf{R}\widehat{\mathcal{S}}(\mathcal{F}) = \mathbf{R}p_{\widehat{X}*}(p_A^*\mathcal{F} \otimes P)$. We denote $\mathbf{R}\mathcal{S} : \mathbf{D}(\widehat{X}) \rightarrow \mathbf{D}(X)$ the functor in the other direction defined analogously. For a coherent sheaf \mathcal{F} on X , we will consider for each $i \geq 0$ its i -th cohomological support locus

$$V^i(\mathcal{F}) := \{\alpha \in \widehat{X} \mid h^i(X, \mathcal{F} \otimes \alpha) > 0\}.$$

By base-change, the support of $R^i\widehat{\mathcal{S}}\mathcal{F}$ is contained in $V^i(\mathcal{F})$.

Proposition/Definition 2.1 (GV -sheaf, [PP4]). Given a coherent sheaf \mathcal{F} on X , the following conditions are equivalent:

- (a) $\text{codim Supp}(R^i\widehat{\mathcal{S}}\mathcal{F}) \geq i$ for all $i > 0$.
- (b) $\text{codim } V^i(\mathcal{F}) \geq i$ for all $i > 0$.

If one of the above conditions is satisfied, \mathcal{F} is called a GV -sheaf. (The proof of the equivalence is a standard base-change argument – cf. [PP4] Lemma 3.6.)

Notation/Terminology 2.2. (a) (IT_0 -sheaf). The simplest examples of GV -sheaves are those such that $V^i(\mathcal{F}) = \emptyset$ for every $i > 0$. In this case \mathcal{F} is said to *satisfy the Index Theorem with index 0* (IT_0 for short). If \mathcal{F} is IT_0 then $\mathbf{R}\widehat{\mathcal{S}}\mathcal{F} = R^0\widehat{\mathcal{S}}\mathcal{F}$, which is a locally free sheaf.

(b) (Weak Index Theorem). Let \mathcal{G} be an object in $\mathbf{D}(X)$ and $k \in \mathbf{Z}$. \mathcal{G} is said to *satisfy the Weak Index Theorem with index k* (WIT_k for short), if $R^i \widehat{\mathcal{S}}\mathcal{G} = 0$ for $i \neq k$. In this case we denote $\widehat{\mathcal{G}} = R^k \widehat{\mathcal{S}}\mathcal{G}$. Hence $\mathbf{R}\widehat{\mathcal{S}}\mathcal{G} = \widehat{\mathcal{G}}[-k]$.

(c) The same terminology and notation holds for sheaves on \widehat{X} , or more generally objects in $\mathbf{D}(\widehat{X})$, considering the functor $\mathbf{R}\widehat{\mathcal{S}}$.

We now state a basic result from [PP4] only in the special case of abelian varieties considered in this paper. In this case, with the exception of the implications from (1) to the other parts, it was in fact proved earlier by Hacon [Hac]. We denote $\mathbf{R}\Delta\mathcal{F} := \mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Theorem 2.3. *Let X be an abelian variety and \mathcal{F} a coherent sheaf on X . Then the following are equivalent:*

- (1) \mathcal{F} is a GV-sheaf.
- (2) For any sufficiently positive ample line bundle A on \widehat{X} ,

$$H^i(\mathcal{F} \otimes \widehat{A}^{-1}) = 0, \text{ for all } i > 0.$$

- (3) $\mathbf{R}\Delta\mathcal{F}$ satisfies WIT_g .

Proof. This is Corollary 3.10 of [PP4], with the slight difference that conditions (1), (2) and (3) are all stated with respect to the Poincaré line bundle P , while condition (3) of Corollary 3.10 of *loc. cit.* holds with respect to P^\vee . This can be done since, on abelian varieties, the Poincaré bundle satisfies the symmetry relation $P^\vee \cong ((-1_X) \times 1_{\widehat{X}})^* P$. Therefore Grothendieck duality (cf. Lemma 2.5 below) gives that the Fourier-Mukai functor defined by P^\vee on $X \times \widehat{X}$ is the same as $(-1_X)^* \circ \mathbf{R}\widehat{\mathcal{S}}$. We can also assume without loss of generality that the ample line bundle A on \widehat{X} considered below is symmetric. \square

Remark 2.4. The above Theorem holds in much greater generality ([PP4], Corollary 3.10). Moreover, in [PP5] we will show that the equivalence between (1) and (3) holds in a local setting as well. Condition (2) is a Kodaira-Kawamata-Viehweg-type vanishing criterion. This is because, up to an étale cover of X , the vector bundle \widehat{A}^{-1} is a direct sum of copies of an ample line bundle (cf. [Hac], and also [PP4] and the proof of Theorem 4.1 in the sequel).

Lemma 2.5 ([Muk1] 3.8). *The Fourier-Mukai and duality functors satisfy the exchange formula:*

$$\mathbf{R}\Delta \circ \mathbf{R}\widehat{\mathcal{S}} \cong (-1_{\widehat{X}})^* \circ \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\Delta[g].$$

A useful immediate consequence of the equivalence of (a) and (c) of Theorem 2.3, together with Lemma 2.5, is the following (cf. [PP4], Remark 3.11.):

Corollary 2.6. *If \mathcal{F} is a GV-sheaf on X then*

$$R^i \widehat{\mathcal{S}}\mathcal{F} \cong \mathcal{E}xt^i(\mathbf{R}\Delta\mathcal{F}, \mathcal{O}_{\widehat{X}}).$$

M -regular sheaves and their characterization. We now recall the M -regularity condition, which is simply a stronger (by one) generic vanishing condition, and relate it to the notion of GV-sheaf. The reason for the different terminology is that the notion of M -regularity was discovered – in connection with many geometric applications – before fully appreciating its relationship with generic vanishing theorems (see [PP1], [PP2], [PP3]).

Proposition/Definition 2.7. Let \mathcal{F} be a coherent sheaf on an abelian variety X . The following conditions are equivalent:

- (a) $\text{codim Supp}(R^i \widehat{S}\mathcal{F}) > i$ for all $i > 0$.
- (b) $\text{codim } V^i(\mathcal{F}) > i$ for all $i > 0$.

If one of the above conditions is satisfied, \mathcal{F} is called an *M-regular sheaf*.

The proof is identical to that of Proposition/Definition 2.1. By definition, every *M-regular sheaf* is a *GV-sheaf*. Non-regular *GV-sheaves* are those whose support loci have dimension as big as possible. As shown by the next result, as a consequence of the Auslander-Buchsbaum theorem, this is equivalent to the presence of torsion in the Fourier transform of the Grothendieck dual object.

Proposition 2.8. *Let X be an abelian variety of dimension g , and let \mathcal{F} be a GV-sheaf on X . The following conditions are equivalent:*

- (1) \mathcal{F} is *M-regular*.
- (2) $\widehat{\mathbf{R}\Delta\mathcal{F}} = \mathbf{R}\widehat{S}(\mathbf{R}\Delta\mathcal{F})[g]$ is a torsion-free sheaf.¹

Proof. By Corollary 2.6, \mathcal{F} is *M-regular* if and only if for each $i > 0$

$$\text{codim Supp}(\mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \mathcal{O}_{\widehat{X}})) > i.$$

The theorem is then a consequence of the following commutative algebra fact, which is surely known to the experts. \square

Lemma 2.9. *Let \mathcal{G} be a coherent sheaf on a smooth variety X . Then \mathcal{G} is torsion-free if and only if $\text{codim Supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_X)) > i$ for all $i > 0$.*

Proof. If \mathcal{G} is torsion free then it is a subsheaf of a locally free sheaf \mathcal{E} . From the exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{G} \longrightarrow 0$$

it follows that, for $i > 0$, $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{X}}) \cong \mathcal{E}xt^{i+1}(\mathcal{E}/\mathcal{G}, \mathcal{O}_{\widehat{X}})$. But then a well-known consequence of the Auslander-Buchsbaum Theorem applied to \mathcal{E}/\mathcal{G} implies that

$$\text{codim Supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{X}})) > i, \text{ for all } i > 0.$$

Conversely, since X is smooth, the functor $\mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_X)$ is an involution on $\mathbf{D}(X)$. Thus there is a spectral sequence

$$E_2^{ij} := \mathcal{E}xt^i((\mathcal{E}xt^j(\mathcal{G}, \mathcal{O}_X), \mathcal{O}_X)) \Rightarrow H^{i-j} = \mathcal{H}^{i-j}\mathcal{G} = \begin{cases} \mathcal{G} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

If $\text{codim Supp}(\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_X)) > i$ for all $i > 0$, then $\mathcal{E}xt^i(\mathcal{E}xt^j(\mathcal{G}, \mathcal{O}_X), \mathcal{O}_X) = 0$ for all i, j such that $j > 0$ and $i - j \leq 0$, so the only E_∞^i term which might be non-zero is E_∞^{00} . But the differentials coming into E_p^{00} are always zero, so we get a sequence of inclusions

$$\mathcal{F} = H^0 = E_\infty^{00} \subset \dots \subset E_3^{00} \subset E_2^{00}.$$

The extremes give precisely the injectivity of the natural map $\mathcal{G} \rightarrow \mathcal{G}^{**}$. Hence \mathcal{G} is torsion free. \square

¹Note that it is a sheaf by Theorem 2.3.

Remark 2.10. It is worth noting that in the previous proof, the fact that we are working on an abelian varieties is of no importance. In fact, an extension of Proposition 2.8 holds in the generality of [PP4], and even in a local setting, as it will be shown in [PP5].

3. TENSOR PRODUCTS OF GV AND M -REGULAR SHEAVES

We now address the issue of preservation of bounds on the codimension of support loci under tensor products. Our main result in this direction is (2) of Theorem 3.2 below, namely that the tensor product of two M -regular sheaves on an abelian variety is M -regular, provided that one of them is locally free. Note that the same result holds for Castelnuovo-Mumford regularity on projective spaces ([La2], Proposition 1.8.9). We do not know whether the same holds if one removes the local freeness condition on \mathcal{E} (in the case of Castelnuovo-Mumford regularity it does not).

