

# DERIVED EQUIVALENCE, BGG CORRESPONDENCE, AND COHOMOLOGY

MIHNEA POPA

ABSTRACT. These notes are concerned with the behavior of various cohomology groups of the structure sheaf or the canonical bundle of a smooth projective variety, or of its Picard variety, under the action of Fourier-Mukai transforms and the BGG correspondence. They are prepared for lectures to be delivered at IHP Paris, and in Brasilia, in the summer of 2010. I thank the organizers of the respective events for the opportunity.

## CONTENTS

1. Part I: Derivative complex, BGG correspondence, and cohomology of compact Kähler manifolds	1
1.1. Generic vanishing	2
1.2. Derivative complexes	3
1.3. A brief introduction to the BGG correspondence	6
1.4. BGG and the canonical cohomology module	8
1.5. Inequalities for numerical invariants	11
2. Part II: Derived equivalence and the Picard variety	14
2.1. Basics on equivalences of derived categories and birational geometry	14
2.2. Fourier-Mukai functors and special autoequivalences of derived categories	15
2.3. Hochschild (co)homology	16
2.4. Actions of non-affine algebraic groups	17
2.5. Behavior of the Picard variety under derived equivalence	19
References	22

## 1. PART I: DERIVATIVE COMPLEX, BGG CORRESPONDENCE, AND COHOMOLOGY OF COMPACT KÄHLER MANIFOLDS

In the first part of the lectures I will present results from [LP], together with the necessary background.

---

*Date:* July 23, 2010.

**1.1. Generic vanishing.** Let  $X$  be a smooth projective variety over  $\mathbb{C}$ . If  $L$  is an ample line bundle on  $X$ , then the celebrated Kodaira Vanishing theorem says that

$$H^i(X, \omega_X \otimes L) = 0 \text{ for all } i > 0.$$

This has of course a myriad of applications. Some of them are numerical: as a simple example, it implies that  $\chi(\omega_X \otimes L) \geq 0$ . However, one hopes that some of these consequences hold even under weaker positivity hypotheses. So a natural question arises: are there any modified vanishing theorems involving just  $\omega_X$ , or more generally  $\omega_X \otimes L$ , with  $L$  say a nef line bundle?

**Example 1.1.** Let  $C$  be a smooth projective curve of genus  $g \geq 1$ . Then we have  $H^1(C, \omega_C) \cong \mathbb{C}$ , but also  $H^1(C, \omega_C \otimes \alpha) = 0$  for any  $\alpha \in \text{Pic}^0(C) - \{0\}$ . Hence the higher cohomology of a general (and in fact any nontrivial) deformation of the canonical line bundle by a topologically trivial line bundle vanishes. This is an instance of “generic vanishing”. (A key related point here, as we’ll see below, is that the Albanese map of  $C$ , i.e. the Abel-Jacobi map  $C \rightarrow J(C)$ , is an embedding.)

This example is a very special case of the following general phenomenon discovered by Green and Lazarsfeld. To state the result, we introduce the following notation:

$$V^i(\omega_X) := \{\alpha \in \text{Pic}^0(X) \mid H^i(X, \omega_X \otimes \alpha) \neq 0\} \subseteq \text{Pic}^0(X).$$

By the semicontinuity theorem this is a closed subset for each  $i$ , called the  $i$ -th cohomological support locus of  $\omega_X$ . E.g. for a curve  $C$  as above we have  $V^0(\omega_C) = \text{Pic}^0(C)$  and  $V^1(\omega_C) = \{0\}$ .

**Theorem 1.2** ([GL1], Theorem 1). *Let  $X$  be a compact Kähler manifold of dimension  $d$ , with Albanese map  $a : X \rightarrow \text{Alb}(X)$ . Denote  $k := d - \dim a(X)$ . Then for all  $i > 0$*

$$\text{codim } V^i(\omega_X) \geq i - k.$$

In particular, if  $k = 0$ , i.e. when  $a$  is generically finite (we say that  $X$  is of *maximal Albanese dimension*), we have that  $V^i(\omega_X \otimes L)$  are proper subsets for  $i > 0$ . This implies that

$$\chi(\omega_X \otimes L) = \chi(\omega_X \otimes L \otimes \alpha) = h^0(X, \omega_X \otimes L \otimes \alpha) \geq 0.$$

where  $\alpha \in \text{Pic}^0(X)$  is taken general and we note that the Euler characteristic is invariant under deformations. This is first example showing that sometimes one needs less than Kodaira Vanishing to reach similar conclusions.

**Remark 1.3** (Generalizations). Theorem 1.2 was extended to all sheaves of the form  $R^i f_* \omega_Y$ , with  $f : Y \rightarrow X$  a surjective morphism from a smooth projective variety  $Y$  by Hacon [Hac]. Furthermore, it was extended to all line bundles of the form  $\omega_X \otimes L$  with  $L$  nef, and similarly to  $R^i f_* \omega_Y \otimes L$  with  $f$  as above and  $L$  nef, by Pareschi and the author [PP1].

We can consider the same objects from the point of view of integral transforms. Denote by  $p_1$  and  $p_2$  the projections from  $X \times \text{Pic}^0(X)$  to  $X$  and  $\text{Pic}^0(X)$  respectively, and by  $P$  a normalized Poincaré line bundle on  $X \times \text{Pic}^0(X)$ . We denote by

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0(X)), \quad \mathbf{R}\Phi_P \mathcal{E} = \mathbf{R}p_{2*}(p_1^* \mathcal{E} \otimes^{\mathbf{L}} P)$$

the integral functor given by  $P$ , where  $\mathbf{D}(X)$  is the bounded derived category of coherent sheaves.

Using the notation  $\mathbf{R}\Delta \mathcal{F} := \mathbf{R}\text{Hom}_X(\mathcal{F}, \omega_X)$ , under the hypothesis of Theorem 1.2 we have the following (which also follows from results of Kashiwara [Ka], as explained in [Po]):

**Theorem 1.4** ([PP2], Theorem 2.2). *For a coherent sheaf  $\mathcal{F}$  on  $X$ , the following are equivalent:*

- (i)  $\text{codim } V^i(\mathcal{F}) \geq i - k$ , for all  $i > 0$ .
- (ii)  $R^i \Phi_P(\mathbf{R}\Delta \mathcal{F}) = 0$  for all  $i \notin [d - k, d]$ .

In the particular case  $\mathcal{F} = \omega_X$  we have that  $\mathbf{R}\Delta\omega_X \cong \mathcal{O}_X$ , and therefore Theorem 1.2 becomes equivalent to the following vanishing result (first conjectured in [GL2], and proved in the smooth projective case in [Hac] and [Pa]):

**Theorem 1.5.** *With the notation in Theorem 1.2, we have*

$$R^i\Phi_P\mathcal{O}_X = R^ip_{2*}P = 0, \text{ for all } i < d - k.$$

Write now  $V = H^1(X, \mathcal{O}_X)$  and  $W = V^\vee$ , and let  $\mathbf{A} = \text{Spec}(\text{Sym}(W))$  be the affine space corresponding to  $V$ , viewed as an algebraic variety. Thus a point in  $\mathbf{A}$  is the same as a vector in  $V$ . Then there is a natural complex  $\mathcal{K}^\bullet$  of trivial algebraic vector bundles on  $\mathbf{A}$ :

$$0 \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^1(X, \mathcal{O}_X) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{A}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0,$$

with maps given at each point of  $\mathbf{A}$  by wedging with the corresponding element of  $V = H^1(X, \mathcal{O}_X)$ . Now let  $\mathbf{V}$  be the vector space  $V$ , considered as a complex manifold, so that  $\mathbf{V} = \mathbb{C}^q$ , where  $q = h^1(X, \mathcal{O}_X)$  is the irregularity of  $X$ . Then on  $\mathbf{V}$  we can form as above a complex  $(\mathcal{K}^\bullet)^{an}$

$$0 \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^1(X, \mathcal{O}_X) \longrightarrow \dots \longrightarrow \mathcal{O}_{\mathbf{V}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0,$$

of coherent analytic sheaves, which is just the complex of analytic sheaves determined by the algebraic complex  $\mathcal{K}^\bullet$ . This analytic complex was called the *derivative complex*  $D_{\mathcal{O}_X}^\bullet$  of  $\mathcal{O}_X$  in [GL2]. Since passing from a coherent algebraic to a coherent analytic sheaf is an exact functor (cf. [GAGA, 3.10]), one has  $\mathcal{H}^i((\mathcal{K}^\bullet)^{an}) = \mathcal{H}^i((\mathcal{K}^\bullet))^{an}$ . The main result of [GL2], Theorem 3.2, says that via the exponential map  $\exp : V \rightarrow \text{Pic}^0(X)$  we have the identification of the analytic stalks at the origin

$$(1) \quad \mathcal{H}^i((\mathcal{K}^\bullet)^{an})_0 \cong (R^ip_{2*}P)_0.$$

Green and Lazarsfeld showed in [GL2] that this implies Theorem 1.2, the fact that all irreducible components of the cohomological support loci  $V^i(\omega_X)$  are (translated) subtori of  $\text{Pic}^0(X)$ , and further results. Note that (1) and Theorem 1.5 imply that  $\mathcal{K}^\bullet$  is exact in the first  $d - k$  terms from the left.

**1.2. Derivative complexes.** Let  $X$  be a compact Kähler manifold of dimension  $d$ , with  $V = H^1(X, \mathcal{O}_X) \neq 0$ , and let  $\mathbf{P} = \mathbf{P}_{\text{sub}}(V)$  be the projective space of one-dimensional subspaces of  $V$ . Thus a point in  $\mathbf{P}$  is given by a non-zero vector  $v \in H^1(X, \mathcal{O}_X)$ , defined up to scalars. Pointwise cup product with  $v$  determines a complex  $\underline{\mathbf{L}}_X$  of vector bundles on  $\mathbf{P}$ :

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0.$$

Since the differential of the complex  $\mathcal{K}^\bullet$  on  $V$  constructed in the previous section scales linearly in the radial directions, one sees that  $\mathcal{K}^\bullet$  descends from  $V$  to the complex  $\underline{\mathbf{L}}_X$  on  $\mathbf{P}$ .

Letting  $S = \text{Sym}(W)$  be the symmetric algebra on the vector space  $W = V^\vee$ , taking global sections in  $\underline{\mathbf{L}}_X$  gives rise to a linear complex  $\mathbf{L}_X$  of graded  $S$ -modules in homological degrees 0 to  $d$ :

$$(3) \quad 0 \longrightarrow S \otimes_{\mathbb{C}} H^0(X, \mathcal{O}_X) \longrightarrow S \otimes_{\mathbb{C}} H^1(X, \mathcal{O}_X) \longrightarrow \dots \longrightarrow S \otimes_{\mathbb{C}} H^d(X, \mathcal{O}_X) \longrightarrow 0.$$

We shall also be interested in the coherent sheaf  $\mathcal{F} = \mathcal{F}_X$  on  $\mathbf{P}$  arising as the cokernel of the right-most map in the complex  $\underline{\mathbf{L}}_X$ , so that one has an exact sequence:

$$(4) \quad \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow \mathcal{F} \longrightarrow 0.$$

For reasons that will become apparent shortly, we call  $\underline{\mathbf{L}}_X$  and  $\mathbf{L}_X$  the BGG-complexes of  $X$ , and  $\mathcal{F}_X$  its BGG-sheaf.

We will first be concerned with the exactness properties of  $\underline{\mathbf{L}}_X$  and  $\mathbf{L}_X$ . Let

$$\text{alb}_X : X \longrightarrow \text{Alb}(X)$$

be the Albanese mapping of  $X$ , and let

$$k = k(X) = \dim X - \dim \text{alb}_X(X)$$

be the dimension of the general fiber of  $\text{alb}_X$ . We say that  $X$  carries an *irregular fibration* if it admits a surjective morphism  $X \rightarrow Y$  with connected positive dimensional fibres onto a normal analytic variety  $Y$  with the property that (any smooth model of)  $Y$  has maximal Albanese dimension. These are the higher-dimensional analogues of irrational pencils in the case of surfaces. The behavior of  $\mathbf{L}_X$  and  $\underline{\mathbf{L}}_X$  is summarized in the following technical statement:

**Theorem 1.6.** (i). *The complexes  $\mathbf{L}_X$  and  $\underline{\mathbf{L}}_X$  are exact in the first  $d - k$  terms from the left, but  $\mathbf{L}_X$  has non-trivial homology at the next term to the right.*

(ii). *Assume that  $X$  does not carry any irregular fibrations. Then the BGG sheaf  $\mathcal{F}$  is a vector bundle on  $\mathbf{P}$  with  $\text{rk}(\mathcal{F}) = \chi(\omega_X)$ , and  $\underline{\mathbf{L}}_X$  is a resolution of  $\mathcal{F}$ .*

The proof proceeds in the form of three propositions.