Unlike the previous section, the proof of the result is quite specific to abelian varieties. One of the essential ingredients is Mukai's main inversion result (cf. [Muk1], Theorem 2.2), which states that the functor $\mathbf{R}\widehat{\mathcal{S}}$ is an equivalence of derived categories and, more precisely,

$$(1) \quad \mathbf{R}\mathcal{S} \circ \mathbf{R}\widehat{\mathcal{S}} \cong (-1_A)^*[-g] \quad \text{and} \quad \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\mathcal{S} \cong (-1_{\widehat{A}})^*[-g].$$

Besides this, the argument uses the characterization of M -regularity among GV -sheaves given by Proposition 2.8.

Proposition 3.1. *Let \mathcal{F} be a GV -sheaf and H a locally free sheaf satisfying IT_0 on an abelian variety X . Then $\mathcal{F} \otimes H$ satisfies IT_0 .*

Proof. Consider any $\alpha \in \text{Pic}^0(X)$. Note that $H \otimes \alpha$ also satisfies IT_0 , so $\mathbf{R}\widehat{\mathcal{S}}(H \otimes \alpha) = R^0\widehat{\mathcal{S}}(H \otimes \alpha)$ is a vector bundle N_α on \widehat{X} . By Mukai's inversion theorem (1) N_α satisfies WIT_g with respect to $\mathbf{R}\mathcal{S}$ and $H \otimes \alpha \cong \mathbf{R}\mathcal{S}((-1_X)^*N_\alpha)[g]$. Consequently for all i we have

$$(2) \quad H^i(X, \mathcal{F} \otimes H \otimes \alpha) \cong H^i(X, \mathcal{F} \otimes \mathbf{R}\mathcal{S}((-1_{\widehat{X}})^*N_\alpha)[g]).$$

But a basic exchange formula for integral transforms ([PP4], Lemma 2.1) states, in the present context, that

$$(3) \quad H^i(X, \mathcal{F} \otimes \mathbf{R}\mathcal{S}((-1_{\widehat{X}})^*N_\alpha)[g]) \cong H^i(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha[g]).$$

Putting (2) and (3) together, we get that

$$(4) \quad H^i(X, \mathcal{F} \otimes H \otimes \alpha) \cong H^i(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha[g]) = H^{g+i}(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha).$$

The hypercohomology groups on the right hand side are computed by the spectral sequence

$$E_2^{jk} := H^j(Y, R^k\mathcal{S}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha[g]) \Rightarrow H^{j+k}(Y, \mathbf{R}\widehat{\mathcal{S}}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha[g]).$$

Since \mathcal{F} is GV , we have the vanishing of $H^j(Y, R^k\mathcal{S}\mathcal{F} \otimes (-1_{\widehat{X}})^*N_\alpha[g])$ for $j+k > g$, and from this it follows that the hypercohomology groups in (4) are zero for $i > 0$. \square

Theorem 3.2. *Let X be an abelian variety, and \mathcal{F} and \mathcal{E} two coherent sheaves on X , with \mathcal{E} locally free.*

- (1) *If \mathcal{F} and \mathcal{E} are GV -sheaves, then $\mathcal{F} \otimes \mathcal{E}$ is a GV -sheaf.*
- (2) *If \mathcal{F} and \mathcal{E} are M -regular, then $\mathcal{F} \otimes \mathcal{E}$ is M -regular.*

Proof. (1) Let A be a sufficiently ample line bundle on \widehat{X} . Then, by Theorem 2.3(2), $\mathcal{E} \otimes \widehat{A}^{-1}$ satisfies IT_0 . By Proposition 3.1, this implies that $(\mathcal{F} \otimes \mathcal{E}) \otimes \widehat{A}^{-1}$ also satisfies IT_0 . Applying Theorem 2.3(2) again, we deduce that $\mathcal{F} \otimes \mathcal{E}$ is GV .

(2) Both \mathcal{F} and \mathcal{E} are GV , so (1) implies that $\mathcal{F} \otimes \mathcal{E}$ is also a GV . We use Proposition 2.8. This implies to begin with that $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F})$ and $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{E}) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathcal{E}^\vee)$ are torsion-free sheaves (we harmlessly forget about what degree they live in). Going backwards, it also implies that we are done if we show that $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta(\mathcal{F} \otimes \mathcal{E}))$ is torsion free. But note that

$$\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta(\mathcal{F} \otimes \mathcal{E})) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F} \otimes \mathcal{E}^\vee) \cong \mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F}) \ast \mathbf{R}\widehat{\mathcal{S}}(\mathcal{E}^\vee)$$

where \ast denotes the (derived) Pontrjagin product of sheaves on abelian varieties, and the last isomorphism is the exchange of Pontrjagin and tensor products under the Fourier-Mukai functor (cf. [Muk1] (3.7)). Note that this derived Pontrjagin product is in fact an honest Pontrjagin product, as we know that all the objects above are sheaves. Recall that by definition the Pontrjagin product of two sheaves \mathcal{G} and \mathcal{H} is simply $\mathcal{G} \ast \mathcal{H} := m_*(p_1^*\mathcal{G} \otimes p_2^*\mathcal{H})$, where $m : \widehat{X} \times \widehat{X} \rightarrow \widehat{X}$ is the group law on \widehat{X} . Since m is a surjective morphism, if \mathcal{G} and \mathcal{H} are torsion-free, then so is $p_1^*\mathcal{G} \otimes p_2^*\mathcal{H}$ and its push-forward $\mathcal{G} \ast \mathcal{H}$. \square

Remark 3.3. As mentioned in §2, Generic Vanishing conditions can be naturally defined for objects in the derived category, rather than sheaves (see [PP4]). In this more general setting, (1) of Theorem 3.2 holds for $\mathcal{F} \otimes \mathcal{G}$, where \mathcal{F} is any GV -object and \mathcal{E} any GV -sheaf, while (2) holds for \mathcal{F} any M -regular object and \mathcal{E} any M -regular locally free sheaf. The proof is the same.

4. NEFNESS OF GV -SHEAVES

Debarre has shown in [De2] that every M -regular sheaf on an abelian variety is ample. We deduce from this and Theorem 2.3 that GV -sheaves satisfy the analogous weak positivity.

Theorem 4.1. *Every GV -sheaf on an abelian variety is nef.*

Proof. Step 1. We first reduce to the case when the abelian variety X is principally polarized. For this, consider A any ample line bundle on \widehat{X} . By Theorem 2.3 we know that the GV -condition is equivalent to the vanishing

$$H^i(\mathcal{F} \otimes \widehat{A}^{-m}) = 0, \text{ for all } i > 0, \text{ and all } m \gg 0.$$

But A is the pullback $\hat{\psi}^*L$ of a principal polarization L via an isogeny $\hat{\psi} : \widehat{X} \rightarrow \widehat{Y}$ (cf. [LB] Proposition 4.1.2). We then have

$$0 = H^i(\mathcal{F} \otimes \widehat{A}^{-m}) \cong H^i(\mathcal{F} \otimes (\hat{\psi}^*(L^{-m}))) \cong H^i(\mathcal{F} \otimes \psi_*\widehat{M}^{-m}) \cong H^i(\psi^*\mathcal{F} \otimes \widehat{M}^{-m}).$$

Here ψ denotes the dual isogeny. (The only thing that needs an explanation is the next to last isomorphism, which is the commutation of the Fourier-Mukai functor with isogenies, [Muk1] 3.4.) But this implies that $\psi^*\mathcal{F}$ is also GV , and since nefness is preserved by isogenies this completes the reduction step.

Step 2. Assume now that X is principally polarized by Θ . As above, we know that

$$H^i(\mathcal{F} \otimes \widehat{\mathcal{O}(-m\Theta)} \otimes \alpha) = 0, \text{ for all } i > 0, \text{ all } \alpha \in \text{Pic}^0(X) \text{ and all } m \gg 0.$$

If we denote by $\phi_m : X \rightarrow X$ multiplication by m , i.e. the isogeny induced by $m\Theta$, then this implies that

$$H^i(\phi_m^* \mathcal{F} \otimes \mathcal{O}(m\Theta) \otimes \beta) = 0, \text{ for all } i > 0 \text{ and all } \beta \in \text{Pic}^0(X)$$

as $\phi_m^* \mathcal{O}(\widehat{-m\Theta}) \cong \bigoplus \mathcal{O}(m\Theta)$ by [Muk1] Proposition 3.11(1). This means that the sheaf $\phi_m^* \mathcal{F} \otimes \mathcal{O}(m\Theta)$ satisfies IT_0 on X , so in particular it is M -regular. By Debarre's result [De2] Corollary 3.2, it is then ample.

But ϕ_m is a finite cover, and $\phi_m^* \Theta \equiv m^2 \Theta$. The statement above is then same as saying that, in the terminology of [La2] §6.2, the \mathbb{Q} -twisted² sheaf $\mathcal{F} \langle \frac{1}{m} \cdot \Theta \rangle$ on X is ample, since $\phi_m^* (\mathcal{F} \langle \frac{1}{m} \cdot \Theta \rangle)$ is an honest ample sheaf. As m goes to ∞ , we see that \mathcal{F} is a limit of ample \mathbb{Q} -twisted sheaves, and so it is nef by [La2] Proposition 6.2.11. \square

Combining the result above with the fact that higher direct images of canonical bundles are GV (cf. [PP3] Theorem 5.9), we obtain the following result, one well-known instance of which is that the canonical bundle of any smooth subvariety of an abelian variety is nef.

Corollary 4.2. *Let X be a smooth projective variety and $a : Y \rightarrow X$ a (not necessarily surjective) morphism to an abelian variety. Then $R^j a_* \omega_Y$ is a nef sheaf on X for all j .*

One example of an immediate application of Corollary is to integrate a result of Peternell-Sommese in the general picture.