**Proposition 1.7.** *The complexes  $\mathbf{L}_X$  and  $\underline{\mathbf{L}}_X$  are exact in the first  $d - k$  terms from the left.*

*Proof.* It is sufficient to prove the exactness for  $\mathbf{L}_X$ , as this implies the corresponding statement for its sheafified sibling. We take global sections in the complex  $\mathcal{K}^\bullet$  on  $\mathbf{A} = \text{Spec}(\text{Sym}(W))$  constructed in the previous section, where  $\mathbf{A}$  is the affine space corresponding to  $V$ . Recalling that  $\Gamma(\mathbf{A}, \mathcal{O}_{\mathbf{A}}) = S$ , we have the identification

$$\mathbf{L}_X = \Gamma(\mathbf{A}, \mathcal{K}^\bullet).$$

As  $\mathbf{A}$  is affine, to prove the stated exactness properties of  $\mathbf{L}_X$ , it is equivalent to establish the analogous exactness for the complex  $\mathcal{K}^\bullet$ , i.e. we need to show the vanishings  $\mathcal{H}^i(\mathcal{K}^\bullet) = 0$  of the cohomology sheaves of this complex in the range  $i < d - k$ . For this it is in turn equivalent to prove the vanishing

$$(*) \quad \mathcal{H}^i(\mathcal{K}^\bullet)_0 = 0$$

of the stalks at the origin of these homology sheaves in the same range  $i < d - k$ . Indeed, (\*) implies that  $\mathcal{H}^i(\mathcal{K}^\bullet) = 0$  in a neighborhood of the origin, and we noted above that the differential of  $\mathcal{K}^\bullet$  scales linearly in radial directions through the origin.

We saw at the end of §1 that by passing to the corresponding analytic complex we have  $\mathcal{H}^i((\mathcal{K}^\bullet)^{an}) = \mathcal{H}^i((\mathcal{K}^\bullet)^{an})$ . So it is equivalent for (\*) to prove:

$$(**) \quad \mathcal{H}^i((\mathcal{K}^\bullet)^{an})_0 = 0 \text{ for } i < d - k.$$

But this follows immediately from (1) and Theorem 1.5.  $\square$

The next point is the non-exactness of  $\mathbf{L}_X$  beyond the range specified in the previous proposition.

**Proposition 1.8.** *The complex  $\mathbf{L}_X$  is not exact at the term  $S \otimes_{\mathbb{C}} H^{d-k}(X, \mathcal{O}_X)$ .*

**Remark 1.9.** The analogous statement for the sheafified complex  $\underline{\mathbf{L}}_X$  can fail. For example, if  $X = A \times \mathbf{P}^k$ , with  $A$  an abelian variety of dimension  $d - k$ , then  $\underline{\mathbf{L}}_X$  – which in this case is just a Koszul complex on  $\mathbf{P}^{d-k-1}$  – is everywhere exact.

*Proof of Proposition 1.8.* Arguing as in the previous proof, it suffices to check the corresponding statement for the derivative complex around 0, which in turn is equivalent to showing that

$$(*) \quad (R^{d-k} p_{2*} P)_0 \neq 0.$$

The plan is to deduce this from results of Kollár [Ko2] on direct images of dualizing sheaves of smooth projective varieties, extended by Saito [Sa] and Takegoshi [Ta] to the compact Kähler setting. To streamline the formulas, for the remainder of the proof we change notation and write

$$a : X \longrightarrow A = \text{Alb}(X)$$

for the Albanese mapping of  $X$ . First, the papers just cited prove that one has a splitting

$$(5) \quad \mathbf{R}a_*\omega_X \cong \bigoplus_{j=0}^k R^j a_*\omega_X[-j]$$

in the derived category of  $A$ . Recall that

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0(X)), \quad \mathbf{R}\Phi_P \mathcal{E} = \mathbf{R}p_{2*}(p_1^* \mathcal{E} \otimes^{\mathbf{L}} P)$$

is the integral functor given by  $P$ , and analogously for  $\mathbf{R}\Phi_{P^\vee}$ . Following Mukai's notation [Mu2], we also denote by

$$\mathbf{R}\widehat{\mathcal{S}} : \mathbf{D}(A) \rightarrow \mathbf{D}(\text{Pic}^0(X))$$

the standard Fourier-Mukai functor on  $A$ , again given by the respective Poincaré bundle. We have  $\mathbf{R}\Phi_P = \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}a_*$  and  $\mathbf{R}\Phi_{P^\vee} = (-1)^* \circ \mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}a_*$ . Combined with (5), this gives

$$(6) \quad \mathbf{R}\Phi_{P^\vee}\omega_X \cong \bigoplus_{j=0}^k (-1)^* \mathbf{R}\widehat{\mathcal{S}}(R^j a_*\omega_X[-j]).$$

Now by the commutation of Grothendieck duality with integral functors (see for instance [PP1] Lemma 2.2<sup>1</sup>) we have

$$(7) \quad \mathbf{R}\Phi_P \mathcal{O}_X \cong (\mathbf{R}\Phi_{P^\vee}\omega_X)^\vee[-d],$$

where in general we denote  $\mathcal{E}^\vee := \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$ . We obtain

$$\mathbf{R}\Phi_P \mathcal{O}_X \cong \bigoplus_{j=0}^k (-1)^* (\mathbf{R}\widehat{\mathcal{S}}(R^j a_*\omega_X))^\vee[j-d] \cong \bigoplus_{j=0}^k \mathbf{R}\widehat{\mathcal{S}}((R^j a_*\omega_X)^\vee)[q+j-d],$$

where the second identity follows from the analogue of (7) (which in this case is Mukai's [Mu2] (3.8)). We now use the main result in Hacon [Hac], which says that for each  $j$  the object

$$\mathcal{G}_j := \mathbf{R}\widehat{\mathcal{S}}((R^j a_*\omega_X)^\vee)[q]$$

is in fact a sheaf.<sup>2</sup> This gives a decomposition

$$\mathbf{R}\Phi_P \mathcal{O}_X \cong \bigoplus_{j=0}^k \mathcal{G}_j[j-d]$$

and since  $\mathbf{R}\Phi_P \mathcal{O}_X = \mathbf{R}p_{2*}P$ , it suffices to show that  $\mathcal{G}_k \neq 0$  in a neighborhood of the origin. Since

$$R^i \widehat{\mathcal{S}}((R^j a_*\omega_X)^\vee[q]) = 0 \text{ for all } i > 0,$$

we can apply the base change theorem for bounded complexes (see [EGAIII] §7.7) with respect to  $p_2$ , so that the statement is equivalent to showing

$$\mathbf{H}^0(A, (R^k a_*\omega_X)^\vee[q]) \cong H^0(A, R^k a_*\omega_X)^\vee \neq 0,$$

where the isomorphism is a consequence of Grothendieck-Serre duality. Now by passing to a resolution of singularities of the Stein factorization  $X \rightarrow Y \rightarrow A$  of the image  $Z = a(X)$ , we can assume that  $Z$  is smooth and

<sup>1</sup>This is proved in [PP1] in the context of smooth projective varieties, but as indicated in [PP2] the same proof works on complex manifolds, due to the fact that the analogue of Grothendieck duality holds by [RRV].

<sup>2</sup>By [PP2] Theorem 2.2, this is equivalent to the more familiar notion that  $R^j a_*\omega_X$  satisfies generic vanishing.

$a : X \rightarrow Z$  is a fiber space,<sup>3</sup> in which case by [Ko1] Proposition 7.6 we have that  $R^k a_* \omega_X \cong \omega_Z$ . But the canonical bundle of any variety of maximal Albanese dimension has non-zero sections.  $\square$

**Remark 1.10.** A somewhat simpler proof of Proposition 1.8 is possible, which deduces the statement directly from Kollár's theorem by passing through the BGG correspondence. See Remark 1.27 in the next section.

**Remark 1.11** (Converse to the Generic Vanishing theorem [GL1]). According to [PP2] Theorem 2.2, the fact that  $(R^{d-k} p_{2*} \mathcal{P})_0 \neq 0$  in Proposition 1.8 is equivalent to the fact that there exists an  $i > 0$  such that around the origin the cohomological support locus  $V^i(\omega_X)$  satisfies

$$\text{codim}_0 V^i(\omega_X) = i - k,$$

while as mentioned before we know that  $\text{codim} V^i(\omega_X) \geq i - k$  for all  $i > 0$ . Therefore this condition becomes equivalent to  $\dim a(X) \leq k$ , which is a (strong) converse to Theorem 1.2: indeed, in order to determine the dimension of the Albanese image of  $X$ , we only need to know the dimension of the cohomological support loci around the origin.

**Proposition 1.12.** (i). *If  $X$  has maximal Albanese dimension, then  $\underline{\mathbf{L}}_X$  is a resolution of  $\mathcal{F}$ .*

(ii). *Suppose that  $0 \in \text{Pic}^0(X)$  is an isolated point of the cohomological support loci  $V^i(\omega_X)$  for every  $i > 0$ . Then  $\mathcal{F}$  is a vector bundle on  $\mathbf{P}$ , with  $\text{rk}(\mathcal{F}) = \chi(\omega_X)$ .*

(iii). *The hypothesis of (ii) holds in particular if  $X$  does not carry any irregular fibrations.*

*Proof.* The first statement is the case  $k = 0$  of Proposition 1.7, and (iii) follows from [GL2], Theorem 0.1. In general,  $V^i(\mathcal{O}_X) = -V^{d-i}(\omega_X)$  contains the support of the direct image  $R^i p_{2*} P$ . Hence if the  $V^i(\omega_X)$  are finite for  $i > 0$ , then the corresponding direct images are supported at only finitely many points, and this implies that the vector bundle maps appearing in  $\underline{\mathbf{L}}_X$  are everywhere of constant rank. (Compare [EL] or [HP], Proposition 2.11.) Thus  $\mathcal{F}$  is locally free, and its rank is computed from  $\underline{\mathbf{L}}_X$ .  $\square$

**Remark 1.13.** Note that the main result of [GL2] asserts more generally that if  $X$  doesn't admit any irregular fibrations, then in fact  $V^i(\omega_X)$  is finite for every  $i > 0$ .

**1.3. A brief introduction to the BGG correspondence.** Let  $V$  be a  $q$ -dimensional vector space over a field  $k$ , and let  $E = \bigoplus_{i=0}^q \Lambda^i V$  be the exterior algebra over  $V$ . Let  $W = V^\vee$  be the dual vector space, and  $S = \text{Sym}(W)$  the symmetric algebra over  $W$ . Consider dual bases  $e_i$  and  $x_i$  for  $V$  and  $W$  respectively, with  $\deg e_i = -1$  and  $\deg x_i = 1$ , so that  $S \cong k[x_1, \dots, x_q]$ , with the natural grading.

The  $k$ -algebras  $E$  and  $S$  are *dual Koszul algebras*, in the sense that the field  $k$  has linear free resolutions over  $E$  and  $S$ , and

$$\text{Ext}_S^\bullet(k, k) \cong E \text{ and } \text{Ext}_E^\bullet(k, k) \cong S.$$

More explicitly, we can write down the minimal free resolutions of  $k$  over the two algebras. For  $S$ , this is the well-known Koszul complex

$$0 \longrightarrow S \otimes_k \bigwedge^q W \longrightarrow \dots \longrightarrow S \otimes_k \bigwedge^2 W \longrightarrow S \otimes_k W \longrightarrow S \longrightarrow k \longrightarrow 0.$$

The differentials are the compositions

$$S \otimes \bigwedge^p W \longrightarrow S \otimes \bigwedge^{p-1} W \otimes W \longrightarrow S \otimes \bigwedge^{p-1} W$$

induced by  $\bigwedge^p W \rightarrow \bigwedge^{p-1} W \otimes W$ , dual to the wedge map  $V \otimes \bigwedge^{p-1} V \rightarrow \bigwedge^p V$ .

<sup>3</sup>What is used here is that for any composition of proper morphisms  $U \xrightarrow{f} V \xrightarrow{g} W$ , with  $U$  and  $V$  smooth and  $g$  generically finite, we have  $R^i g_* R^j f_* \omega_U = 0$  for all  $i > 0$  and all  $j$  by [Ko2] Theorem 3.4 and its extensions.

**Example 1.14** (Cartan resolution). The field  $k$  has an (infinite) linear minimal free resolution over  $E$ , called the *Cartan resolution*, given by

$$\dots \longrightarrow E \otimes_k S_2^* \longrightarrow E \otimes_k S_1^* \longrightarrow E \otimes_k S_0^* \longrightarrow k \longrightarrow 0.$$

Here we have  $S = \bigoplus S_i$ , and the differentials are induced by the natural operations  $V \otimes E \rightarrow E$  and  $S^p V \rightarrow S^{p-1} V \otimes V$  (dual to multiplication in  $S$ ). In coordinates we can write

$$d_i : E \otimes_k S_i^* \longrightarrow E \otimes_k S_{i-1}^*$$

as being induced by linearity by

$$e \otimes f \mapsto \sum_j e e_j \otimes x_j f.$$

To any graded module  $P = \bigoplus_i P_i$  over  $E$  (so that there exist natural operations  $\bigwedge^i V \otimes P_j \rightarrow P_{j-i}$ ) one can apply the Bernstein-Gel'fand-Gel'fand (BGG) correspondence, to obtain a complex  $\mathbf{L}(P)$  of free modules over the symmetric algebra  $S$  given by

$$\dots \longrightarrow S \otimes_{\mathbb{C}} P_j \longrightarrow S \otimes_{\mathbb{C}} P_{j-1} \longrightarrow \dots$$

with differential induced by  $P_i \rightarrow P_{i-1} \otimes W$ , obtained from module map  $V \otimes P_i \rightarrow P_{i-1}$ . In coordinates, the differential is the extension by linearity of

$$s \otimes p \mapsto \sum_i x_i s \otimes e_i p.$$

One can easily check by direct calculation that this is indeed a differential, and that it is given by a matrix with linear entries in the  $x_i$ , seen as a morphism between free  $S$ -modules. In other words,  $\mathbf{L}(P)$  is a *linear complex* of free  $S$ -modules. The complex  $\mathbf{L}(P)$  can also be sheafified to a linear complex  $\underline{\mathbf{L}}(P)$  of vector bundles on  $\mathbf{P} = \mathbf{P}(V)$ :

$$\dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-j) \otimes P_j \longrightarrow \mathcal{O}_{\mathbf{P}}(-j+1) \otimes P_{j-1} \longrightarrow \dots$$

By passing to the total complex of a double complex,  $\mathbf{L}$  extends to a functor between the categories of (bounded) complexes

$$\mathbf{L} : \text{Kom}^b(E - \text{mod}) \longrightarrow \text{Kom}^b(S - \text{mod}).$$

One can check that this further induces a functor at the level of derived categories, and the main theorem of [BGG] is that one obtains an equivalence of derived categories

$$\mathbf{L} : \mathbf{D}^b(E - \text{mod}) \longrightarrow \mathbf{D}^b(S - \text{mod}).$$

For a modern treatment of this, and further results, cf. [EFS] or [Eis]. Below I will only apply this construction for a single module.

**Definition 1.15** (Dual module). The *dual over  $E$*  of the module  $P$  is defined to be the  $E$ -module  $\widehat{P} := \bigoplus_j P_{-j}^*$  (so positive degrees are switched to negative ones and viceversa).

It turns out that the exactness of  $\mathbf{L}(P)$  has to do with regularity properties of  $\widehat{P}$  over  $E$ . Here regularity is considered in analogy with the notion of Castelnuovo-Mumford regularity for finitely generated graded modules over the symmetric algebra, so it is a measure of complexity. I will restrict to the case of modules concentrated in non-positive degrees.

**Definition 1.16** (Regularity). A finitely generated graded  $E$ -module  $Q$  with no component of positive degree is called  *$m$ -regular* if it is generated in degrees 0 up to  $-m$ , and if its minimal free resolution has at most  $m+1$  linear strands. Specifically, the generators of  $Q$  appear in degrees 0,  $-1, \dots, -m$ , the relations among these generators are in degrees  $-1, \dots, -(m+1)$ , and more generally the  $p^{\text{th}}$  module of syzygies of  $M$  has all its generators in degrees  $\geq -(p+m)$ .

**Lemma 1.17.** *A finitely generated graded  $E$ -module  $Q$  with no component of positive degree is  $m$ -regular if and only if  $\mathrm{Tor}_i^E(Q, k)_{-i-j} = 0$  for all  $i \geq 0$  and all  $j \geq m + 1$ .*

*Proof.* By definition, each graded piece  $\mathrm{Tor}_i^E(Q, k)_{-i-j}$  corresponds precisely to the component of degree  $-i - j$  in the  $i$ -th term in the minimal  $E$ -free resolution of  $Q$ . Hence the last vanishing is equivalent to the fact that  $P$  is generated in degrees between 0 and  $-m$ , and its minimal free resolution has at most  $m + 1$  linear strands.  $\square$

An immediate application of the results of Eisenbud-Fløystad-Schreyer is the following addition to [EFS] Corollary 2.5 (cf. also [Eis] Theorem 7.7).

**Proposition 1.18.** *Consider a finitely generated graded module over  $E$  with no component of negative degree, say  $P = \bigoplus_{i=0}^d P_i$ . Then  $\mathbf{L}(P)$  is exact at the first  $d - m$  steps from the left if and only if  $\widehat{P}$  is  $m$ -regular over  $E$ .*

*Proof.* By [Eis] Theorem 7.8, we have that for all  $i \geq 0$  and all  $j$ :

$$\mathrm{Tor}_i^E(\widehat{P}, k)_{-i-j} \cong H_j(\mathbf{L}(P))_{i+j}^*,$$

which is precisely the cohomology of the complexes induced on the graded pieces<sup>4</sup>

$$P_{i-1} \otimes S^{j-1}W \longrightarrow P_i \otimes S^jW \longrightarrow P_{i+1} \otimes S^{j+1}W$$

Now the exactness of the first  $d - m$  steps in  $\mathbf{L}(P)$  is equivalent to  $H_j(\mathbf{L}(P)) = 0$  for all  $j \geq m + 1$ , which by the above is equivalent to  $\mathrm{Tor}_i^E(\widehat{P}, k)_{-i-j} = 0$  for all  $i \geq 0$  and all  $j \geq m + 1$ . This is equivalent to  $m$ -regularity by Lemma 1.17.  $\square$

**1.4. BGG and the canonical cohomology module.** I will apply the construction in the previous subsection to modules over the exterior algebra naturally arising in a geometric context. Let  $X$  be a compact Kähler manifold of dimension  $d$  and irregularity  $q = h^1(X, \mathcal{O}_X)$ , and  $a : X \rightarrow A = \mathrm{Alb}(X)$  its Albanese map. Set

$$V = H^1(X, \mathcal{O}_X) \text{ and } E = \bigoplus_{i=0}^q \Lambda^i V.$$

We are interested in the graded  $E$ -modules

$$P_X = \bigoplus_{i=0}^d H^i(X, \mathcal{O}_X), \quad Q_X = \bigoplus_{i=0}^d H^i(X, \omega_X),$$

the  $E$ -module structure arising from wedge product with elements of  $H^1(X, \mathcal{O}_X)$ . These become dual modules (i.e.  $Q_X \cong \widehat{P}_X$  with the notation above), thanks to Serre duality, provided that we assign  $H^i(X, \mathcal{O}_X)$  degree  $d - i$ , and  $H^i(X, \omega_X)$  degree  $-i$ . I will be mainly concerned with describing the structure of  $Q_X$  as an  $E$ -module.

The BGG functor applied to  $P_X$  gives a complex  $\mathbf{L}(P_X)$  of graded  $S$ -modules in homological degrees 0 to  $d$ , which here takes the form:

$$(8) \quad 0 \longrightarrow S \otimes_{\mathbb{C}} H^0(X, \mathcal{O}_X) \longrightarrow S \otimes_{\mathbb{C}} H^1(X, \mathcal{O}_X) \longrightarrow \dots \longrightarrow S \otimes_{\mathbb{C}} H^d(X, \mathcal{O}_X) \longrightarrow 0.$$

The following is the main observation relating derivative and BGG complexes.

**Lemma 1.19.** *The complex  $\mathbf{L}(P_X)$  coincides with the complex  $\mathbf{L}_X$  appearing in (3).*

<sup>4</sup>In analogy with the Koszul cohomology of modules over  $S$ , computed with the help of the Koszul resolution of  $k$ , a natural name for this is *Cartan cohomology*.

*Proof.* Using the notation at the end of §1 and in the proof of Proposition 1.7, we write down explicitly the differential of  $\mathcal{K}^\bullet$ . Consider  $\underline{t} = (t_1, \dots, t_q)$  coordinates on  $V$ . Consider also  $\phi_1, \dots, \phi_q$  a basis for the harmonic  $(0, 1)$ -forms on  $X$ , so a basis for  $V$  via the natural identification, and  $f_1, \dots, f_q$  the dual basis of linear forms on  $V$ . The differential of  $\mathcal{K}^\bullet$  is given by wedging with the universal  $(0, 1)$ -form  $\sum_{i=1}^q f_i(\underline{t})\phi_i$ , and a simple calculation shows that on the stalk  $\mathcal{O}_{V, \underline{t}} \otimes H^i(X, \mathcal{O}_X)$  it acts as

$$g \otimes \alpha \rightarrow \sum_{i=1}^q f_i(\underline{t})g \otimes (\alpha \wedge \phi_i).$$

The complex  $\mathbf{L}_X = \Gamma(\mathbf{A}, \mathcal{K}^\bullet)$  then has differential acting as

$$s \otimes \alpha \rightarrow \sum_{i=1}^q f_i s \otimes (\alpha \wedge \phi_i),$$

which shows that it can be identified with the BGG complex  $\mathbf{L}(P_X)$ .  $\square$

Writing  $\mathbf{P} = \mathbf{P}(V)$  for the projective space of one-dimensional subspaces of  $V$ , this complex sheafifies to yield a linear complex  $\underline{\mathbf{L}}(P_X)$  of vector bundles on  $\mathbf{P}$ :

$$(9) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow 0.$$

As above, this can be identified with the complex  $\underline{\mathbf{L}}_X$  in (2). Recall that we shall also be interested in the coherent sheaf  $\mathcal{F}$  on  $\mathbf{P}$  arising as the cokernel of the right-most map in the complex  $\underline{\mathbf{L}}(P_X)$ :

$$\mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \rightarrow \mathcal{F} \rightarrow 0.$$

**Example 1.20.** (1) If  $X$  is itself a torus of dimension  $g$ , then  $P_X \cong E(-g) \cong \widehat{E}$ . Then  $\mathbf{L}(P_X) = \mathbf{L}(\widehat{E})$  is the (dual) Koszul complex over  $S$

$$0 \longrightarrow S \longrightarrow S \otimes V \longrightarrow S \otimes \bigwedge^2 V \longrightarrow \dots \longrightarrow S \otimes \bigwedge^g V \longrightarrow 0$$

while  $Q_X \cong E$ . Note that here  $\mathcal{F} = 0$ .

(2) Let  $C$  be a smooth projective curve of genus  $g \geq 1$ . Then  $\underline{\mathbf{L}}(P_C)$  consists of the first two terms in the dual of the Euler sequence on  $\mathbf{P}^{g-1}$ :

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \mathcal{F} = T_{\mathbf{P}}(-1) \longrightarrow 0.$$

(3) More generally, consider  $X = C_m$  to be the  $m$ -th symmetric product of a curve  $C$  as in (2). Then  $\underline{\mathbf{L}}(P_X)$  consists of the truncation of a twisted Koszul complex at the  $m$ -th step from the left:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{O}_{\mathbf{P}}(-m+1) \otimes V \longrightarrow \mathcal{O}_{\mathbf{P}}(-m+2) \otimes \Lambda^2 V \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes \Lambda^m V \longrightarrow \mathcal{F} = (\Lambda^m T_{\mathbf{P}})(-m) \longrightarrow 0$$

By passing to the graded pieces in  $\mathbf{L}(P_X)$  one obtains:

**Lemma 1.21.** *With the notation above, the following are equivalent, and imply that  $\underline{\mathbf{L}}(P_X)$  is exact.*

(1)  $\mathbf{L}(P_X)$  is exact.

(2) The (Cartan) complexes

$$H^{i-1}(X, \mathcal{O}_X) \otimes S^{j-1}W \longrightarrow H^i(X, \mathcal{O}_X) \otimes S^j W \longrightarrow H^{i+1}(X, \mathcal{O}_X) \otimes S^{j+1}W$$

are exact for all  $i \geq 0$  and all  $j \geq 1$ .

**Example 1.22.** Let  $C$  be a smooth projective curve of genus  $g \geq 1$ . We have seen that  $\mathbf{L}(P_C)$  is the dual Euler sequence, so it is exact. Hence  $Q_C = H^0(C, \omega_C) \oplus H^1(C, \omega_C)$  is 0-regular as a module over  $E = \bigoplus_{i=0}^g \Lambda^i V$ , where  $V = H^1(C, \mathcal{O}_C)$  by Proposition 1.18. Its minimal  $E$ -resolution is an example of resolving a power of the maximal ideal  $\mathfrak{m} \subset E$ . We have  $Q_C \cong \mathbb{C} \oplus W \cong \Lambda^g V \oplus \Lambda^{g-1} V$ , in degrees  $-1$  and  $0$  respectively, which in turn can be identified with  $\mathfrak{m}^{g-1}(1)$  as a module over  $E$ . Resolutions of powers of the maximal ideal are computed in [EFS] §5. In this case we have an infinite resolution

$$\dots \longrightarrow E(2) \otimes U_2 \longrightarrow E(1) \otimes U_1 \longrightarrow E \otimes U_0 \longrightarrow \mathfrak{m}^{g-1}(1) \longrightarrow 0$$

where the vector spaces  $U_i$  are given by

$$U_i := \text{Im}(\Lambda^2 W \otimes S^{i+1} V \longrightarrow W \otimes S^i V).$$

where via the Koszul complex one can identify  $U_i \cong H^0(\mathbf{P}(V), T_{\mathbf{P}(V)}(i-1))^*$ . (Hint: use the formula  $\text{Tor}_i^E(Q_C, \mathbb{C})_{-i} \cong H_0(\mathbf{L}(P_C))_i^*$ .)