Corollary 4.3 ([PS], Theorem 1.17). *Let $a : Y \rightarrow X$ be a finite surjective morphism of smooth projective varieties, with X an abelian variety. Then the vector bundle E_a is nef.*

Proof. By duality we have $a_* \omega_Y \cong \mathcal{O}_X \oplus E_a$. Thus E_a is a quotient of $a_* \omega_Y$, so by Corollary 4.2 it is nef. \square

5. GENERATION PROPERTIES OF M -REGULAR SHEAVES ON ABELIAN VARIETIES

The interest in the notion of M -regularity comes from the fact that M -regular sheaves on abelian varieties have strong generation properties. In this respect, M -regularity on abelian varieties parallels the notion of Castelnuovo-Mumford regularity on projective spaces (cf. the survey [PP3]). In this section we survey the basic results about generation properties of M -regular sheaves. The presentation is somewhat new, since the proof of the basic result (the implication $(a) \Rightarrow (b)$ of Theorem 5.1 below) makes use of the relationship between M -regularity and GV -sheaves (Proposition 2.8). The argument in this setting turns out to be more natural, and provides as a byproduct the reverse implication $(b) \Rightarrow (a)$, which is new.

Another characterization of M -regularity. M -regular sheaves on abelian varieties are characterized as follows:

Theorem 5.1. ([PP1], Theorem 2.5) *Let \mathcal{F} be a GV -sheaf on an abelian variety X . Then the following conditions are equivalent:*

(a) \mathcal{F} is M -regular.

²Note that the twist is indeed only up to numerical equivalence.

(b) For every locally free sheaf H on X satisfying IT_0 , and for every non-empty Zariski open set $U \subset \widehat{X}$, the sum of multiplication maps of global sections

$$\mathcal{M}_U : \bigoplus_{\alpha \in U} H^0(X, \mathcal{F} \otimes \alpha) \otimes H^0(X, H \otimes \alpha^{-1}) \xrightarrow{\oplus m_\alpha} H^0(X, \mathcal{F} \otimes H)$$

is surjective.

Proof. Since \mathcal{F} is a GV-sheaf, by Theorem 2.3 the transform of $\mathbf{R}\Delta\mathcal{F}$ is a sheaf in degree g , i.e. $\mathbf{R}\widehat{\mathcal{S}}(\mathbf{R}\Delta\mathcal{F}) = \widehat{\mathbf{R}\Delta\mathcal{F}}[-g]$. If H is a coherent sheaf satisfying IT_0 then $\mathbf{R}\widehat{\mathcal{S}}H = \widehat{H}$, a locally free sheaf in degree 0. It turns out that the following natural map is an isomorphism

$$(5) \quad \mathrm{Ext}^g(H, \mathbf{R}\Delta\mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}).$$

This simply follows from Mukai's Theorem (1), which yields that

$$\mathrm{Ext}^g(H, \mathbf{R}\Delta\mathcal{F}) = \widehat{\mathrm{Hom}}_{\mathbf{D}(X)}(H, \mathbf{R}\Delta\mathcal{F}[g]) \cong \mathrm{Hom}_{\mathbf{D}(\widehat{X})}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) = \mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}).$$

Proof of (a) \Rightarrow (b). Since $\widehat{\mathbf{R}\Delta\mathcal{F}}$ is torsion-free by Proposition 2.8, the evaluation map at the fibres

$$(6) \quad \mathrm{Hom}(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \rightarrow \prod_{\alpha \in U} \mathcal{H}om(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes_{\mathcal{O}_{\widehat{X}, \alpha}} k(\alpha)$$

is injective for all open sets $U \subset \mathrm{Pic}^0(X)$. Therefore, composing with the isomorphism (5), we get an injection

$$(7) \quad \mathrm{Ext}^g(H, \mathbf{R}\Delta\mathcal{F}) \rightarrow \prod_{\alpha \in U} \mathcal{H}om(\widehat{H}, \widehat{\mathbf{R}\Delta\mathcal{F}}) \otimes_{\mathcal{O}_{\widehat{X}, \alpha}} k(\alpha).$$

By base-change, this is the dual map of the map in (b), which is therefore surjective.

Proof of (b) \Rightarrow (a). Let A be an ample symmetric line bundle on \widehat{X} . From Mukai's Theorem (1), it follows that $A^{-1} = \widehat{H}_A$, where H_A is a locally free sheaf on X satisfying IT_0 and such that $\widehat{H}_A = A^{-1}$. We have that (b) is equivalent to the injectivity of (7). We now take $H = H_A$ in both (5) and (7). The facts that (5) is an isomorphism and that (7) is injective yield the injectivity, for all open sets $U \subset \mathrm{Pic}^0(X)$, of the evaluation map at fibers

$$H^0(\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A) \xrightarrow{ev_U} \prod_{\alpha \in U} (\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A) \otimes_{\mathcal{O}_{\widehat{X}, \alpha}} k(\alpha).$$

Letting A be sufficiently positive so that $\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A$ is globally generated, this is equivalent to the torsion-freeness of $\widehat{\mathbf{R}\Delta\mathcal{F}}^3$ and hence, by Proposition 2.8, to the M -regularity of \mathcal{F} . \square

Continuous global generation and global generation. Recall first the following:

Definition 5.2 ([PP1], Definition 2.10). Let Y be a variety equipped with a morphism $a : Y \rightarrow X$ to an abelian variety X .

(a) A sheaf \mathcal{F} on Y is *continuously globally generated with respect to a* if the sum of evaluation maps

$$\mathcal{E}v_U : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes a^*\alpha) \otimes a^*\alpha^{-1} \longrightarrow \mathcal{F}$$

³Note that the kernel of ev_U generates a torsion subsheaf of $\widehat{\mathbf{R}\Delta\mathcal{F}} \otimes A$ whose support is contained in the complement of U .

is surjective for every non-empty open subset $U \subset \text{Pic}^0(X)$.

(b) More generally, let T be a proper subvariety of Y . The sheaf \mathcal{F} is said to be *continuously globally generated with respect to a away from T* if $\text{Supp}(\text{Coker } \mathcal{E}v_U) \subset T$ for every non-empty open subset $U \subset \text{Pic}^0(X)$.

(c) When a is the Albanese morphism, we will suppress a from the terminology, speaking of *continuously globally generated* (resp. *continuously globally generated away from T*) sheaves.

In Theorem 5.1, taking H to be a sufficiently positive line bundle on X easily yields (cf [PP1], Proposition 2.13):

Corollary 5.3. *An M -regular sheaf on X is continuously globally generated.*

The relationship between continuous global generation and global generation comes from:

Proposition 5.4 ([PP1], Proposition 2.12). *(a) In the setting of Definition 5, let \mathcal{F} (resp. A) be a coherent sheaf on Y (resp. a line bundle, possibly supported on a subvariety Z of Y), both continuously globally generated. Then $\mathcal{F} \otimes A \otimes a^*\alpha$ is globally generated for all $\alpha \in \text{Pic}^0(X)$.*

(b) More generally, let \mathcal{F} and A as above. Assume that \mathcal{F} is continuously globally generated away from T and that A is continuously globally generated away from W . Then $\mathcal{F} \otimes A \otimes a^\alpha$ is globally generated away from $T \cup W$ for all $\alpha \in \text{Pic}^0(X)$.*

The proposition is proved via the classical method of *reducible sections*, i.e. those sections of the form $s_\alpha \cdot t_{-\alpha}$, where s_α (resp. $t_{-\alpha}$) belongs to $H^0(F \otimes a^*\alpha)$ (resp. $H^0(A \otimes a^*\alpha^{-1})$).

Generation properties on varieties of maximal Albanese dimension via Generic Vanishing. The above results give effective generation criteria once one has effective *Generic Vanishing* criteria ensuring that the dimension of the cohomological support loci is not too big. The main example of such a criterion is the Green-Lazarsfeld Generic Vanishing Theorem for the canonical line bundle of an irregular variety, proved in [GL1] and further refined in [GL2] using the deformation theory of cohomology groups.⁴ For the purposes of this paper, it is enough to state the Generic Vanishing Theorem in the case of varieties Y of *maximal Albanese dimension*, i.e. such that the Albanese map $a : Y \rightarrow \text{Alb}(Y)$ is generically finite onto its image. More generally, we consider a morphism $a : Y \rightarrow X$ to an abelian variety X . Then, as in §1 one can consider the *cohomological support loci* $V_a^i(\omega_Y) = \{\alpha \in \text{Pic}^0(X) \mid h^i(\omega_Y \otimes a^*\alpha) > 0\}$. (In case a is the Albanese map we will suppress a from the notation.)

The result of Green-Lazarsfeld (see also [EL] Remark 1.6) states that, *if the morphism a is generically finite, then*

$$\text{codim } V_a^i(\omega_Y) \geq i \text{ for all } i > 0.$$

Moreover, in [GL2] it is proved that $V_a^i(\omega_Y)$ are unions of translates of subtori. Finally, an argument of Ein-Lazarsfeld [EL] yields that, *if there exists an $i > 0$ such that $\text{codim } V_a^i(\omega_Y) = i$, then the image of a is ruled by subtori of X* . All of this implies the following typical application of the concept of M -regularity.

Proposition 5.5. *Assume that $\dim Y = \dim a(Y)$ and that $a(Y)$ is not ruled by tori. Let Z be the exceptional locus of a , i.e. the inverse image via a of the locus of points in $a(Y)$ having*

⁴More recently, Hacon [Hac] has given a different proof, based on the Fourier-Mukai transform and Kodaira Vanishing. Building in part on Hacon's ideas, several extensions of this result are given in [PP4].

non-finite fiber. Then:

(i) $a_*\omega_Y$ is an M -regular sheaf on X .

(ii) $a_*\omega_Y$ is continuously globally generated.

(iii) ω_Y is continuously globally generated away from Z .

(iv) For all $k \geq 2$, $\omega_Y^{\otimes k} \otimes a^*\alpha$ is globally generated away from Z for any $\alpha \in \text{Pic}^0(X)$.