There has been a considerable amount of recent work in the commutative algebra community aimed at extending to modules over an exterior algebra aspects of the classical theory of graded modules over a polynomial ring. Here is a first example of geometric interest:

**Example 1.23** (Hyperplane arrangements). Eisenbud-Popescu-Yuzvinsky [EPY] have proved that if  $X$  is the complement of a hyperplane arrangement in  $\mathbb{C}^n$ , then the singular homology module  $Q = H_*(X, \mathbb{C})$  has a linear free resolution (though it is not generated in degree 0) over  $E$ , where  $E$  is the exterior algebra on the vector space  $V = H^1(X, \mathbb{C})$  and the module structure is given by cap product.

In our present context, it is natural to ask whether one can say anything in general about the algebraic properties of the modules  $P_X$  and  $Q_X$  canonically associated to a Kähler manifold  $X$ : for instance, in what degrees do generators and relations live?

**Example 1.24.** Here is an elementary example that suggests what one might expect. Consider an abelian variety  $A$  of dimension  $d + 1$ , and let  $X \subseteq A$  be a smooth hypersurface of very large degree. Then by the Lefschetz theorem one has

$$\begin{aligned} H^i(X, \mathcal{O}_X) &= H^i(A, \mathcal{O}_A) = \Lambda^i H^1(X, \mathcal{O}_X) \text{ for } i < d \\ H^d(X, \mathcal{O}_X) &\not\cong H^d(A, \mathcal{O}_A) = \Lambda^d H^1(X, \mathcal{O}_X). \end{aligned}$$

Thus  $P_X$  has generators as an  $E$ -module in two degrees:  $1 \in H^0(X, \mathcal{O}_X)$ , and many new generators in  $H^d(X, \mathcal{O}_X)$ . If one takes the viewpoint that the simplest  $E$ -modules are those whose generators appear in a single degree, this means that  $P_X$  is rather complicated: (an appropriate shift of)  $P_X$  has worst-possible regularity  $= d$ . On the other hand, the situation with the dual module  $Q_X$  is quite different. Here  $H^0(X, \omega_X)$  is big, and the maps

$$H^0(X, \omega_X) \otimes \Lambda^i H^1(X, \mathcal{O}_X) \longrightarrow H^i(X, \omega_X)$$

are surjective, i.e.  $Q_X$  is generated in degree 0. This suggests that  $Q_X$  behaves more predictably as an  $E$ -module than does  $P_X$ .

The following is the main result in this direction, asserting that the regularity of  $Q_X$  is computed by the Albanese fiber-dimension of  $X$ . According to Proposition 1.18 and Lemma 1.19, it is equivalent to Theorem 1.6 (i).

**Theorem 1.25.** *As above, let  $X$  be a compact Kähler manifold, and let  $k = \dim X - \dim \text{alb}_X(X)$ . Then*

$$\text{reg}(Q_X) = k,$$

*i.e.  $Q_X$  is  $k$ -regular, but not  $(k - 1)$ -regular as an  $E$ -module. In particular,  $X$  has maximal Albanese dimension (i.e.  $k = 0$ ) if and only if  $Q_X$  is generated in degree 0 and has a linear free resolution.*

**Remark 1.26** (Exterior Betti numbers). The exterior graded Betti numbers of  $Q_X$  are computed as the dimensions of the vector spaces  $\mathrm{Tor}_i^E(Q, \mathbb{C})_{-i-j}$ . When  $X$  is of maximal Albanese dimension and  $q(X) > \dim X$ , Theorem 1.25 implies that these vanish for  $i \geq 0$  and  $j \geq 1$ , and the  $i$ -th Betti number in the linear resolution of  $Q_X$  is

$$b_i = \dim_{\mathbb{C}} \mathrm{Tor}_i^E(Q, \mathbb{C})_{-i} = h^0(\mathbf{P}, \mathcal{F}(i))$$

where  $\mathcal{F}$  is the BGG sheaf defined in (4). (The last equality follows from general machinery, cf. [Eis] Theorem 7.8.) On the other hand  $\mathcal{F}$  is 0-regular in the sense of Castelnuovo-Mumford by virtue of having a linear resolution, so the higher cohomology of its nonnegative twists vanishes. Hence  $b_i = \chi(\mathbf{P}, \mathcal{F}(i))$ , i.e. the exterior Betti numbers are computed by the Hilbert polynomial of  $\mathcal{F}$ .

**Remark 1.27** (Streamlined proof of Proposition 1.8). One can use the BGG correspondence to deduce the non-exactness statement of Proposition 1.8 directly from (5), avoiding the arguments with the Fourier-Mukai transform. In fact, in view of Proposition 1.18, it is equivalent to prove that  $Q_X$  is not  $(k-1)$ -regular. To this end, observe that the splitting  $\mathbf{R}a_*\omega_X \cong \bigoplus_{j=0}^k R^j a_*\omega_X[-j]$  of [Ko2], [Sa] and [Ta] implies that  $Q_X$  can be expressed as a direct sum

$$Q_X = \bigoplus_{j=0}^k Q^j[j], \quad \text{with } Q^j = H^*(A, R^j a_*\omega_X).$$

Moreover this is a decomposition of  $E$ -modules:  $E$  acts on  $H^*(A, R^j a_*\omega_X)$  via cup product through the identification  $H^1(X, \mathcal{O}_X) = H^1(A, \mathcal{O}_A)$ , and we again consider  $H^i(A, R^j a_*\omega_X)$  to live in degree  $-i$ . We claim next that  $Q^k \neq 0$ . In fact, each of the  $R^j a_*\omega_X$  is supported on the  $(d-k)$ -dimensional Albanese image of  $X$ , and hence has vanishing cohomology in degrees  $> d-k$ . Therefore  $H^d(X, \omega_X) = H^{d-k}(A, R^k a_*\omega_X)$ , which shows that  $Q^k \neq 0$ . On the other hand,  $Q^k[k]$  is concentrated in degrees  $\leq -k$ , and therefore  $Q_X$  must have generators in degrees  $\leq -k$ .

**Remark 1.28.** Keeping the notation of the previous Remark, together with R. Lazarsfeld and C. Schnell we have shown (details forthcoming) that each of the modules  $Q^j$  just introduced is 0-regular. Thus the minimal  $E$ -resolution of  $Q_X$  splits into the direct sum of the (shifted)-linear resolutions of the modules  $Q^j[j]$ .

**1.5. Inequalities for numerical invariants.** The second line of application for Theorem 1.6 is as a mechanism for generating inequalities on numerical invariants of  $X$ . The search for relations among the Hodge numbers of an irregular variety has a long history, going back at least as far as the classical inequality of Castelnuovo and de Franchis giving a lower bound on the holomorphic Euler characteristic of surfaces  $S$  without irrational pencils, namely

$$p_g(S) = h^{0,2}(S) \geq 2q - 3, \quad \text{or equivalently } \chi(\omega_S) \geq q - 2$$

(see [BPV] Proposition 4.2 and [Be1] C.8). This was extended by several authors; most significantly Catanese [Ca] showed that for a compact Kähler manifold with no irregular fibrations, the natural maps

$$(10) \quad \phi_k : \bigwedge^k H^i(X, \mathcal{O}_X) \longrightarrow H^k(X, \mathcal{O}_X)$$

are injective on primitive forms  $\omega_1 \wedge \dots \wedge \omega_k$ , for  $k \leq \dim X$ . Since these correspond to the Plücker embedding of the Grassmannian  $\mathbf{G}(k, V)$ , one obtains the bounds  $h^{0,k}(X) \geq k(q(X) - k) + 1$ . For  $h^{0,2}$ , still based on Catanese's results, one can show the even stronger inequality:

$$(11) \quad h^{0,2}(X) \geq 4q(X) - 10$$

provided that  $\dim X \geq 3$  (this is explained in an upcoming revision of [LP]). These results admit vast improvements or extensions by making use of the BGG sheaf and BGG complexes in the previous sections.

Assuming that  $X$  does not have any irregular fibrations, statement (ii) of Theorem 1.6 implies that the BGG-sheaf  $\mathcal{F}$  is a vector bundle on the projective space  $\mathbf{P}$ , whose invariants are determined by  $\underline{\mathbf{L}}_X$ . Geometric facts about

vector bundles on projective spaces then give rise to inequalities for  $X$ . Specifically, consider the formal power series:

$$(12) \quad \gamma(X; t) =_{\text{def}} \prod_{j=1}^d (1 - jt)^{(-1)^j h^{d,j}} \in \mathbb{Z}[[t]],$$

where  $h^{i,j} = h^{i,j}(X)$ . Write  $q = h^1(X, \mathcal{O}_X)$  for the irregularity of  $X$  (so that  $q = h^{d,d-1}$ ), and for  $1 \leq i \leq q-1$  denote by

$$\gamma_i = \gamma_i(X) \in \mathbb{Z}$$

the coefficient of  $t^i$  in  $\gamma(X; t)$ . Thus  $\gamma_i$  is a polynomial in the  $h^{d,j}$ .

**Theorem 1.29.** *Assume that  $X$  does not carry any irregular fibrations (so that in particular  $X$  itself has maximal Albanese dimension). Then*

(i). *Any Schur polynomial of weight  $\leq q-1$  in the  $\gamma_i$  is non-negative. In particular*

$$\gamma_i(X) \geq 0$$

*for every  $1 \leq i \leq q-1$ .*

(ii). *If  $i$  is any index with  $\chi(\omega_X) < i < q$ , then  $\gamma_i(X) = 0$ .*

(iii). *One has  $\chi(\omega_X) \geq q-d$ .*

*Proof.* Thanks to Proposition 1.12, the hypothesis guarantees that  $\mathcal{F}$  is locally free, and has a linear resolution:

$$(13) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \longrightarrow \dots \\ \dots \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbf{P}} \otimes H^d(X, \mathcal{O}_X) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Identifying as usual cohomology classes on  $\mathbf{P}^n$  with integers,  $\gamma(X; t)$  is then just the Chern polynomial of  $\mathcal{F}$ . On the other hand, as  $\mathcal{F}$  is globally generated, the Chern classes  $c_i(\mathcal{F})$  – as well as the Schur polynomials in these – and represented by effective cycles. Thus

$$\gamma_i(X) = \deg c_i(\mathcal{F}) \geq 0.$$

The second statement follows from the fact that  $c_i(\mathcal{F}) = 0$  for  $i > \text{rank}(\mathcal{F})$ .

Turning to (iii), we may assume that  $q > d$  since in any event  $\chi \geq 0$  by generic vanishing. If  $q-d=1$ , then the issue is to show that  $\chi = \text{rank}(\mathcal{F}) \geq 1$ , or equivalently that  $\mathcal{F} \neq 0$ . But this is clear, since there are no non-trivial exact complexes of length  $n$  on  $\mathbf{P}^n$  whose terms are sums of line bundles of the same degree. So we may suppose finally that  $q-1 = n > d$ . The quickest argument is note that chasing through (13) implies that  $\mathcal{F}$  and its twists have vanishing cohomology in degrees  $0 < j < n-d-1$ . But if  $\chi \leq n-d$  this means by a result of Evans–Griffith that  $\mathcal{F}$  is a direct sum of line bundles, which as before is impossible: see [La], Example 7.3.10, for a quick proof of this splitting criterion due to Ein based on Castelnuovo–Mumford regularity and vanishing theorems for vector bundles.