Proof. By Grauert-Riemenschneider vanishing, $R^i a_*\omega_Y = 0$ for all $i \neq 0$. By the Projection Formula we get $V_a^i(\omega_Y) = V^i(a_*\omega_Y)$. Combined with the Ein-Lazarsfeld result, (i) follows. Part (ii) follows from Corollary 5.3. For (iii) note that, as with global generation (and by a similar argument), continuous global generation is preserved by finite maps: if a is finite and $a_*\mathcal{F}$ is continuously globally generated, then \mathcal{F} is continuously globally generated. (iv) for $k = 2$ follows from (iii) and Proposition 5.4. For arbitrary $k \geq 2$ it follows in the same way by induction (note that if a sheaf \mathcal{F} is such that $\mathcal{F} \otimes a^*\alpha$ is globally generated away from Z for every $\alpha \in \text{Pic}^0(X)$, then it is continuously globally generated away from Z). \square

6. PLURICANONICAL MAPS OF IRREGULAR VARIETIES OF MAXIMAL ALBANESE DIMENSION

One of the most elementary results about projective embeddings is that every curve of general type can be embedded in projective space by the tricanonical line bundle. This is sharp for curves of genus two. It turns out that this result can be generalized to arbitrary dimension, namely to varieties of maximal Albanese dimension. In fact, using Vanishing and Generic Vanishing Theorems and the Fourier-Mukai transform, Chen and Hacon proved that *for every smooth complex variety of general type and maximal Albanese dimension Y such that $\chi(\omega_Y) > 0$, the tricanonical line bundle $\omega_Y^{\otimes 3}$ gives a birational map* (cf. [CH], Theorem 4.4). The main point of this section is that the concept of M -regularity (combined of course with vanishing results) provides a quick and conceptually simple proof of on one hand a slightly more explicit version of the Chen-Hacon Theorem, but on the other hand under a slightly more restrictive hypothesis. We show the following:

Theorem 6.1. *Let Y be a smooth projective complex variety of general type and maximal Albanese dimension. If the Albanese image of Y is not ruled by tori, then $\omega_Y^{\otimes 3}$ is very ample away from the exceptional locus of the Albanese map.*

Here the *exceptional locus* of the Albanese map $a : Y \rightarrow \text{Alb}(Y)$ is $Z = a^{-1}(T)$, where T is the locus of points in $\text{Alb}(Y)$ over which the fiber of a has positive dimension.

Remark 6.2. A word about the hypothesis of the Chen-Hacon Theorem and of Theorem 6.1 is in order. As a consequence of the Green-Lazarsfeld Generic Vanishing Theorem (end of §3), it follows that $\chi(\omega_Y) \geq 0$ for every variety Y of maximal Albanese dimension. Moreover, Ein-Lazarsfeld [EL] prove that for Y of maximal Albanese dimension, *if $\chi(\omega_Y) = 0$, then $a(Y)$ is ruled by subtori of $\text{Alb}(Y)$* . In dimension ≥ 3 there exist examples of varieties of general type and maximal Albanese dimension with $\chi(\omega_Y) = 0$ (cf. *loc. cit.*).

In the course of the proof we will invoke $\mathcal{J}(Y, \|L\|)$, the *asymptotic multiplier ideal sheaf* associated to a complete linear series $|L|$ (cf. [La2] §11). One knows that, given a line bundle L of non-negative Iitaka dimension,

$$(8) \quad H^0(Y, L \otimes \mathcal{J}(\|L\|)) = H^0(Y, L),$$

i.e. the zero locus of $\mathcal{J}(\| L \|)$ is contained in the base locus of $|L|$ ([La2], Proposition 11.2.10). Another basic property we will use is that, for every k ,

$$(9) \quad \mathcal{J}(\| L^{\otimes(k+1)} \|) \subseteq \mathcal{J}(\| L^{\otimes k} \|).$$

(Cf. [La2], Theorem 11.1.8.) A first standard result is

Lemma 6.3. *Let Y be a smooth projective complex variety of general type. Then:*

(a) $h^0(\omega_Y^{\otimes m} \otimes \alpha)$ is constant for all $\alpha \in \text{Pic}^0(Y)$ and for all $m > 1$.

(b) The zero locus of $\mathcal{J}(\| \omega_Y^{\otimes(m-1)} \|)$ is contained in the base locus of $\omega_Y^{\otimes m} \otimes \alpha$, for all $\alpha \in \text{Pic}^0(Y)$.

Proof. Since bigness is a numerical property, all line bundles $\omega_Y \otimes \alpha$ are big, for $\alpha \in \text{Pic}^0(Y)$. By Nadel Vanishing for asymptotic multiplier ideals ([La2], Theorem 11.2.12)

$$H^i(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) = 0$$

for all $i > 0$ and all $\alpha, \beta \in \text{Pic}^0(X)$. Therefore, by the invariance of the Euler characteristic,

$$h^0(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) = \text{constant} = \lambda_\alpha$$

for all $\beta \in \text{Pic}^0(Y)$. Now

$$h^0(Y, \omega_Y^{\otimes m} \otimes \beta \otimes \mathcal{J}(\| (\omega_Y \otimes \alpha)^{\otimes(m-1)} \|)) \leq h^0(Y, \omega_Y^{\otimes m} \otimes \beta)$$

for all $\beta \in \text{Pic}^0(X)$ and, because of (8) and (9), equality holds for $\beta = \alpha^m$. By semicontinuity it follows that $h^0(Y, \omega_Y^{\otimes m} \otimes \beta) = \lambda_\alpha$ for all β contained in a Zariski open set U_α of $\text{Pic}^0(X)$ which contains α^m . Since this is true for all α , the statement follows. Part (b) follows from the previous argument. \square

Lemma 6.4. *Let Y be a smooth projective complex variety of general type and maximal Albanese dimension, such that its Albanese image is not ruled by tori. Let Z be the exceptional locus of its Albanese map. Then, for every $\alpha \in \text{Pic}^0(Y)$:*

(a) the zero-locus of $\mathcal{J}(\| \omega_Y \otimes \alpha \|)$ is contained (set-theoretically) in Z .

(b) $\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\| \omega_Y \|)$ is globally generated away from Z .

Proof. (a) By (8) and (9) the zero locus of $\mathcal{J}(\| \omega \otimes \alpha \|)$ is contained in the base locus of $\omega^{\otimes 2} \otimes \alpha^2$. By Proposition 5.5, the base locus of $\omega^{\otimes 2} \otimes \alpha^2$ is contained in Z . (b) Again by Proposition 5.5, the base locus of $\omega^{\otimes 2} \otimes \alpha$ is contained in Z . By Lemma 6.3(b), the zero locus of $\mathcal{J}(\| \omega_Y \|)$ is contained in Z . \square

Proof. (of Theorem 6.1) As above, let $a : Y \rightarrow \text{Alb}(Y)$ be the Albanese map and let Z be the exceptional locus of a . As in the proof of Prop. 5.5, the Ein-Lazarsfeld result at the end of §3 (see also Remark 6.2), the hypothesis implies that $a_*\omega_Y$ is M -regular, so ω_Y is continuously globally generated away from Z . We make the following:

Claim. *For every $y \in Y - Z$, the sheaf $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is M -regular.*

We first see how the Claim implies Theorem 6.1. The statement of the Theorem is equivalent to the fact that, for any $y \in Y - Z$, the sheaf $I_y \otimes \omega_Y^{\otimes 3}$ is globally generated away from Z . By Corollary 5.3, the Claim yields that $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|))$ is continuously globally generated. Therefore $I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\| \omega_Y \|)$ is continuously globally generated away from Z . Hence, by

Proposition 5.4, $I_y \otimes \omega_Y^{\otimes 3} \otimes \mathcal{J}(\|\omega_Y\|)$ is globally generated away from Z . Since the zero locus of $\mathcal{J}(\|\omega_Y\|)$ is contained in Z (by Lemma 6.4(a)), the Theorem follows from the Claim.

Proof of the Claim. We consider the standard exact sequence

$$(10) \quad 0 \rightarrow I_y \otimes \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|) \rightarrow \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|) \rightarrow (\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|))|_y \rightarrow 0.$$

(Note that y does not lie in the zero locus of $\mathcal{J}(\|\omega_Y\|)$.) By Nadel Vanishing for asymptotic multiplier ideals, $H^i(Y, \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|)) = 0$ for all $i > 0$ and $\alpha \in \text{Pic}^0(Y)$. Since, by Lemma 6.4, y is not in the base locus of $\omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|)$, taking cohomology in (10) it follows that

$$(11) \quad H^i(Y, I_y \otimes \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|)) = 0$$

for all $i > 0$ and $\alpha \in \text{Pic}^0(X)$ as well. Since y does not belong to the exceptional locus of a , the map $a_*(\omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|)) \rightarrow a_*((\omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|))|_y)$ is surjective. On the other hand, since a is generically finite, by a well-known extension of Grauert-Riemenschneider vanishing, $R^i a_*(\omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|))$ vanishes for all $i > 0$.⁵ Therefore (10) implies also that for all $i > 0$

$$(12) \quad R^i a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|)) = 0.$$

Combining (11) and (12) one gets, by projection Formula, that the sheaf $a_*(I_y \otimes \omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|))$ is IT_0 on X , hence M -regular. \square

Remark 6.5. It follows from the proof that $\omega_Y^{\otimes 3} \otimes \alpha$ is very ample away from Z for all $\alpha \in \text{Pic}^0(Y)$ as well.

Remark 6.6 (The Chen-Hacon Theorem). The reader might wonder why, according to the above quoted theorem of Chen-Hacon, the tricanonical bundle of varieties of general type and maximal Albanese dimension is birational (but not necessarily very ample outside the Albanese exceptional locus) even under the weaker assumption that $\chi(\omega_Y)$ is positive, which does not ensure the continuous global generation of $a_*\omega_Y$. The point is that, according to Generic Vanishing, if the Albanese dimension is maximal, then $\chi(\omega_Y) > 0$ implies $h^0(\omega_Y \otimes \alpha) > 0$ for all $\alpha \in \text{Pic}^0(Y)$. Hence, even if ω_Y is not necessarily continuously globally generated away of some subvariety of Y , the following condition holds: for *general* $y \in Y$, there is a Zariski open set $U_y \subset \text{Pic}^0(Y)$ such that y is not a base point of $\omega_Y \otimes \alpha$ for all $\alpha \in U_y$. Using the same argument of Proposition 5.4 – based on reducible sections – it follows that such y is not a base point of $\omega_Y^{\otimes 2} \otimes \alpha$ for *all* $\alpha \in \text{Pic}^0(Y)$. Then the Chen-Hacon Theorem follows by an argument analogous to that of Theorem 6.1.

To complete the picture, it remains to analyze the case of varieties Y of maximal Albanese dimension and $\chi(\omega_Y) = 0$. Chen and Hacon prove that if the Albanese dimension is maximal, then $\omega_Y^{\otimes 6}$ is always birational (and $\omega_Y^{\otimes 6} \otimes \alpha$ as well). The same result can be made slightly more precise as follows, extending also results in [PP1] §5:

Theorem 6.7. *If Y is a smooth projective complex variety of maximal Albanese dimension then, for all $\alpha \in \text{Pic}^0(Y)$, $\omega_Y^{\otimes 6} \otimes \alpha$ is very ample away from the exceptional locus of the Albanese map. Moreover, if L a big line bundle on Y , then $(\omega_Y \otimes L)^{\otimes 3} \otimes \alpha$ gives a birational map.*

The proof is similar to that of Theorem 6.1, and left to the interested reader. For example, for the first part the point is that, by Nadel Vanishing for asymptotic multiplier ideals,

⁵The proof of this is identical to that of the usual Grauert-Riemenschneider vanishing theorem in [La2] §4.3.B, replacing Kawamata-Viehweg vanishing with Nadel vanishing.

$H^i(Y, \omega_Y^{\otimes 2} \otimes \alpha \otimes \mathcal{J}(\|\omega_Y\|)) = 0$ for all $\alpha \in \text{Pic}^0(Y)$. Hence, by the same argument using Grauert-Riemenschneider vanishing, $a_*(\omega_Y^{\otimes 2} \otimes \mathcal{J}(\|\omega_Y\|))$ is M -regular.

Finally, we remark that in [CH], Chen-Hacon also prove effective birationality results for pluricanonical maps of irregular varieties of arbitrary Albanese dimension (in function of the minimal power for which the corresponding pluricanonical map on the general Albanese fiber is birational). It is likely that the methods above apply to this context as well.

7. FURTHER APPLICATIONS OF M -REGULARITY

7.1. M -regularity indices and Seshadri constants. Here we express a natural relationship between Seshadri constants of ample line bundles on abelian varieties and the M -regularity indices of those line bundles as defined in [PP2]. This result is a theoretical improvement of the lower bound for Seshadri constants proved in [Nak]. In the opposite direction, combined with the results of [La1], it provides bounds for controlling M -regularity. For a general overview of Seshadri constants, in particular the statements used below, one can consult [La2] Ch.I §5.

We start by recalling the basic definition from [PP2] and by also looking at a slight variation. We will denote by X an abelian variety of dimension g over an algebraically closed field and by L an ample line bundle on X .

Definition 7.1. The M -regularity index of L is defined as

$$m(L) := \max\{l \mid L \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p} \text{ is } M\text{-regular for all distinct } x_1, \dots, x_p \in X \text{ with } \sum k_i = l\}.$$

Definition 7.2. We also define a related invariant, associated to just one given point $x \in X$:

$$p(L, x) := \max\{l \mid L \otimes m_x^l \text{ is } M\text{-regular}\}.$$

The definition does not depend on x because of the homogeneity of X , so we will denote this invariant simply by $p(L)$.

Our main interest will be in the asymptotic versions of these indices, which turn out to be related to the Seshadri constant associated to L .

Definition 7.3. The *asymptotic M -regularity index* of L and its punctual counterpart are defined as

$$\rho(L) := \sup_n \frac{m(L^n)}{n} \quad \text{and} \quad \rho'(L) := \sup_n \frac{p(L^n)}{n}.$$

The main result of this section is:

Theorem 7.4. *We have the following inequalities:*

$$\epsilon(L) = \rho'(L) \geq \rho(L) \geq 1.$$

In particular $\epsilon(L) \geq \max\{m(L), 1\}$.

This improves a result of Nakamaye (cf. [Nak] and the references therein). Nakamaye also shows that $\epsilon(L) = 1$ for some line bundle L if and only if X is the product of an elliptic curve with another abelian variety. As explained in [PP2] §3, the value of $m(L)$ is reflected in the geometry of the map to projective space given by L . Here is a basic example:

Example 7.5. If L is very ample – or more generally gives a birational morphism outside a codimension 2 subset – then $m(L) \geq 2$, so by the Theorem above $\epsilon(L) \geq 2$. Note that on an arbitrary smooth projective variety very ampleness implies in general only that $\epsilon(L, x) \geq 1$ at each point.

The proof of Theorem 7.4 is a simple application Corollary 5.3 and Proposition 5.4, via the results of [PP2] §3. We use the relationship with the notions of k -jet ampleness and separation of jets. Denote by $s(L, x)$ the largest number $s \geq 0$ such that L separates s -jets at x . Recall also the following:

Definition 7.6. A line bundle L is called *k -jet ample*, $k \geq 0$, if the restriction map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_X/m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p})$$

is surjective for any distinct points x_1, \dots, x_p on X such that $\sum k_i = k + 1$. Note that if L is k -jet ample, then it separates k -jets at every point.

Proposition 7.7 ([PP2] Theorem 3.8 and Proposition 3.5). (i) L^n is $(n + m(L) - 2)$ -jet ample, so in particular $s(L^n, x) \geq n + m(L) - 2$.
(ii) If L is k -jet ample, then $m(L) \geq k + 1$.

This points in the direction of local positivity, since one way to interpret the Seshadri constant of L is (independently of x):

$$\epsilon(L) = \sup_n \frac{s(L^n, x)}{n}.$$

To establish the connection with the asymptotic invariants above we also need the following:

Lemma 7.8. For any $n \geq 1$ and any $x \in X$ we have $s(L^{n+1}, x) \geq m(L^n)$.

Proof. This follows immediately from Corollary 5.3 and Proposition 5.4: if $L^n \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}$ is M -regular, then $L^{n+1} \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}$ is globally generated, and so by [PP2] Lemma 3.3, L^{n+1} is $m(L)$ -jet ample. \square

Proof. (of Theorem 7.4.) Note first that for every $p \geq 1$ we have

$$(13) \quad m(L^n) \geq m(L) + n - 1,$$

which follows immediately from the two parts of Proposition 7.7. In particular $m(L^n)$ is always at least $n - 1$, and so $\rho(L) \geq 1$. Putting together the definitions, (13) and Lemma 7.8, we obtain the main inequality $\epsilon(L) \geq \rho(L)$. Finally, the asymptotic punctual index computes precisely the Seshadri constant. Indeed, by completely similar arguments as above, we have that for any ample line bundle L and any $p \geq 1$ one has

$$p(L^n) \geq s(L^n, x) \text{ and } s(L^{n+1}, x) \geq p(L^n, x).$$

The statement follows then from the definition. \square

Remark 7.9. What the proof above shows is that one can give an interpretation for $\rho(L)$ similar to that for $\epsilon(L)$ in terms of separation of jets. In fact $\rho(L)$ is precisely the “asymptotic jet ampleness” of L , namely:

$$\rho(L) = \sup_n \frac{a(L^n)}{n},$$

where $a(M)$ is the largest integer k for which a line bundle M is k -jet ample.

Question 7.10. Do we always have $\epsilon(L) = \rho(L)$? Can one give independent lower bounds for $\rho(L)$ or $\rho'(L)$ (which would then bound Seshadri constants from below)?

In the other direction, there are numerous bounds on Seshadri constants, which in turn give bounds for the M -regularity indices that (at least to us) are not obvious from the definition. All of the results in [La2] Ch.I §5 gives some sort of bound. Let's just give a couple of examples:

Corollary 7.11. *If $(J(C), \Theta)$ is a principally polarized Jacobian, then $m(n\Theta) \leq \sqrt{g} \cdot n$. On an arbitrary abelian variety, for any principal polarization Θ we have $m(n\Theta) \leq (g!)^{\frac{1}{g}} \cdot n$.*

Proof. It is shown in [La1] that $\epsilon(\Theta) \leq \sqrt{g}$. We then apply Theorem 7.4. For the other bound we use the usual elementary upper bound for Seshadri constants, namely $\epsilon(\Theta) \leq (g!)^{\frac{1}{g}}$. \square

Corollary 7.12. *If (A, Θ) is a very general PPAV, then there exists at least one n such that $p(n\Theta) \geq \frac{1}{4}(g!)^{\frac{1}{g}} \cdot n$.*

Proof. Here we use the lower bound given in [La1] via a result of Buser-Sarnak. \square

There exist more specific results on $\epsilon(\Theta)$ for Jacobians (cf. [De1], Theorem 7), each giving a corresponding result for $m(n\Theta)$. We can ask however:

Question 7.13. Can we calculate $m(n\Theta)$ individually on Jacobians, at least for small n , in terms of the geometry of the curve?

Example 7.14 (Elliptic curves). As a simple example, the question above has a clear answer for elliptic curves. We know that on an elliptic curve E a line bundle L is M -regular if and only if $\deg(L) \geq 1$, i.e. if and only if L is ample. From the definition of M -regularity we see then that if $\deg(L) = d > 0$, then $m(L) = d - 1$. This implies that on an elliptic curve $m(n\Theta) = n - 1$ for all $n \geq 1$. This is misleading in higher genus however; in the simplest case we have the following general statement: *If (X, Θ) is an irreducible principally polarized abelian variety of dimension at least 2, then $m(2\Theta) \geq 2$.* This is an immediate consequence of the properties of the Kummer map. The linear series $|2\Theta|$ induces a $2 : 1$ map of X onto its image in \mathbf{P}^{2g-1} , with injective differential. Thus the cohomological support locus for $\mathcal{O}(2\Theta) \otimes m_x \otimes m_y$ consists of a finite number of points, while the one for $\mathcal{O}(2\Theta) \otimes m_x^2$ is empty.