For a more direct argument in the case at hand that avoids Evans–Griffith, let  $s \in H^0(\mathbf{P}, \mathcal{F})$  be a general section, and let  $Z = \text{Zeroes}(s)$ . We may suppose that  $Z$  is non-empty – or else we could construct a vector bundle  $\mathcal{F}'$  of smaller rank having a linear resolution as in (13) – and smooth of dimension  $n - \chi$ . Splicing together the sequence (13) and the Koszul complex determined by  $s$ , we arrive at a long exact sequence having the shape:

$$(*) \quad 0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-d) \longrightarrow \oplus \mathcal{O}_{\mathbf{P}}(-d+1) \longrightarrow \dots \longrightarrow \oplus \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \\ \longrightarrow \oplus \mathcal{O}_{\mathbf{P}} \longrightarrow \Lambda^2 \mathcal{F} \longrightarrow \dots \longrightarrow \Lambda^{\chi-1} \mathcal{F} \longrightarrow \mathcal{O}_{\mathbf{P}}(c_1) \longrightarrow \mathcal{O}_Z(c_1) \longrightarrow 0,$$

where  $c_1 = c_1(\mathcal{F})$ . Observe that  $\omega_Z = \mathcal{O}_Z(c_1 - n - 1)$  by adjunction. Since  $\mathcal{F}$  is globally generated, a variant of the Le Potier vanishing theorem<sup>5</sup> yields that

$$H^i(\mathbf{P}, \Lambda^a \mathcal{F} \otimes \omega_{\mathbf{P}}(\ell)) = 0 \text{ for } i > \chi - a, \ell > 0.$$

Now twist through in (\*) by  $\mathcal{O}_{\mathbf{P}}(d - n - 1)$ . Chasing through the resulting long exact sequence, one finds that  $H^{n-d-(\chi-1)}(Z, \omega_Z(d)) \neq 0$ . But if  $\chi \leq n - d$ , this contradicts Kodaira vanishing on  $Z$ .  $\square$

**Remark 1.30** (Evans–Griffith Theorem). Note that part (iii) extends to arbitrary dimension the Castelnuovo–de Franchis inequality for surfaces mentioned above. A somewhat more general form was first proved in [PP2] and can be deduced with the present methods as well: one can assume only that there are no irregular fibrations  $f : X \rightarrow Y$  such that  $Y$  is generically finite onto a *proper* subvariety of a complex torus (i.e.  $X$  has no higher irrational pencils in Catanese’s terminology [Ca].) The argument in [PP2] involved applying the Evans–Griffith syzygy theorem to the Fourier–Mukai transform of the Poincaré bundle on  $X \times \text{Pic}^0(X)$ . The possibility mentioned in the previous proof of applying the Evans–Griffith–Ein splitting criterion to the BGG bundle  $\mathcal{F}$  is related to both that and the proof above, but quicker (this is what I will present during the lectures).

I finish by giving some examples and variants of the inequalities appearing in the first assertions of Theorem 1.29. More on this and on the borderline cases in the higher dimensional Castelnuovo–de Franchis inequality in (iii) can be found in [LP] §3.

**Example 1.31** (Theorem 1.29 in small dimensions). We unwind a few of the inequalities in statement (i) of Theorem 1.29. We assume that  $X$  has dimension  $d$ , and that it carries no irregular fibrations. For compactness, write  $h^{0,j} = h^{0,j}(X)$  and  $q = q(X)$ . To begin with, the condition  $\gamma_1 \geq 0$  gives a linear inequality among  $q, h^{0,2}, \dots, h^{0,d-1}$ . In small dimensions this becomes:

$$(14) \quad \begin{aligned} h^{0,2} &\geq -3 + 2q && \text{when } d = 3; \\ h^{0,3} &\geq 4 - 3q + 2h^{0,2} && \text{when } d = 4; \\ h^{0,4} &\geq -5 + 4q - 3h^{0,2} + 2h^{0,3} && \text{when } d = 5. \end{aligned}$$

Similarly,  $\gamma_2$  is a quadratic polynomial in the same invariants, and one may solve  $\gamma_2 \geq 0$  to deduce the further and stronger inequalities:

$$(15) \quad \begin{aligned} h^{0,2} &\geq -\frac{7}{2} + 2q + \frac{\sqrt{8q-23}}{2} && \text{when } d = 3; \\ h^{0,3} &\geq \frac{7}{2} - 3q + 2h^{0,2} + \frac{\sqrt{49-24q+8h^{0,2}}}{2} && \text{when } d = 4; \\ h^{0,4} &\geq -\frac{11}{2} + 4q - 3h^{0,2} + 2h^{0,3} + \frac{\sqrt{71+48q-24h^{0,2}+8h^{0,3}}}{2} && \text{when } d = 5, \end{aligned}$$

where in the last case we assume that the expression under the square root is non-negative. (This is automatic when  $d = 3$  since  $q \geq 3$ , and when  $d = 4$  thanks to (11).) Note that equality holds in (14) when  $X$  is an abelian variety (in which case  $\mathcal{F} = 0$ ). Similarly, when  $X$  is a theta divisor in an abelian variety of dimension  $d + 1$ , then  $\text{rank } \mathcal{F} = 1$ , so equality holds in each of the three instances of (15).

When  $d = 3$  or  $d = 4$ , we can combine the various inequalities in play to get an asymptotic statement:

<sup>5</sup> The statement we use is that if  $\mathcal{E}$  is a nef vector bundle of rank  $e$  on a smooth projective variety  $V$  of dimension  $n$ , then

$$H^i(V, \Lambda^a \mathcal{E} \otimes \omega_V \otimes L) = 0$$

for  $i > e - a$  and any ample line bundle  $L$ . In fact, it is equivalent to show that  $H^j(V, \Lambda^a \mathcal{E}^* \otimes L^*) = 0$  for  $j < n + a - e$ . For this, after passing to a suitable branched covering as in [La], proof of Theorem 4.2.1, we may assume that  $L = M^{\otimes a}$ , in which case the statement follows from Le Potier vanishing in its usual form: see [La], Theorem 7.3.6.

**Corollary 1.32.** (i). *If  $X$  is an irregular compact Kähler threefold with no irregular fibration, then*

$$h^{0,3}(X) \geq h^{0,2}(X) - 2,$$

*so asymptotically*

$$h^{0,2}(X) \succeq 4q(X) \quad \text{and} \quad h^{0,3}(X) \succeq 4q(X).$$

(ii). *If  $X$  is an irregular compact Kähler fourfold with no irregular fibration, then asymptotically*

$$h^{0,2}(X) \succeq 4q(X), \quad h^{0,3}(X) \succeq 5q(X) + \sqrt{2q(X)}, \quad \text{and} \quad h^{0,4}(X) \succeq 3q(X) + \sqrt{2q(X)}.$$

*Proof.* (i). The first inequality is equivalent to the statement  $\chi(\omega_X) \geq q - 3$  from statement (iii) of Theorem 1.29, and the other inequalities follow from this and (11).

(ii). The inequality  $\chi(\omega_X) \geq q - 4$  is equivalent to

$$h^{0,4}(X) \geq (2q - 5) + (h^{0,3}(X) - h^{0,2}(X)).$$

The statement then follows by using (15) to bound  $h^{0,3} - h^{0,2}$ , and invoking the inequality  $h^{0,2} \succeq 4q$  coming from (11).  $\square$

## 2. PART II: DERIVED EQUIVALENCE AND THE PICARD VARIETY

The goal of the second part of my lectures is to describe the results in the preprint [PS]. The main result is the fact that Fourier-Mukai partners have isogenous Albanese and Picard varieties. This gives rise in particular to some numerical applications.

**2.1. Basics on equivalences of derived categories and birational geometry.** Let  $X$  be a smooth projective variety over an algebraically closed field. I will denote

$$\mathbf{D}(X) := \mathbf{D}^b(\text{Coh}(X))$$

the bounded derived category of coherent sheaves on  $X$ . I will assume basics on derived categories, in particular those of coherent sheaves, for which an excellent introduction is [Hu]. Motivated by mirror symmetry and subsequently by birational geometry, we have a

*Basic question:* Given two smooth projective varieties  $X$  and  $Y$ , what are the consequences of an exact equivalence<sup>6</sup> of triangulated categories  $\mathbf{D}(X) \cong \mathbf{D}(Y)$  with respect to the geometry or numerical invariants of  $X$  and  $Y$ ?

One has a first collection of fundamental such consequences, due mainly to Bondal [BO], Orlov [Or] and Kawamata [Ka] (cf. [Hu] §6):

**Theorem 2.1.** *Assuming that  $X$  and  $Y$  are smooth projective varieties with  $\mathbf{D}(X) \cong \mathbf{D}(Y)$ , we have*

(i)  $\dim X = \dim Y$ ,  $\kappa(X) = \kappa(Y)$  (Kodaira dimension) and  $\nu(X) = \nu(Y)$  (numerical dimension).

(ii)  $\omega_X$  and  $\omega_Y$  have the same (possibly infinite) order.

(iii)  $\omega_X$  is nef if and only if  $\omega_Y$  is nef.

(iv) *There is an isomorphism of canonical rings  $R(X) \cong R(Y)$  as  $k$ -algebras. In particular all the plurigeners satisfy  $P_m(X) = P_m(Y)$ .<sup>7</sup> (Note that  $P_1(X) = h^{\dim X, 0}(X)$ .)*

<sup>6</sup>Recall that this means that it commutes with the shift functor and it maps distinguished triangles to distinguished triangles.

<sup>7</sup>Recall that the canonical ring is defined as  $R(X) := \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$ , while the  $m$ -th plurigenus is  $P_m(X) = h^0(X, \omega_X^{\otimes m})$ .

For varieties of general type, Kawamata has shown that derived equivalence implies a strong type of birationality, which in particular implies the invariance of all Hodge numbers of type  $h^{p,0}$  (and of the Albanese and Picard varieties).

**Theorem 2.2** ([Ka], Theorem 1.4(2)). *Let  $X$  and  $Y$  be smooth projective varieties of general type such that  $\mathbf{D}(X) \cong \mathbf{D}(Y)$ . Then  $X$  and  $Y$  are  $K$ -equivalent, i.e. there exist birational morphisms  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  from a smooth projective variety  $Z$  such that  $f^*\omega_X \cong g^*\omega_Y$ .*

On the other hand, in general one cannot deduce numerical consequences of derived equivalence simply based on birationality; there are many examples of derived equivalent but non-birational varieties. For instance:

- An abelian variety  $A$  and its dual  $\widehat{A}$  (Mukai [Mu2]).
- A  $K3$  surface  $S$  and a certain moduli space  $M_S$  of semistable sheaves on  $S$ , which is again a  $K3$  surface (Mukai [Mu3]).
- There exist Calabi-Yau (in the strong sense) threefolds which are derived equivalent, but not birational (see [BC], [Ku]).

The main purpose of this lecture is to explain the invariance of the number of holomorphic one-forms (the irregularity)  $q(X) = h^{1,0}(X) = h^0(X, \Omega_X^1)$ , and more generally the behavior of the Picard variety  $\text{Pic}^0(X)$ . Note first that the classical result of Mukai in the case of abelian varieties suggests that one cannot expect more than isogeny (or perhaps derived equivalence) between  $\text{Pic}^0(X)$  and  $\text{Pic}^0(Y)$ . Moreover, a first hint of what happens is suggested by the following more precise result of Orlov on abelian varieties:

**Theorem 2.3** ([Or], Theorem 5.3.13). *Let  $A$  and  $B$  be two abelian varieties. Then  $\mathbf{D}(A) \cong \mathbf{D}(B)$  if and only if there exists an isometric isomorphism  $\Psi : A \times \widehat{A} \cong B \times \widehat{B}$ , i.e. with  $\tilde{\Psi} = \Psi^{-1}$ , where if  $\Psi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ , then  $\tilde{\Psi} := \begin{pmatrix} \widehat{\delta} & -\widehat{\beta} \\ -\widehat{\gamma} & \widehat{\alpha} \end{pmatrix}$  (using the identification  $\widehat{\widehat{A}} \cong A$  and the same for  $B$ ).*

Given this, the most we can hope for in general is

**Question 2.4.** If  $\mathbf{D}(X) \cong \mathbf{D}(Y)$ , is it true that  $\mathbf{D}(\text{Pic}^0(X)) \cong \mathbf{D}(\text{Pic}^0(Y))$ ?

At the moment I do not have an answer to this question. Note however that a positive answer would imply by Theorem 2.3 that  $\text{Pic}^0(X)$  and  $\text{Pic}^0(Y)$  are isogenous; this is the main result of the notes, Theorem 2.16.

**2.2. Fourier-Mukai functors and special autoequivalences of derived categories.** A much used result about equivalences of derived categories is the following theorem of Orlov (see [Hu] Ch.V).

**Theorem 2.5** (Orlov). *If  $\Phi : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  is an equivalence, then there exists an object  $\mathcal{E} \in \mathbf{D}(X \times Y)$  (the "kernel"), unique up to isomorphism, such that  $\Phi = \Phi_{\mathcal{E}}$ , where  $\Phi_{\mathcal{E}}$  is the integral functor*

$$\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(Y), \quad \Phi_{\mathcal{E}}(\cdot) = \mathbf{R}p_{2*}(p_1^*(\cdot) \overset{\mathbf{L}}{\otimes} \mathcal{E}).$$

In what follows, given an automorphism  $\varphi : X \rightarrow X$ , I will use the notation

$$(\text{id}, \varphi) : X \rightarrow X \times X, \quad x \mapsto (x, \varphi(x)).$$

**Theorem 2.6** (Rouquier, [Ro] Théorème 4.18). *Let  $\Phi = \Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  be an equivalence, induced by  $\mathcal{E} \in \mathbf{D}(X \times Y)$ . Then  $\Phi$  induces an isomorphism of algebraic groups*

$$(16) \quad F : \text{Aut}^0(X) \times \text{Pic}^0(X) \simeq \text{Aut}^0(Y) \times \text{Pic}^0(Y)$$

defined by:

$$F(\varphi, L) = (\psi, M) \iff \Phi_{\mathcal{E}} \circ \Phi_{(\text{id}, \varphi)_* L} \cong \Phi_{(\text{id}, \psi)_* M} \circ \Phi_{\mathcal{E}}.$$

Let me give a brief explanation for the context and the idea behind this result (cf. also [Hu] p.217–218). The crucial point is to consider the group  $\text{Aut}(\mathbf{D}(X))$  of autoequivalences of  $\mathbf{D}(X)$ . Among these, there are some obvious ones:

- If  $\varphi \in \text{Aut}(X)$ , then  $\varphi_* \in \text{Aut}(\mathbf{D}(X))$ .
- If  $L \in \text{Pic}(X)$ , then  $L \otimes \cdot \in \text{Aut}(\mathbf{D}(X))$ .

So for any pair  $(\varphi, L) \in \text{Aut}(X) \times \text{Pic}(X)$  one has an autoequivalence  $(L \otimes \cdot) \circ \varphi_* \in \text{Aut}(\mathbf{D}(X))$ . The kernel of this autoequivalence is easily determined to be  $(\text{id}, \varphi)_* L$  (cf. [Hu] 5.4); in other words

$$\Phi_{(\varphi, L)} := \Phi_{(\text{id}, \varphi)_* L} = (L \otimes \cdot) \circ \varphi_*$$

Let's denote now by  $\text{Aut}^0(X)$  the connected component of the identity in  $\text{Aut}(X)$ . Note that the action of  $\text{Aut}^0(X)$  on  $\text{Pic}^0(X)$  is trivial. Indeed,  $\text{Aut}^0(X)$  acts on  $\text{Pic}^0(X)$  by elements in  $\text{Aut}^0(\text{Pic}^0(X))$ , which are translations. Since the origin is fixed, these must be trivial. This shows in particular that  $\text{Aut}^0(X)$  and  $\text{Pic}^0(X)$  commute as subgroups of  $\text{Aut}(\mathbf{D}(X))$ . Therefore we have an inclusion of the direct product

$$\text{Aut}^0(X) \times \text{Pic}^0(X) \hookrightarrow \text{Aut}(\mathbf{D}(X)).$$

Going back to Rouquier's theorem, note first that every equivalence  $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  induces a homomorphism

$$\tilde{F} : \text{Aut}(\mathbf{D}(X)) \longrightarrow \text{Aut}(\mathbf{D}(Y)), \quad \Psi \mapsto \Phi_{\mathcal{E}} \circ \Psi \circ \Phi_{\mathcal{E}}^{-1}$$

and of course  $\tilde{F}(\text{id}) = \text{id}$ , or in other words  $\tilde{F}(\Phi_{(\text{id}, \mathcal{O}_X)}) = \Phi_{(\text{id}, \mathcal{O}_Y)}$ . We would like to understand the kernels  $\mathcal{P}$  given by general  $\tilde{F}(\Phi_{(\varphi, L)}) = \Phi_{\mathcal{P}}$ . If we now consider pairs  $(\varphi, L)$  in a neighborhood of the identity in  $\text{Aut}(X) \times \text{Pic}(X)$ , the corresponding kernels move in a continuous family of complexes such that over the central fiber we have the structure sheaf of the diagonal (so a line bundle supported on the graph of an automorphism of  $X$ ). It is then not hard to see by semicontinuity arguments that the same must happen on a neighborhood – but if  $\mathcal{P}$  is a line bundle supported on the graph of an automorphism, it must be of the form  $(\text{id}, \psi)_* M$  for some  $\psi \in \text{Aut}(Y)$  and  $M \in \text{Pic}(Y)$ . But now any neighborhood of the identity in  $\text{Aut}(X) \times \text{Pic}(X)$  generates  $\text{Aut}^0(X) \times \text{Pic}^0(X)$  as a group, and using the inverse  $\Phi_{\mathcal{E}}^{-1}$  we can also go in the other direction in order to obtain the isomorphism in Theorem 2.6.

**2.3. Hochschild (co)homology.** Following an idea of Kontsevich, derived equivalences are known to preserve certain cohomological invariants that do not have an immediately obvious birational geometric interpretation, namely Hochschild homology and cohomology. This is shown in [Că] and [Or]; I am following the presentation in [Hu] §6.1. More generally, one shows

$$\mathbf{D}(X) \cong \mathbf{D}(Y) \implies HH(X) \cong HH(Y),$$

where

$$HH(X) := \bigoplus_{i, l} \text{Ext}_{X \times X}^i(i_* \mathcal{O}_X, i_* \omega_X^{\otimes l})$$

and the induced isomorphism preserves the natural bigrading on  $HH$ . Here  $i$  denotes the diagonal embedding on  $X$ . This contains the following statements:

- When  $i = 0$ , we obtain the canonical ring  $R(X) = \bigoplus_{l \geq 0} HH_{0, l}(X)$ . This gives the statement in Theorem 2.1 (iv).

- When  $l = 0$  we obtain the invariance of the *Hochschild cohomology*

$$HH^i(X) := \text{Ext}_{X \times X}^i(i_* \mathcal{O}_X, i_* \mathcal{O}_X) \cong \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X),$$

where the last isomorphism is the well-known Hochschild-Kostant-Rosenberg isomorphism for cohomology (which does not respect the multiplicative structure on the two sides, if we consider  $\bigoplus_i HH^i(X)$ ).

- When  $l = 1$  we obtain the invariance of the *Hochschild homology*

$$HH_i(X) := \text{Ext}_{X \times X}^i(i_* \mathcal{O}_X, i_* \omega_X) \cong \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X \otimes \omega_X),$$

where the last isomorphism is the Hochschild-Kostant-Rosenberg isomorphism for homology.

**Example 2.7.** (1) The isomorphism  $HH^1(X) \cong HH^1(Y)$  is equivalent to

$$H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) \cong H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y),$$

which is the infinitesimal version of Theorem 2.6.

(2) Via Serre duality, the isomorphism  $HH_i(X) \cong HH_i(Y)$  is equivalent to

$$\bigoplus_{p-q=i} H^p(X, \Omega_X^q) \cong \bigoplus_{p-q=i} H^p(Y, \Omega_Y^q).$$

**Corollary 2.8.** *If  $\mathbf{D}(X) \cong \mathbf{D}(Y)$ , then the sum of the Hodge numbers on the columns in the Hodge diamond is constant, i.e. for all  $i$*

$$\sum_{p-q=i} h^{p,q}(X) = \sum_{p-q=i} h^{p,q}(Y).$$

An immediate calculation shows then the following:

**Corollary 2.9.** *Assume that  $\mathbf{D}(X) \cong \mathbf{D}(Y)$ .*

(i) *If  $X$  and  $Y$  are surfaces, then  $h^{p,q}(X) = h^{p,q}(Y)$  for all  $p$  and  $q$ .*

(ii) *If  $X$  and  $Y$  are threefolds, the same thing holds, except for*

$$2h^{1,0}(X) + h^{2,1}(X) = 2h^{1,0}(Y) + h^{2,1}(Y).$$

So we see for instance that the invariance of  $h^{1,0}$  would imply the invariance of all Hodge numbers for threefolds. But while the invariance of  $h^{n,0}$  with  $n = \dim X$  is always known, in general even the invariance of any of the other  $h^{p,0}$  with  $p > 0$  is not clear.

**2.4. Actions of non-affine algebraic groups.** For this section I recommend Brion [Br1], [Br2], and the references therein. Let  $G$  be a connected algebraic group over a field. According to Chevalley's theorem (see e.g. [Br1] p.1),  $G$  has a unique maximal connected affine subgroup  $\text{Aff}(G)$ , and the quotient  $G/\text{Aff}(G)$  is an abelian variety. We denote this abelian variety by  $\text{Alb}(G)$ , since the map  $G \rightarrow \text{Alb}(G)$  is the Albanese map of  $G$ , i.e. the universal morphism to an abelian variety (see [Se2]). So we have an exact sequence

$$1 \longrightarrow \text{Aff}(G) \longrightarrow G \longrightarrow \text{Alb}(G) \longrightarrow 1$$

and  $G \rightarrow \text{Alb}(G)$  is a homogeneous fiber bundle with fiber  $\text{Aff}(G)$ .

**Lemma 2.10** ([Br2], Lemma 2.2). *The map  $G \rightarrow \text{Alb}(G)$  is locally trivial in the Zariski topology.*

Now let  $X$  be a smooth projective variety, and assume that  $G$  acts faithfully on  $X$ , i.e.  $G \subset \text{Aut}(X)$ . Note that  $G$  then acts naturally on  $\text{Alb}(X)$ . A crucial fact in what follows is the following theorem of Nishi and Matsumura (cf. also [Br1]).

**Theorem 2.11** ([Ma], Theorem 2). *The group  $G$  acts on  $\text{Alb}(X)$  by translations, and the kernel of the induced homomorphism  $G \rightarrow \text{Alb}(X)$  is affine. Consequently, the induced map  $\text{Alb}(G) \rightarrow \text{Alb}(X)$  has finite kernel.*

Here is a more detailed explanation of the result above.

**Lemma 2.12.** *The action of  $G$  on  $\text{Alb}(X)$  induces a map of abelian varieties*

$$\text{Alb}(G) \rightarrow \text{Alb}(X),$$

*whose image is contained in the Albanese image  $\text{alb}_X(X)$ . More precisely, the composition  $G \rightarrow \text{Alb}(X)$  is given by the formula  $g \mapsto \text{alb}_X(gx_0 - x_0)$ , where  $x_0 \in X$  is an arbitrary point.*

*Proof.* From  $G \times X \rightarrow X$ , we obtain a map of abelian varieties

$$\text{Alb}(G) \times \text{Alb}(X) \simeq \text{Alb}(G \times X) \rightarrow \text{Alb}(X).$$

It is clearly the identity on  $\text{Alb}(X)$ , and therefore given by a map of abelian varieties  $\text{Alb}(G) \rightarrow \text{Alb}(X)$ . To see what it is, fix a base-point  $x_0 \in X$ , and write the Albanese map of  $X$  in the form  $X \rightarrow \text{Alb}(X)$ ,  $x \mapsto \text{alb}_X(x - x_0)$ . Let  $g \in G$  be an automorphism of  $X$ . By the universal property of  $\text{Alb}(X)$ , it induces an automorphism  $\tilde{g} \in \text{Aut}^0(\text{Alb}(X))$ , making the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ \text{Alb}(X) & \xrightarrow{\tilde{g}} & \text{Alb}(X) \end{array}$$

commute; in other words,  $\tilde{g}(\text{alb}_X(x - x_0)) = \text{alb}_X(gx - x_0)$ . Any such automorphism is translation by an element of  $\text{Alb}(X)$ , and the formula shows that this element has to be  $\text{alb}_X(gx_0 - x_0)$ . It follows that the map  $G \rightarrow \text{Alb}(X)$  is given by  $g \mapsto \text{alb}_X(gx_0 - x_0)$ . By Chevalley's theorem, it factors through  $\text{Alb}(G)$ .  $\square$

The Nishi-Matsumura theorem states that the kernel of the induced morphism  $\text{Alb}(G) \rightarrow \text{Alb}(X)$  in the Lemma is finite.

We take now for  $G$  the connected component of the identity  $G_X := \text{Aut}^0(X)$  in  $\text{Aut}(X)$ , and let  $a(X)$  be the dimension of the abelian variety  $\text{Alb}(G_X)$ . By Theorem 2.11 the image of  $\text{Alb}(G_X)$  is an abelian subvariety of  $\text{Alb}(X)$  of dimension  $a(X)$ . This implies the inequality  $a(X) \leq q(X)$ . Brion observed that  $X$  can always be fibered over an abelian variety which is a quotient of  $\text{Alb}(G_X)$  of the same dimension  $a(X)$ ; the following proof is taken from [Br1] §3.