7.2. Regularity of Picard bundles and vanishing on symmetric products. In this subsection we study the regularity of Picard bundles over the Jacobian of a curve, twisted by positive multiples of the theta divisor. Some applications to the degrees of equations cutting out special subvarieties of Jacobians are drawn in the second part. Let C be a smooth curve of genus $g \geq 2$, and denote by $J(C)$ the Jacobian of C . The objects we are interested in are the Picard bundles on $J(C)$: a line bundle L on C of degree $n \geq 2g - 1$ – seen as a sheaf on $J(C)$ via an Abel-Jacobi embedding of C into $J(C)$ – satisfies IT_0 , and the Fourier-Mukai transform $E_L = \hat{L}$ is called an n -th Picard bundle. When possible, we omit the dependence on L and write simply E . Note that any other such n -th Picard bundle E_M , with $M \in \text{Pic}^n(C)$, is a translate of E_L . The line bundle L induces an identification between $J(C)$ and $\text{Pic}^n(C)$, so that the projectivization of E – seen as a vector bundle over $\text{Pic}^n(C)$ – is the symmetric product C_n (cf. [ACGH] Ch.VII §2).

The following theorem is the main cohomological result we are aiming for. It is worth noting that Picard bundles are known to be negative (i.e with ample dual bundle), so vanishing theorems

are not automatic. To be very precise, everything that follows holds if n is assumed to be at least $4g - 4$. (However the value of n does not affect the applications.)

Theorem 7.15. *For every $1 \leq k \leq g - 1$, $\otimes^k E \otimes \mathcal{O}(\Theta)$ satisfies IT_0 .*

Before proving the Theorem, we record the following preliminary:

Lemma 7.16. *For any $k \geq 1$, let $\pi_k : C^k \rightarrow J(C)$ a desymmetrized Abel-Jacobi mapping and let L be a line bundle on C of degree $n \gg 0$ as above. Then $\pi_{k*}(L \boxtimes \dots \boxtimes L)$ satisfies IT_0 , and*

$$(\pi_{k*}(L \boxtimes \dots \boxtimes L))^\wedge = \otimes^k E,$$

where E is the n -th Picard bundle of C .

Proof. The first assertion is clear. Concerning the second assertion note that, by definition, $\pi_{k*}(L \boxtimes \dots \boxtimes L)$ is the Pontrjagin product $L * \dots * L$. By the exchange of Pontrjagin and tensor product under the Fourier-Mukai transform ([Muk1] (3.7)), it follows that $(L * \dots * L)^\wedge \cong \widehat{L} \otimes \dots \otimes \widehat{L} = \otimes^k E$. \square

*Proof. (of Theorem 7.15)*⁶ We will use loosely the notation Θ for any translate of the canonical theta divisor. The statement of the theorem becomes then equivalent to the vanishing

$$h^i(\otimes^k E \otimes \mathcal{O}(\Theta)) = 0, \forall i > 0, \forall 1 \leq k \leq g - 1.$$

To prove this vanishing we use the Fourier-Mukai transform. The first point is that Lemma 7.16 above, combined with Grothendieck duality (Theorem 2.5 above), tells us precisely that $\otimes^k E$ satisfies WIT_g , and, by Mukai inversion theorem (Theorem 1) its Fourier transform is

$$\widehat{\otimes^k E} = (-1_J)^* \pi_{k*}(L \boxtimes \dots \boxtimes L).$$

Using, once again, the fact that the Fourier-Mukai transform is an equivalence, we have the following sequence of isomorphisms:

$$\begin{aligned} H^i(\otimes^k E \otimes \mathcal{O}(\Theta)) &\cong \text{Ext}^i(\mathcal{O}(-\Theta), \otimes^k E) \cong \text{Ext}^i(\widehat{\mathcal{O}(-\Theta)}, \widehat{\otimes^k E}) \\ &\cong \text{Ext}^i(\mathcal{O}(\Theta), (-1_J)^* \pi_{k*}(L \boxtimes \dots \boxtimes L)) \cong H^i((-1_J)^* \pi_{k*}(L \boxtimes \dots \boxtimes L) \otimes \mathcal{O}(-\Theta)) \end{aligned}$$

(here we use the fact that both $\mathcal{O}(-\Theta)$ and $\otimes^k E$ satisfy WIT_g and that $\widehat{\mathcal{O}(-\Theta)} = \mathcal{O}(\Theta)$).

As we are loosely writing Θ for any translate, multiplication by -1 does not influence the vanishing, so the result follows if we show:

$$h^i(\pi_{k*}(L \boxtimes \dots \boxtimes L) \otimes \mathcal{O}(-\Theta)) = 0, \forall i > 0.$$

Now the image W_k of the Abel-Jacobi map $u_k : C^k \rightarrow J(C)$ has rational singularities (cf. [Ke2]), so we only need to prove the vanishing:

$$h^i(u_k^*(\pi_{k*}(L \boxtimes \dots \boxtimes L) \otimes \mathcal{O}(-\Theta))) = 0, \forall i > 0.$$

Thus we are interested in the skew-symmetric part of the cohomology group $H^i(C^k, (L \boxtimes \dots \boxtimes L) \otimes \pi_k^* \mathcal{O}(-\Theta))$, or, by Serre duality that of

$$H^i(C^k, ((\omega_C \otimes L^{-1}) \boxtimes \dots \boxtimes (\omega_C \otimes L^{-1})) \otimes \pi_k^* \mathcal{O}(\Theta)), \text{ for } i < k.$$

⁶We are grateful to Olivier Debarre for pointing out a numerical mistake in the statement, in a previous version of this paper.

At this stage we can essentially invoke a Serre vanishing type argument, but it is worth noting that the computation can be in fact made very concrete. For the identifications used next we refer to [Iz] Appendix 3.1. As $k \leq g - 1$, we can write

$$\pi_k^* \mathcal{O}(\Theta) \cong ((\omega_C \otimes A^{-1}) \boxtimes \dots \boxtimes (\omega_C \otimes A^{-1})) \otimes \mathcal{O}(-\Delta),$$

where Δ is the union of all the diagonal divisors in C^k and A is a line bundle of degree $g - k - 1$. Then the skew-symmetric part of the cohomology groups we are looking at is isomorphic to

$$S^i H^1(C, \omega_C^{\otimes 2} \otimes A^{-1} \otimes L^{-1}) \otimes \wedge^{k-i} H^0(C, \omega_C^{\otimes 2} \otimes A^{-1} \otimes L^{-1}),$$

and since for $1 \leq k \leq g - 1$ and $n \geq 4g - 4$ the degree of the line bundle $\omega_C^{\otimes 2} \otimes A^{-1} \otimes L^{-1}$ is negative, this vanishes precisely for $i < k$. \square

An interesting consequence of the vanishing result for Picard bundles proved above is a new – and in some sense more classical – way to deduce Theorem 4.1 of [PP1] on the M-regularity of twists of ideal sheaves \mathcal{I}_{W_d} on the Jacobian $J(C)$. This theorem has a number of applications to the equations of the W_d 's inside $J(C)$, and also to vanishing results for pull-backs of theta divisors to symmetric products. For this circle of ideas we refer the reader to [PP1] §4. For any $1 \leq d \leq g - 1$, $g \geq 3$, consider $u_d : C_d \rightarrow J(C)$ to be an Abel-Jacobi mapping of the symmetric product (depending on the choice of a line bundle of degree d on C), and denote by W_d the image of u_d in $J(C)$.

Theorem 7.17. *For every $1 \leq d \leq g - 1$, $\mathcal{I}_{W_d}(2\Theta)$ satisfies IT_0 .*

Proof. We have to prove that:

$$h^i(\mathcal{I}_{W_d} \otimes \mathcal{O}(2\Theta) \otimes \alpha) = 0, \quad \forall i > 0, \quad \forall \alpha \in \text{Pic}^0(J(C)).$$

In the rest of the proof, by Θ we will understand generically any translate of the canonical theta divisor, and so α will disappear from the notation.

It is well known that W_d has a natural determinantal structure, and its ideal is resolved by an Eagon-Northcott complex. We will chase the vanishing along this complex. This setup is precisely the one used by Fulton and Lazarsfeld in order to prove for example the existence theorem in Brill-Noether theory – for explicit details on this cf. [ACGH] Ch.VII §2. Concretely, W_d is the "highest" degeneracy locus of a map of vector bundles

$$\gamma : E \rightarrow F,$$

where $\text{rk} F = m$ and $\text{rk} E = n = m + d - g + 1$, with $m \gg 0$ arbitrary. The bundles E and F are well understood: E is the n -th Picard bundle of C , discussed above, and F is a direct sum of topologically trivial line bundles. (For simplicity we are again moving the whole construction on $J(C)$ via the choice of a line bundle of degree n .) In other words, W_d is scheme theoretically the locus where the dual map

$$\gamma^* : F^* \rightarrow E^*$$

fails to be surjective. This locus is resolved by an Eagon-Northcott complex (cf. [Ke1]) of the form:

$$0 \rightarrow \wedge^m F^* \otimes S^{m-n} E \otimes \det E \rightarrow \dots \rightarrow \wedge^{n+1} F^* \otimes E \otimes \det E \rightarrow \wedge^n F^* \rightarrow \mathcal{I}_{W_d} \rightarrow 0.$$

As it is known that the determinant of E is nothing but $\mathcal{O}(-\Theta)$, and since F is a direct sum of topologically trivial line bundles, the statement of the theorem follows by chopping this into short exact sequences, as long as we prove:

$$h^i(S^k E \otimes \mathcal{O}(\Theta)) = 0, \quad \forall i > 0, \quad \forall 1 \leq k \leq m - n = g - d - 1.$$

Since we are in characteristic zero, $S^k E$ is naturally a direct summand in $\otimes^k E$, and so it is sufficient to prove that:

$$h^i(\otimes^k E \otimes \mathcal{O}(\Theta)) = 0, \forall i > 0, \forall 1 \leq k \leq g - d - 1.$$

But this follows from Theorem 7.15. \square

Remark 7.18. Using [PP1] Proposition 2.9, it follows that $\mathcal{I}_{W_d}(k\Theta)$ satisfies IT_0 for all $k \geq 2$.

Remark 7.19. It is conjectured, based on a connection with minimal cohomology classes (cf. [PP7] for a discussion), that the only nondegenerate subvarieties Y of a principally polarized abelian variety (A, Θ) such that $\mathcal{I}_Y(2\Theta)$ satisfies IT_0 are precisely the W_d 's above, in Jacobians, and the Fano surface of lines in the intermediate Jacobian of the cubic threefold.