**Lemma 2.13.** *There is an affine subgroup  $\text{Aff}(G_X) \subseteq H \subseteq G_X$  with  $H/\text{Aff}(G_X)$  finite, such that  $X$  admits a  $G_X$ -equivariant map  $\psi: X \rightarrow G_X/H$ . Consequently,  $X$  is isomorphic to the equivariant fiber bundle  $G_X \times^H Z$  with fiber  $Z = \psi^{-1}(0)$ .*

*Proof.* By the Poincaré complete reducibility theorem, the map  $\text{Alb}(G_X) \rightarrow \text{Alb}(X)$  splits up to isogeny. This means that we can find a subgroup  $H$  containing  $\text{Aff}(G_X)$ , such that there is a surjective map  $\text{Alb}(X) \rightarrow G_X/H$  with  $\text{Alb}(G_X) \rightarrow G_X/H$  an isogeny. It follows that  $H/\text{Aff}(G_X)$  is finite, and hence that  $H$  is an affine subgroup of  $G_X$  whose identity component is  $\text{Aff}(G_X)$ . Let  $\psi: X \rightarrow G_X/H$  be the resulting map; it is equivariant by construction. Since  $G_X$  acts transitively on  $G_X/H$ , we conclude that  $\psi$  is an equivariant fiber bundle over  $G_X/H$  with fiber  $Z = \psi^{-1}(0)$ , and therefore isomorphic to

$$G_X \times^H Z = (G_X \times Z)/H,$$

where  $H$  acts on the product by  $(g, z) \cdot h = (g \cdot h, h^{-1} \cdot z)$ .  $\square$

Note that the group  $H$  naturally acts on  $Z$ ; the proof shows that we obtain  $X$  from the principal  $H$ -bundle  $G_X \rightarrow G_X/H$  by replacing the fiber  $H$  by  $Z$  (see [Se1], §3.2). While  $X \rightarrow G_X/H$  is not necessarily locally trivial, it is at least locally isotrivial, and hence a fiber bundle in the étale topology.

**Lemma 2.14.** *Both  $G_X \rightarrow G_X/H$  and  $X \rightarrow G_X/H$  are locally isotrivial.*

*Proof.* Consider the pullback of  $X$  along the étale map  $\text{Alb}(G_X) \rightarrow G_X/H$ ,

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Alb}(G_X) & \longrightarrow & G_X/H. \end{array}$$

One notes that  $X' \rightarrow \text{Alb}(G_X)$  is associated to the principal bundle  $G_X \rightarrow \text{Alb}(G_X)$ . The latter is locally trivial in the Zariski topology by Lemma 2.10.  $\square$

**Corollary 2.15.** *If  $a(X) > 0$  (i.e.  $G_X$  is not affine), then  $\chi(\mathcal{O}_X) = 0$ .*

*Proof.* Clearly  $\chi(\mathcal{O}_{X'}) = 0$  since  $X'$  is locally isomorphic to the product of  $Z$  and  $\text{Alb}(G_X)$ . But  $\chi(\mathcal{O}_{X'}) = \deg(X'/X) \cdot \chi(\mathcal{O}_X)$ .  $\square$

**2.5. Behavior of the Picard variety under derived equivalence.** I will now discuss the main result of the notes.

**Theorem 2.16.** *Let  $X$  and  $Y$  be smooth projective varieties such that  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ . Then*

(1)  $\text{Pic}^0(X)$  and  $\text{Pic}^0(Y)$  are isogenous.

(2)  $\text{Pic}^0(X) \simeq \text{Pic}^0(Y)$  unless  $X$  and  $Y$  are étale locally trivial fibrations over isogenous positive dimensional abelian varieties (hence  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$ ).

(3) In particular

$$h^0(X, \Omega_X^1) = h^0(Y, \Omega_Y^1) \quad \text{and} \quad h^0(X, T_X) = h^0(Y, T_Y).$$

Note that the first identity in (3) follows from (1), while the second follows by combining this with Example 2.7(1). Combining the invariance of  $h^{1,0}$  with Corollary 2.9, we obtain the invariance of Hodge numbers for derived equivalent threefolds.

**Corollary 2.17.** *Let  $X$  and  $Y$  be smooth projective threefolds with  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ . Then*

$$h^{p,q}(X) = h^{p,q}(Y)$$

for all  $p$  and  $q$ .

In the case of fourfolds, in addition to the Hodge numbers that are equal due to the general invariance of Hochschild homology (namely  $h^{3,0}$  and  $h^{4,0}$ ), Theorem 2.16 implies:

**Corollary 2.18.** *Let  $X$  and  $Y$  be smooth projective fourfolds with  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ . Then  $h^{2,1}(X) = h^{2,1}(Y)$ . If in addition  $\text{Aut}^0(X)$  is not affine, then  $h^{2,0}(X) = h^{2,0}(Y)$  and  $h^{3,1}(X) = h^{3,1}(Y)$ .*

*Proof.* Corollary 2.8 applied to fourfolds implies that  $h^{2,1}$  is invariant if and only if  $h^{1,0}$  is invariant, and  $h^{2,0}$  is invariant if and only if  $h^{3,1}$  is invariant. On the other hand, if  $\text{Aut}^0(X)$  is not affine, then  $\chi(\mathcal{O}_X) = 0$  (cf. Lemma 2.15), which implies that  $h^{2,0}$  is invariant if and only if  $h^{1,0}$  is invariant. We apply Theorem 2.16.  $\square$

**Example 2.19.** Here is an example of how Theorem 2.16 can help in verifying the invariance of classification properties that are characterized numerically, namely a quick proof of the fact that if  $\mathbf{D}(X) \simeq \mathbf{D}(Y)$ , and  $X$  is an abelian variety, then so is  $Y$  ([HN] Proposition 3.1). Indeed, by Theorem 2.1(iv) and Theorem 2.16 we have that  $P_1(Y) = P_2(Y) = 1$  and  $q(Y) = \dim Y$ . The main result of [CH] implies that  $Y$  is birational, so it actually has a birational morphism, to an abelian variety  $B$ . But  $\omega_X \simeq \mathcal{O}_X$ , and so by Theorem 2.1(ii)  $\omega_Y \simeq \mathcal{O}_Y$  as well, hence  $Y \simeq B$ .

The rest of the section is devoted to the proof of Theorem 2.16. Let  $\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$  be the equivalence, given by Orlov's theorem by an object  $\mathcal{E} \in \mathbf{D}(X \times Y)$ . By Theorem 2.6, there is an isomorphism of algebraic groups

$$(17) \quad F : \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \simeq \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$

and we have the following (this was proved by Orlov [Or], Corollary 5.1.10, in the case of abelian varieties):

**Lemma 2.20.** *One has  $F(\varphi, L) = (\psi, M)$  if and only if*

$$p_1^*L \otimes (\varphi \times \mathrm{id})^*\mathcal{E} \simeq p_2^*M \otimes (\mathrm{id} \times \psi)_*\mathcal{E}.$$

*Proof.* By construction,  $F(\varphi, L) = (\psi, M)$  is equivalent to the relation

$$\Phi_{\mathcal{E}} \circ \Phi_{(\mathrm{id}, \varphi)_*L} = \Phi_{(\mathrm{id}, \psi)_*M} \circ \Phi_{\mathcal{E}}.$$

Since both sides are equivalences, their kernels have to be isomorphic. Mukai's formula for the kernel of the composition of two integral functors (see [Hu], Proposition 5.10) gives

$$(18) \quad p_{13*}(p_{12}^*(\mathrm{id}, \varphi)_*L \otimes p_{23}^*\mathcal{E}) \simeq p_{13*}(p_{12}^*\mathcal{E} \otimes p_{23}^*(\mathrm{id}, \psi)_*M).$$

To compute the left-hand side of (18), let  $\lambda : X \times Y \rightarrow X \times X \times Y$  be given by  $\lambda(x, y) = (x, \varphi(x), y)$ , making the following diagram commutative:

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\lambda} & X \times X \times Y & \xrightarrow{p_{13}} & X \times Y \\ p_1 \downarrow & & p_{12} \downarrow & & \\ X & \xrightarrow{(\mathrm{id}, \varphi)} & X \times X & & \end{array}$$

By the base-change formula,  $p_{12}^*(\mathrm{id}, \varphi)_*L \simeq \lambda_*p_1^*L$ ; using the projection formula and the identities  $p_{13} \circ \lambda = \mathrm{id}$  and  $p_{23} \circ \lambda = \varphi \times \mathrm{id}$ , we then have

$$p_{13*}(p_{12}^*(\mathrm{id}, \varphi)_*L \otimes p_{23}^*\mathcal{E}) \simeq p_1^*L \otimes \lambda^*p_{23}^*\mathcal{E} \simeq p_1^*L \otimes (\varphi \times \mathrm{id})^*\mathcal{E}.$$

To compute the right-hand side of (18), we similarly define  $\mu : X \times Y \rightarrow X \times Y \times Y$  by the formula  $\mu(x, y) = (x, y, \psi(y))$ , to fit into the diagram

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\mu} & X \times Y \times Y & \xrightarrow{p_{13}} & X \times Y \\ p_2 \downarrow & & p_{23} \downarrow & & \\ Y & \xrightarrow{(\mathrm{id}, \psi)} & Y \times Y & & \end{array}$$

Since  $p_{13} \circ \mu = (\mathrm{id} \times \psi)$  and  $p_{12} \circ \mu = \mathrm{id}$ , the same calculation as above shows that

$$p_{13*}(p_{12}^*\mathcal{E} \otimes p_{23}^*(\mathrm{id}, \psi)_*M) \simeq (\mathrm{id} \times \psi)_*(\mathcal{E} \otimes p_2^*M) \simeq (\mathrm{id} \times \psi)_*\mathcal{E} \otimes p_2^*M,$$

where the last step uses that the action of  $\mathrm{Aut}^0(Y)$  on  $\mathrm{Pic}^0(Y)$  is trivial, so  $(\mathrm{id} \times \psi)^*p_2^*M \simeq p_2^*M$ .  $\square$

To prove the theorem, we start with the numerical part (3). Note that Example 2.7(1) implies the invariance of the quantity  $h^0(X, \Omega_X^1) + h^0(X, T_X)$ . Hence it suffices to show that  $q(X) = q(Y)$ , where we set  $q(X) = h^0(X, \Omega_X^1)$ , and similarly for  $Y$ .

Continuing to write  $G_X = \text{Aut}^0(X)$  and  $G_Y = \text{Aut}^0(Y)$ , consider the induced map

$$\beta: \text{Pic}^0(X) \rightarrow G_Y, \quad \beta(L) = p_1(F(\text{id}, L)),$$

and let  $B = \text{Im } \beta$ . Similarly, we define

$$\alpha: \text{Pic}^0(Y) \rightarrow G_X, \quad \alpha(M) = p_1(F^{-1}(\text{id}, M)),$$

and let  $A = \text{Im } \alpha$ . One easily verifies that  $F$  induces an isomorphism

$$F: A \times \text{Pic}^0(X) \rightarrow B \times \text{Pic}^0(Y).$$

If both  $A$  and  $B$  are trivial, we immediately obtain  $\text{Pic}^0(X) \simeq \text{Pic}^0(Y)$ . Excluding this case from now on, we let the abelian variety  $A \times B$  act on  $X \times Y$  by automorphisms. Take a point  $(x, y)$  is the support of the kernel  $\mathcal{E}$ , and consider the orbit map

$$f: A \times B \rightarrow X \times Y, \quad (\varphi, \psi) \mapsto (\varphi(x), \psi(y)).$$

By Lemma 2.12 and the Nishi-Matsumura Theorem 2.11, the induced map  $A \times B \rightarrow \text{Alb}(X) \times \text{Alb}(Y)$  has finite kernel. Consequently, the dual map  $f^*: \text{Pic}^0(X) \times \text{Pic}^0(Y) \rightarrow \widehat{A} \times \widehat{B}$  is surjective.