Question 7.20. What is the minimal k such that $\mathcal{I}_{W_d^r}(k\Theta)$ is M -regular, for r and d arbitrary?

We describe below one case in which the answer can already be given, namely that of the singular locus of the Riemann theta divisor on a non-hyperelliptic jacobian. It should be noted that in this case we do not have that $\mathcal{I}_{W_{g-1}^1}(2\Theta)$ satisfies IT_0 any more (but rather $\mathcal{I}_{W_{g-1}^1}(3\Theta)$ does, by the same [PP1] Proposition 2.9).

Proposition 7.21. $\mathcal{I}_{W_{g-1}^1}(2\Theta)$ is M -regular.

Proof. It follows from the results of [vGI] that

$$h^i(\mathcal{I}_{W_{g-1}^1} \otimes \mathcal{O}(2\Theta) \otimes \alpha) = \begin{cases} 0 & \text{for } i \geq g - 2, \forall \alpha \in \text{Pic}^0(J(C)) \\ 0 & \text{for } 0 < i < g - 2, \forall \alpha \in \text{Pic}^0(J(C)) \text{ such that } \alpha \neq \mathcal{O}_{J(C)}. \end{cases}$$

For the reader's convenience, let us briefly recall the relevant points from Section 7 of [vGI]. We denote for simplicity, via translation, $\Theta = W_{g-1}$, (so that $W_{g-1}^1 = \text{Sing}(\Theta)$). In the first place, from the exact sequence

$$0 \rightarrow \mathcal{O}(2\Theta) \otimes \alpha \otimes \mathcal{O}(-\Theta) \rightarrow \mathcal{I}_{W_{g-1}^1}(2\Theta) \otimes \alpha \rightarrow \mathcal{I}_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha \rightarrow 0$$

it follows that

$$h^i(J(C), \mathcal{I}_{W_{g-1}^1}(2\Theta) \otimes \alpha) = h^i(\Theta, \mathcal{I}_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha) \text{ for } i > 0.$$

Hence one is reduced to a computation on Θ . It is a standard fact (see e.g. [vGI], 7.2) that, via the Abel-Jacobi map $u = u_{g-1} : C_{g-1} \rightarrow \Theta \subset J(C)$,

$$h^i(\Theta, \mathcal{I}_{W_{g-1}^1/\Theta}(2\Theta) \otimes \alpha) = h^i(C_{g-1}, L^{\otimes 2} \otimes \beta \otimes \mathcal{I}_Z),$$

where $Z = u^{-1}(W_{g-1}^1)$, $L = u^*\mathcal{O}_X(\Theta)$ and $\beta = u^*\alpha$. We now use the standard exact sequence ([ACGH], p.258):

$$0 \rightarrow T_{C_{g-1}} \xrightarrow{du} H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{C_{g-1}} \rightarrow L \otimes \mathcal{I}_Z \rightarrow 0.$$

Tensoring with $L \otimes \beta$, we see that it is sufficient to prove that

$$H^i(C_{g-1}, T_{C_{g-1}} \otimes L \otimes \beta) = 0, \forall i \geq 2, \forall \beta \neq \mathcal{O}_{C_{g-1}}.$$

To this end we use the well known fact (cf. *loc. cit.*) that

$$T_{C_{g-1}} \cong p_*\mathcal{O}_D(D)$$

where $D \subset C_{g-1} \times C$ is the universal divisor and p is the projection onto the first factor. As p_{1D} is finite, the degeneration of the Leray spectral sequence and the projection formula ensure that

$$h^i(C_{g-1}, T_{C_{g-1}} \otimes L \otimes \beta) = h^i(D, \mathcal{O}_D(D) \otimes p^*(L \otimes \beta)),$$

which are zero for $i \geq 2$ and β non-trivial by [vGI], Lemma 7.24. \square

7.3. Numerical study of semihomogeneous vector bundles. An idea that originated in work of Mukai is that on abelian varieties the class of vector bundles to which the theory of line bundles should generalize naturally is that of *semihomogeneous* bundles (cf. [Muk1], [Muk3], [Muk4]). These vector bundles are semistable, behave nicely under isogenies and Fourier transforms, and have a Mumford type theta group theory as in the case of line bundles (cf. [Um]). The purpose of this section is to show that this analogy can be extended to include effective global generation and normal generation statements dictated by specific numerical invariants measuring positivity. Recall that normal generation is Mumford's terminology for the surjectivity of the multiplication map $H^0(E) \otimes H^0(E) \rightarrow H^0(E^{\otimes 2})$.

In order to set up a criterion for normal generation, it is useful to introduce the following notion, which parallels the notion of Castelnuovo-Mumford regularity.

Definition 7.22. A coherent sheaf \mathcal{F} on a polarized abelian variety (X, Θ) is called *m - Θ -regular* if $\mathcal{F}((m-1)\Theta)$ is M -regular.

The relationship with normal generation comes from (3) of the following ‘‘abelian’’ Castelnuovo-Mumford Lemma. Note that (1) is Corollary 5.3 plus Proposition 5.4.

Theorem 7.23 ([PP1], Theorem 6.3). *Let \mathcal{F} be a 0- Θ -regular coherent sheaf on X . Then:*

- (1) \mathcal{F} is globally generated.
- (2) \mathcal{F} is m - Θ -regular for any $m \geq 1$.
- (3) The multiplication map

$$H^0(\mathcal{F}(\Theta)) \otimes H^0(\mathcal{O}(k\Theta)) \longrightarrow H^0(\mathcal{F}((k+1)\Theta))$$

is surjective for any $k \geq 2$.

Basics on semihomogeneous bundles. Let X be an abelian variety of dimension g over an algebraically closed field. As a general convention, for a numerical class α we will use the notation $\alpha > 0$ to express the fact that α is ample. If the class is represented by an effective divisor, then the condition of being ample is equivalent to $\alpha^g > 0$. For a line bundle L on X , we denote by ϕ_L the isogeny defined by L :

$$\begin{aligned} \phi_L : X &\longrightarrow \text{Pic}^0(X) \cong \widehat{X} \\ x &\rightsquigarrow t_x^* L \otimes L^{-1}. \end{aligned}$$

Definition 7.24. ([Muk3]) A vector bundle E on X is called *semihomogeneous* if for every $x \in X$, $t_x^* E \cong E \otimes \alpha$, for some $\alpha \in \text{Pic}^0(X)$.

Mukai shows in [Muk3] §6 that the semihomogeneous bundles are Gieseker semistable (while the simple ones – i.e. with no nontrivial automorphisms – are in fact stable). Moreover, any semihomogeneous bundle has a Jordan-Hölder filtration in a strong sense.

Proposition 7.25. ([Muk3] Proposition 6.18) *Let E be a semihomogeneous bundle on X , and let δ be the equivalence class of $\frac{\det(E)}{\text{rk}(E)}$ in $NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then there exist simple semihomogeneous bundles F_1, \dots, F_n whose corresponding class is the same δ , and semihomogeneous bundles E_1, \dots, E_n , satisfying:*

- $E \cong \bigoplus_{i=1}^n E_i$.
- Each E_i has a filtration whose factors are all isomorphic to F_i .

Since the positivity of E is carried through to the factors of a Jordan-Hölder filtration as in the Proposition above, standard inductive arguments allow us to immediately reduce the study below to the case of simple semihomogeneous bundles, which we do freely in what follows.

Lemma 7.26. *Let E be a simple semihomogeneous bundle of rank r on X .*

(1) ([Muk3], Proposition 7.3) *There exists an isogeny $\pi : Y \rightarrow X$ and a line bundle M on Y such that $\pi^*E \cong \bigoplus_r M$.*

(2) ([Muk3], Theorem 5.8(iv)) *There exists an isogeny $\phi : Z \rightarrow X$ and a line bundle L on Z such that $\phi_*L = E$.*

Lemma 7.27. *Let E be a nondegenerate (i.e. $\chi(E) \neq 0$) simple semihomogeneous bundle on X . Then exactly one cohomology group $H^i(E)$ is nonzero, i.e. E satisfies the Index Theorem.*

Proof. This follows immediately from the similar property of the line bundle L in Lemma 7.26(2). \square

Lemma 7.28. *A semihomogeneous bundle E is m - Θ -regular if and only if $E((m-1)\Theta)$ satisfies IT_0 .*

Proof. The more general fact that an M -regular semihomogeneous bundle satisfies IT_0 follows quickly from Lemma 7.26(1) above. More precisely the line bundle M in its statement is forced to be ample since it has a twist with global sections and positive Euler characteristic. \square

A numerical criterion for normal generation. The main result of this section is that the normal generation of a semihomogeneous vector bundle is dictated by an explicit numerical criterion. We assume all throughout that all the semihomogeneous vector bundles involved satisfy the minimal positivity condition, namely that they are 0 - Θ -regular, which in particular is a criterion for global generation by Theorem 7.23. We will in fact prove a criterion which guarantees the surjectivity of multiplication maps for two arbitrary semihomogeneous bundles. This could be seen as an analogue of Butler's theorem [Bu] for semistable bundles on curves.

Theorem 7.29. *Let E and F be semihomogeneous bundles on (X, Θ) , both 0 - Θ -regular. Then the multiplication maps*

$$H^0(E) \otimes H^0(t_x^*F) \longrightarrow H^0(E \otimes t_x^*F)$$

are surjective for all $x \in X$ if the following holds:

$$\frac{1}{r_F} \cdot c_1(F(-\Theta)) + \frac{1}{r'_E} \cdot \phi_\Theta^* c_1(\widehat{E(-\Theta)}) > 0,$$

where $r_F := \text{rk}(F)$ and $r'_E := \text{rk}(\widehat{E(-\Theta)})$. (Recall that ϕ_Θ is the isogeny induced by Θ .)