Now let  $\mathcal{F} := \mathbf{L}f^*\mathcal{E} \in \mathbf{D}(A \times B)$ ; it is nontrivial by our choice of  $(x, y)$ . For  $F(\varphi, L) = (\psi, M)$ , the formula in Lemma 2.20 can be rewritten in the more symmetric form (again using the fact that  $\psi^*M \simeq M$ ):

$$(19) \quad (\varphi \times \psi)^*\mathcal{E} \simeq (L^{-1} \boxtimes M) \otimes \mathcal{E}.$$

For  $(\varphi, \psi) \in A \times B$ , let  $t_{(\varphi, \psi)} \in \text{Aut}^0(A \times B)$  denote translation by  $(\varphi, \psi)$ . The identity in (19) implies that  $t_{(\varphi, \psi)}^*\mathcal{F} \simeq f^*(L^{-1} \boxtimes M) \otimes \mathcal{F}$ , whenever  $F(\varphi, L) = (\psi, M)$ . We introduce the map

$$\pi = (\pi_1, \pi_2): A \times \text{Pic}^0(X) \rightarrow (A \times B) \times (\widehat{A} \times \widehat{B}), \quad \pi(\varphi, L) = (\varphi, \psi, L^{-1}|_A, M|_B),$$

where we write  $L^{-1}|_A$  for the pull-back from  $\text{Alb}(X)$  to  $A$ , and same for  $M$ . We can then write the identity above as

$$(20) \quad t_{\pi_1(\varphi, L)}^*\mathcal{F} \simeq \pi_2(\varphi, L) \otimes \mathcal{F}.$$

Using the fact that  $F$  is a group homomorphism, one can show without much difficulty that  $\pi_1: A \times \text{Pic}^0(X) \rightarrow A \times B$  is surjective. It follows that each cohomology object  $H^i(\mathcal{F})$  is a semi-homogeneous<sup>8</sup> vector bundle on  $A \times B$ , and that  $\dim(\text{Im } \pi) \geq \dim A + \dim B$ . On the other hand Mukai [Mu1], Proposition 5.1, shows that the semi-homogeneity of  $H^i(\mathcal{F})$  is equivalent to the fact that the closed subset

$$\Phi(H^i(\mathcal{F})) := \{(x, \alpha) \in (A \times B) \times (\widehat{A} \times \widehat{B}) \mid t_x^*H^i(\mathcal{F}) \simeq H^i(\mathcal{F}) \otimes \alpha\}$$

has dimension precisely  $\dim A + \dim B$ . This implies that  $\dim(\text{Im } \pi) = \dim A + \dim B$  (and in fact that  $\text{Im } \pi = \Phi^0(H^i(\mathcal{F}))$ , the neutral component, for any  $i$ , though we will not use this; note that  $\Phi$  is denoted  $\Phi^0$ , and  $\Phi^0$  is denoted  $\Phi^{00}$  in [Mu1]). Furthermore, we have

$$\begin{aligned} \text{Ker}(\pi) &= \{(\text{id}, L) \in A \times \text{Pic}^0(X) \mid F(\text{id}, L) = (\text{id}, M) \text{ and } L|_A \simeq \mathcal{O}_A \text{ and } M|_B \simeq \mathcal{O}_B\} \\ &\subseteq \{L \in \text{Pic}^0(X) \mid L|_A \simeq \mathcal{O}_A\} = \text{Ker}(\text{Pic}^0(X) \rightarrow \widehat{A}). \end{aligned}$$

Now the surjectivity of  $f^*$  implies in particular that the restriction map  $\text{Pic}^0(X) \rightarrow \widehat{A}$  is surjective, so we get  $\dim(\text{Ker } \pi) \leq q(X) - \dim A$ , and therefore

$$\dim A + \dim B = \dim A + q(X) - \dim(\text{Ker } \pi) \geq 2 \dim A.$$

<sup>8</sup>Recall that a semihomogeneous sheaf on an abelian variety  $A$  is a sheaf  $E$  such that for each  $a \in A$  there exists  $\alpha \in \text{Pic}^0(A)$  such that  $t_a^*E \simeq E \otimes \alpha$ . It follows from the definition that  $E$  must be a vector bundle.

Thus  $\dim A \leq \dim B$ ; by symmetry,  $\dim A = \dim B$ , and finally,  $q(X) = q(Y)$ . This concludes the proof of the fact that  $\text{Pic}^0(X)$  and  $\text{Pic}^0(Y)$  have the same dimension.

We now use this to show that they are in fact isogenous. Let  $d = \dim A = \dim B$ . The reasoning above proves that  $\text{Im } \pi$  is an abelian subvariety of  $(A \times B) \times (\widehat{A} \times \widehat{B})$ , with  $\dim(\text{Im } \pi) = 2d$ . For dimension reasons, we also have

$$(21) \quad (\text{Ker } \pi)^0 \simeq (\text{Ker}(\text{Pic}^0(X) \rightarrow \widehat{A}))^0 \simeq (\text{Ker}(\text{Pic}^0(Y) \rightarrow \widehat{B}))^0,$$

where the superscripts indicate neutral components. We claim that the projection  $p: \text{Im } \pi \rightarrow A \times \widehat{A}$  is an isogeny (likewise for  $B \times \widehat{B}$ ). Indeed, a point in  $p^{-1}(\text{id}, \mathcal{O}_A)$  is of the form  $(\text{id}, \psi, \mathcal{O}_A, M|_B)$ , where  $F(\text{id}, L) = (\psi, M)$  and  $L|_A \simeq \mathcal{O}_A$ . By (21), a fixed multiple of  $(\text{id}, L)$  belongs to  $\text{Ker } \pi$ , and so  $\text{Ker } p$  is a finite set. It follows that  $\text{Im } \pi$  is isogenous to both  $A \times \widehat{A}$  and  $B \times \widehat{B}$ ; consequently,  $A$  and  $B$  are themselves isogenous.

To conclude the proof of part (3), note that we have extensions

$$0 \rightarrow \text{Ker } \beta \rightarrow \text{Pic}^0(X) \rightarrow B \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Ker } \alpha \rightarrow \text{Pic}^0(Y) \rightarrow A \rightarrow 0.$$

By definition,  $\text{Ker } \beta$  consists of those  $L \in \text{Pic}^0(X)$  for which  $F(\text{id}, L) = (\text{id}, M)$ ; obviously,  $F$  now induces an isomorphism  $\text{Ker } \beta \simeq \text{Ker } \alpha$ , and therefore  $\text{Pic}^0(X)$  and  $\text{Pic}^0(Y)$  are isogenous.

It remains to prove part (2). Note that by Rouquier's isomorphism (17) and the uniqueness of  $\text{Aff}(G)$  in Chevalley's theorem we have  $\text{Aff}(G_X) \simeq \text{Aff}(G_Y)$  and

$$\text{Alb}(G_X) \times \text{Pic}^0(X) \simeq \text{Alb}(G_Y) \times \text{Pic}^0(Y).$$

On the other hand, we have seen that  $q(X) = q(Y)$ , and therefore  $a(X) = a(Y)$ . If  $a(X) = 0$ , we obviously have  $\text{Pic}^0(X) \simeq \text{Pic}^0(Y)$ . On the other hand, if  $a(X) > 0$ , Lemmas 2.13 and 2.14 show that  $X$  can be written as an étale locally trivial fiber bundle over a quotient of  $\text{Alb}(G_X)$  by a finite subgroup, so an abelian variety isogenous to  $\text{Alb}(G_X)$ . The same holds for  $Y$  by symmetry. Note that in this case we have  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$  by Corollary 2.15.

## REFERENCES

- [BGG] I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, Algebraic vector bundles on  $P^n$  and problems of linear algebra, *Funktional. Anal. i Prilozhen.* **12** (1978), no. 3, 66–67.
- [BO] A. Bondal and D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, *Comp. Math.* **125** (2001), 327–344.
- [BC] L. Borisov and A. Căldăraru, The Pfaffian-Grassmanian derived equivalence, *J. Alg. Geom.* **18** (2009), 210–222.
- [Br1] M. Brion, Some basic results on actions of non-affine algebraic groups, preprint arXiv:math/0702518.
- [Br2] M. Brion, On the geometry of algebraic groups and homogeneous spaces, preprint arXiv:math/09095014.
- [Că] A. Căldăraru, The Mukai pairing, I: The Hochschild structure, preprint arXiv:math/0308079.
- [Ca] F. Catanese, Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations, *Invent. Math.* **104** (1991), 389–407.
- [CH] H. Clemens and Ch. Hacon, Deformations of the trivial line bundle and vanishing theorems, *Amer. J. Math.* **124** (2002), no. 4, 769–815.
- [EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, *J. Amer. Math. Soc.* **10** (1997), 243–258.
- [Eis] D. Eisenbud, *The geometry of syzygies*, Springer 2005.
- [EFS] D. Eisenbud, G. Fløystad and F.-O. Schreyer, Sheaf cohomology and free resolutions over the exterior algebra, *Trans. Amer. Math. Soc.* **355** (2003), no. 11, 4397–4426.
- [EPY] D. Eisenbud, S. Popescu and S. Yuzvinsky, Hyperplane arrangement cohomology and monomials in the exterior algebra, *Trans. Amer. Math. Soc.* **355** (2003), no.11, 4365–4383.
- [GL1] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, *Invent. Math.* **90** (1987), 389–407.
- [GL2] M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, *J. Amer. Math. Soc.* **1** (1991), no.4, 87–103.
- [EGAI] A. Grothendieck, *Eléments de Géométrie Algébrique II*, Publ. Res. Math. Inst. Hautes Études Sci., **8**, 1961.
- [EGAIII] A. Grothendieck, *Eléments de Géométrie Algébrique III*, Publ. Res. Math. Inst. Hautes Études Sci., **11**, 1961 and **17**, 1963.

- [Hac] Ch. Hacon, A derived category approach to generic vanishing, *J. Reine Angew. Math.* **575** (2004), 173–187.
- [HP] Ch. Hacon and R. Pardini, On the birational geometry of varieties of maximal Albanese dimension, *J. Reine Angew. Math.* **546** (2002), 177–199.
- [Hu] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford 2006.
- [HN] D. Huybrechts and M. Nieper-Wisskirchen, Remarks on derived equivalences of Ricci-flat manifolds, preprint arXiv:0801.4747, to appear in *Math. Z.*
- [Ka] M. Kashiwara,  $T$ -structures on the derived category of holonomic  $D$ -modules and coherent  $O$ -modules, *Moscow Math. J.* **4** (2004), no.4, 847–868.
- [Ka] Y. Kawamata,  $D$ -equivalence and  $K$ -equivalence, *J. Diff. Geom.* **61** (2002), 147–171.
- [Ko1] J. Kollár, Higher direct images of dualizing sheaves I, *Ann. of Math.* **123** (1986), 11–42.
- [Ko2] J. Kollár, Higher direct images of dualizing sheaves II, *Ann. of Math.* **124** (1986), 171–202.
- [Ku] A. Kuznetsov, Lefschetz decompositions and categorical resolution of singularities, *Selecta Math. (N.S.)* **13** (2008), no. 4, 661–696.
- [La] R. Lazarsfeld, *Positivity in algebraic geometry I & II*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, vol. **48 & 49**, Springer-Verlag, Berlin, 2004.
- [LP] R. Lazarsfeld and M. Popa, Derivative complex, BGG correspondence, and numerical inequalities for compact Kähler manifolds, preprint arXiv:0907.0651, to appear in *Invent. Math.*
- [Ma] H. Matsumura, On algebraic groups of birational transformations, *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Serie VIII*, **34** (1963), 151–155.
- [Mu1] S. Mukai, Semihomogeneous vector bundles on an abelian variety, *J. Math. Kyoto Univ.* **18**, no.2 (1978), 239–272.
- [Mu2] S. Mukai, Duality between  $D(X)$  and  $D(\widehat{X})$  with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175.
- [Mu3] S. Mukai, On the moduli space of bundles on  $K3$  surfaces, I, in *Vector bundles on algebraic varieties*, Bombay (1984).
- [Or] D. Orlov, Derived categories of coherent sheaves and the equivalences between them, *Russian Math. Surveys* **58** (2003), 511–591.
- [Pa] G. Pareschi, Generic vanishing, Gaussian maps, and Fourier-Mukai transform, preprint math.AG/0310026.
- [PP1] G. Pareschi and M. Popa,  $GV$ -sheaves, Fourier-Mukai transform, and Generic Vanishing, preprint math/0608127, to appear in *Amer. J. Math.*
- [PP2] G. Pareschi and M. Popa, Strong Generic Vanishing and a higher dimensional Castelnuovo-de Franchis inequality, *Duke Math. J.* **150** no.2 (2009), 269–285.
- [Po] M. Popa, Generic vanishing filtrations and perverse objects in derived categories of coherent sheaves, preprint arXiv:0911.3648.
- [PS] M. Popa and C. Schnell, Derived invariance of the number of holomorphic 1-forms and vector fields, preprint arXiv:0912.4040, to appear in *Ann. Sci. ENS.*
- [RRV] J.-P. Ramis, G. Rouget and J.-L. Verdier, Dualité relative en géométrie analytique complexe, *Invent. Math.* **13** (1971), 261–283.
- [Ro] R. Rouquier, Automorphismes, graduations et catégories triangulées, available at <http://people.maths.ox.ac.uk/rouquier/papers/autograd.pdf>, preprint 2009.
- [Sa] M. Saito, Decomposition theorem for proper Kähler morphisms, *Tohoku Math. J. (2)* **42**, (1990), no.2, 127–147.
- [GAGA] J. P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* **6** (1955–1956), 1–42.
- [Se1] J.-P. Serre, Espaces fibrés algébriques, Séminaire Claude Chevalley (1958), Exposé No. 1, *Documents Mathématiques* **1**, Soc. Math. France, Paris, 2001.
- [Se2] J.-P. Serre, Morphismes universels et variété d’Albanese, Séminaire Claude Chevalley (1958–1959), Exposé No. 10, *Documents Mathématiques* **1**, Soc. Math. France, Paris, 2001.
- [Ta] K. Takegoshi, Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms, *Math. Ann.* **303** (1995), no.3, 389–416.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN STREET, CHICAGO, IL 60607, USA

*E-mail address:* mpopa@math.uic.edu