Remark 7.30. Although most conveniently written in terms of the Fourier-Mukai transform, the statement of the theorem is indeed a numerical condition intrinsic to E (and F), since by [Muk2] Corollary 1.18 one has:

$$c_1(\widehat{E(-\Theta)}) = -PD_{2g-2}(\text{ch}_{g-1}(E(-\Theta))),$$

where PD denotes the Poincaré duality map

$$PD_{2g-2} : H^{2g-2}(J(X), \mathbb{Z}) \rightarrow H^2(J(X), \mathbb{Z}),$$

and ch_{g-1} the $(g-1)$ -st component of the Chern character. Note also that

$$\text{rk}(\widehat{E(-\Theta)}) = h^0(E(-\Theta)) = \frac{1}{r^{g-1}} \cdot \frac{c_1(E(-\Theta))^g}{g!}$$

by Lemma 7.28 and [Muk1] Corollary 2.8.

We can assume E and F to be simple by the considerations in §2, and we will do so in what follows. We begin with a few technical results. In the first place, it is useful to consider the *skew Pontrjagin product*, a slight variation of the usual Pontrjagin product (see [Pa] §1). Namely, given two sheaves \mathcal{E} and \mathcal{G} on X , one defines

$$\mathcal{E} \hat{*} \mathcal{G} := d_*(p_1^*(\mathcal{E}) \otimes p_2^*(\mathcal{G})),$$

where p_1 and p_2 are the projections from $X \times X$ to the two factors and $d : X \times X \rightarrow X$ is the difference map.

Lemma 7.31. *For all $i \geq 0$ we have:*

$$h^i((E \hat{*} F) \otimes \mathcal{O}_X(-\Theta)) = h^i((E \hat{*} \mathcal{O}_X(-\Theta)) \otimes F).$$

Proof. This follows from Lemma 3.2 in [Pa] if we prove the following vanishings:

- (1) $h^i(t_x^* E \otimes F) = 0, \forall i > 0, \forall x \in X.$
- (2) $h^i(t_x^* E \otimes \mathcal{O}_X(-\Theta)) = 0, \forall i > 0, \forall x \in X.$

We treat them separately:

- (1) By Lemma 7.26(1) we know that there exist isogenies $\pi_E : Y_E \rightarrow X$ and $\pi_F : Y_F \rightarrow X$, and line bundles M on Y_E and N on Y_F , such that $\pi_E^* E \cong \bigoplus_{r_E} M$ and $\pi_F^* F \cong \bigoplus_{r_F} N$. Now on the fiber product $Y_E \times_X Y_F$, the pull-back of $t_x^* E \otimes F$ is a direct sum of line bundles numerically equivalent to $p_1^* M \otimes p_2^* N$. This line bundle is ample and has sections, and so no higher cohomology by the Index Theorem. Consequently the same must be true for $t_x^* E \otimes F$.
- (2) Since E is semihomogeneous, we have $t_x^* E \cong E \otimes \alpha$ for some $\alpha \in \text{Pic}^0(X)$, and so:

$$h^i(t_x^* E \otimes \mathcal{O}_X(-\Theta)) = h^i(E \otimes \mathcal{O}_X(-\Theta) \otimes \alpha) = 0,$$

since $E(-\Theta)$ satisfies IT_0 .

□

Let us assume from now on for simplicity that the polarization Θ is symmetric. This makes the proofs less technical, but the general case is completely similar since everything depends (via suitable isogenies) only on numerical equivalence classes.

Proposition 7.32. *Under the hypotheses above, the multiplication maps*

$$H^0(E) \otimes H^0(t_x^* F) \longrightarrow H^0(E \otimes t_x^* F)$$

are surjective for all $x \in X$ if we have the following vanishing:

$$h^i(\phi_\Theta^*((-1_X)^* E \otimes \mathcal{O}_X(-\Theta)) \hat{\otimes} F(-\Theta)) = 0, \forall i > 0.$$

Proof. By [Pa] Theorem 3.1, all the multiplication maps in the statement are surjective if the skew-Pontrjagin product $E \hat{*} F$ is globally generated, so in particular if $(E \hat{*} F)$ is 0 - Θ -regular. On the other hand, by Lemma 7.31, we can check this 0 -regularity by checking the vanishing of $h^i((E \hat{*} \mathcal{O}_X(-\Theta)) \otimes F)$. To this end, we use Mukai's general Lemma 3.10 in [Muk1] to see that

$$E \hat{*} \mathcal{O}_X(-\Theta) \cong \phi_{\Theta}^*((-1_X)^* E \otimes \mathcal{O}_X(-\Theta)) \widehat{\otimes} \mathcal{O}(-\Theta).$$

This implies the statement. \square

We are now in a position to prove Theorem 7.29: we only need to understand the numerical assumptions under which the cohomological requirement in Proposition 7.32 is satisfied.

Proof. (of Theorem 7.29.) We first apply Lemma 7.26(1) to $G := \phi_{\Theta}^*(-1_X)^* \widehat{E}(-\Theta)$ and $H := F(-\Theta)$: there exist isogenies $\pi_G : Y_G \rightarrow X$ and $\pi_H : Y_H \rightarrow X$, and line bundles M on Y_G and N on Y_H , such that $\pi_G^* G \cong \bigoplus_{r_G} M$ and $\pi_H^* H \cong \bigoplus_{r_H} N$. Consider the fiber product $Z := Y_G \times_X Y_H$, with projections p_G and p_H . Denote by $p : Z \rightarrow X$ the natural composition. By pulling everything back to Z , we see that

$$p^*(G \otimes H) \cong \bigoplus_{r_G \cdot r_H} (p_1^* M \otimes p_2^* N).$$

This implies that our desired vanishing $H^i(G \otimes H) = 0$ (cf. Proposition 7.32) holds as long as

$$H^i(p_G^* M \otimes p_H^* N) = 0, \quad \forall i > 0.$$

Now $c_1(p_G^* M) = p_G^* c_1(M) = \frac{1}{r_G} p^* c_1(G)$ and similarly $c_1(p_H^* N) = p_H^* c_1(N) = \frac{1}{r_H} p^* c_1(H)$. Finally we get

$$c_1(p_G^* M \otimes p_H^* N) = p^* \left(\frac{1}{r_G} \cdot c_1(G) + \frac{1}{r_H} \cdot c_1(H) \right).$$

Thus all we need to have is that the class

$$\frac{1}{r_G} \cdot c_1(G) + \frac{1}{r_H} \cdot c_1(H)$$

be ample, and this is clearly equivalent to the statement of the theorem. \square

(-1) - Θ -regular vector bundles. It can be easily seen that Theorem 7.29 implies that a (-1) - Θ -regular semihomogeneous bundle is normally generated. Under this regularity hypothesis we have however a much more general statement, which works for every vector bundle on a polarized abelian variety.

Theorem 7.33. *For (-1) - Θ -regular vector bundles E and F on X , the multiplication map*

$$H^0(E) \otimes H^0(F) \rightarrow H^0(E \otimes F)$$

is surjective.

Proof. We use an argument exploited in [PP1], inspired by techniques introduced by Kempf. Let us consider the diagram

$$\begin{array}{ccc} \bigoplus_{\xi \in U} H^0(E(-2\Theta) \otimes P_{\xi}) \otimes H^0(2\Theta \otimes P_{\xi}^{\vee}) \otimes H^0(F) & \longrightarrow & H^0(E) \otimes H^0(F) \\ \downarrow & & \downarrow \\ \bigoplus_{\xi \in U} H^0(E(-2\Theta) \otimes P_{\xi}) \otimes H^0(F(2\Theta) \otimes P_{\xi}^{\vee}) & \longrightarrow & H^0(E \otimes F) \end{array}$$

Under the given hypotheses, the bottom horizontal arrow is onto by the general Theorem 5.1. On the other hand, the abelian Castelnuovo-Mumford Lemma Theorem 7.23 insures that each one of the components of the vertical map on the left is surjective. Thus the composition is surjective, which gives the surjectivity of the vertical map on the right. \square

Corollary 7.34. *Every (-1) - Θ -regular vector bundle is normally generated.*

Examples. There are two basic classes of examples of (-1) - Θ -regular bundles, and both turn out to be semihomogeneous. They correspond to the properties of linear series on abelian varieties and on moduli spaces of vector bundles on curves, respectively.

Example 7.35. (Projective normality of line bundles.) For every ample divisor Θ on X , the line bundle $L = \mathcal{O}_X(m\Theta)$ is (-1) - Θ -regular for $m \geq 3$. Thus we recover the classical fact that $\mathcal{O}_X(m\Theta)$ is projectively normal for $m \geq 3$.

Example 7.36. (Verlinde bundles.) Let $U_C(r, 0)$ be the moduli space of rank r and degree 0 semistable vector bundles on a smooth projective curve C of genus $g \geq 2$. This comes with a natural determinant map $\det : U_C(r, 0) \rightarrow J(C)$, where $J(C)$ is the Jacobian of C . To a generalized theta divisor Θ_N on $U_C(r, 0)$ (depending on the choice of a line bundle $N \in \text{Pic}^{g-1}(C)$) one associates for any $k \geq 1$ the (r, k) -Verlinde bundle on $J(C)$, defined by $E_{r,k} := \det_* \mathcal{O}(k\Theta_N)$ (cf. [Po]). It is shown in *loc. cit.* that the numerical properties of $E_{r,k}$ are essential in understanding the linear series $|k\Theta_N|$ on $U_C(r, 0)$. It is noted there that $E_{r,k}$ are polystable and semihomogeneous.

A basic property of these vector bundles is the fact that $r_J^* E_{r,k} \cong \oplus \mathcal{O}_J(kr\Theta_N)$, where r_J denotes multiplication by r on $J(C)$ (cf. [Po] Lemma 2.3). Noting that the pull-back $r_J^* \mathcal{O}_J(\Theta_N)$ is numerically equivalent to $\mathcal{O}(r^2\Theta_N)$, we obtain that $E_{r,k}$ is 0- Θ -regular iff $k \geq r + 1$, and (-1) - Θ -regular iff $k \geq 2r + 1$. This implies by the statements above that $E_{r,k}$ is globally generated for $k \geq r + 1$ and normally generated for $k \geq 2r + 1$. These are precisely the results [Po] Proposition 5.2 and Theorem 5.9(a), the second obtained there by ad-hoc (though related) methods.

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