CONJECTURES ON THE KODAIRA DIMENSION

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To Vyacheslav Shokurov, on his 70th birthday.

1. Introduction. In this note, I propose a few conjectures on the behavior of the Kodaira dimension under morphisms of smooth complex varieties. The statements are of a rather different flavor than the well-known subadditivity conjectured by Iitaka; in some sense they complement it with ideas inspired by the study of the hyperbolicity of parameter spaces. Even though there is one statement that implies them all in §3, I will first discuss in §2 an intermediate conjecture that is already of substantial interest and provides a "superadditivity" counterpart to subaddivity. Concretely, here is essentially the main case of Conjecture 2.1 in the text: if X and Y are smooth projective varieties, and $f\colon X \to Y$ is an algebraic fiber space with general fiber F, which is smooth away from a simple normal crossing divisor $D \subset Y$, then we have

$$\kappa(F) + \kappa\big(\omega_Y(D)\big) \ge \kappa(X).$$

In §4 I will also state a weaker conjecture on the behavior of Kodaira *co*dimension, which is potentially more accessible.

I was inspired to look for such statements while thinking about possible extensions of results for families over abelian varieties in [PS14]. Various cases follow from existing results and techniques, while others have been established since the first announcement and are listed here. For instance, we now know that, just as with Iitaka's conjecture, at least the intermediate conjecture is implied by the main conjectures of the minimal model program.

All the varieties in this note are defined over $\mathbf{C}$.

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2. A superadditivity conjecture for morphisms between smooth projective varieties. Recall that Iitaka's $C_{n,m}$ conjecture predicts that for an algebraic fiber space $f\colon X \to Y$ of smooth projective varieties (meaning f is a surjective morphism, with connected fibers), we have subadditivity for the Kodaira dimension, i.e.

$$\kappa(X) \ge \kappa(F) + \kappa(Y).$$

For surveys on this conjecture, see [Mor87] and [Fuj20].

Here I start by proposing a complementary "superadditivity" statement in this same, most common, setting.

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Conjecture 2.1. *Let $f : X \to Y$ be an algebraic fiber space between smooth projective varieties, and let $V \subseteq Y$ be the open subset over which f is smooth. Then*

$$\kappa(F) + \kappa(V) \geq \kappa(X).$$

Recall that the *log Kodaira dimension* $\kappa(V)$ can be defined as follows: after a birational base change we can assume that the complement $D = Y \smallsetminus V$ is a simple normal crossing (SNC) divisor. One then defines

$$\kappa(V) := \kappa\big(Y, \omega_Y(D)\big),$$

the Iitaka dimension of $\omega_Y(D)$, which is easily checked to be independent of the choice of compactification of V with simple normal crossing boundary.

Conjecture 2.1 will be strengthened later on to one about smooth morphisms between quasi-projective varieties, which provides the correction turning the inequality into an equality; see Conjecture 3.1 below.

Remark 2.2 (**Obvious cases**)**.** The conjecture clearly holds when $\kappa(X) = -\infty$ (so in particular when $\kappa(F) = -\infty$) and when $\kappa(V) = \dim Y$, i.e. V is of log general type. Recall that the Easy Addition lemma, see e.g. [Mor87, Corollary 2.3(iii)], says that for any algebraic fiber space we have

$$\kappa(F) + \dim Y \geq \kappa(X).$$

Remark 2.3. It is worth noting that an important class of fiber spaces for which V is of log general type is that of "moduli" families; more generally, by [PS17, Theorem A], relying also on important work on Viehweg's hyperbolicity conjecture in [VZ02], [CP19], this holds for every family with maximal variation such that F admits a good minimal model. The methods used in these works have also been crucial for solving other cases of Conjecture 2.1.

Remark 2.4 (**Smooth case**)**.** In the smooth case, when combined with Iitaka's subadditivity, Conjecture 2.1 leads to the following additivity formula, also generalized later from a different point of view by Conjecture 3.1:

Conjecture 2.5. *If $f : X \to Y$ is smooth algbraic fiber space between smooth projective varieties, with general fiber F, then*

$$\kappa(X) = \kappa(F) + \kappa(Y).$$

Remark 2.6 (**Domain of general type**)**.** It is also amusing to spell out the special case when X is of general type. When f is smooth (i.e. $V = Y$), or when Y is not uniruled, this was proved in [PS22]; the full statement is a consequence of a more general result by Park, see [Par22, Corollary 1.6].

Theorem 2.7. *With the notation in Conjecture 2.1, if X is of general type, then V is of log general type.*

The non-obvious cases of Conjecture 2.1 and Conjecture 2.5 are summarized in the next two theorems.

Theorem 2.8. *Conjecture 2.1 holds when:*

(1) Y is an abelian variety, or more generally a variety of maximal Albanese dimension.[1]
(2) Y is a curve.
(3) f is smooth, and either the general fiber of the Iitaka fibration of Y admits a good minimal model,[2] *or Y is uniruled. In particular, it holds when f is smooth and Y is a surface or a threefold.*
(4) f is smooth, and $\kappa(Y) \geq \dim Y - 3 \geq 0$.
(5) X is of general type.
(6) F has semiample canonical bundle.
(7) we assume the conjectures of the log MMP.

Proof. Part (1) is shown in [MP21], using techniques from [LPS20].

Part (2) is clear when $g(Y) \geq 2$. When $Y = \mathbf{P}^1$ it follows from [VZ01, Theorem 0.2], while when Y is elliptic it follows from (1).

Part (3) is established in [PS22]; see Theorem C and Corollaries E, F in *loc. cit.*

Part (4) follows by applying Lemma 2.10 below to a model $g\colon Y' \to Z$ of the Iitaka fibration of Y, with Y' and Z smooth. Its general fiber G has dimension at most 3, hence (2) and (3) apply.

Part (5) is Theorem 2.7.

Part (6) follows from a more general result in [Cam22], cited in the next section.

Part (7) is a special case of [Par22, Theorem 1.7 and 1.12]; see *loc. cit.* for a more precise statement. □

Theorem 2.9. *Conjecture 2.5 holds when*

(1) f is a fiber bundle.
(2) Y is of general type.
(3) Y is an abelian variety, or more generally a variety of maximal Albanese dimension.
(4) Y is a curve.
(5) Y is a surface.
(6) Y is uniruled.
(7) Y is a good minimal model with $\kappa(Y) = 0$.
(8) X is a good minimal model with $\kappa(X) = 0$.
(9) X is of general type.
(10) F has semiample canonical bundle.
(11) we assume the conjectures of the MMP.

Proof. Part (1) is one of the original results on the Iitaka conjecture, obtained (in a more general setting) in [NU73].

For (2), (3), (4), and (5), since we know either from Remark 2.2 or from the previous statement that Conjecture 2.1 is settled in these cases, Conjecture 2.5 follows as in

[1]This means that Y admits a generically finite (not necessarily surjective) morphism to an abelian variety.

[2]More precisely one needs to assume a conjecture of Campana-Peternell, which is in turn a consequence of the existence of good minimal models.

Remark 2.4, as in all these cases we know that Iitaka's conjecture holds (see [Kaw82], [Vie83], [Kaw85], [CP17], [HPS18], [Cao18]).

For part (6), see [PS22, Proposition G]; the main input is the case $Y = \mathbf{P}^1$, proved in [VZ01, Theorem 0.2].

Part (7) is [PS22, Theorem H(ii)]. On the other hand, if X is a good minimal model with $\kappa(X) = 0$, then so is Y by [TZ20], hence (8) also follows.

Part (9) is a special case of Theorem 2.7.

Part (10) is a special case of Theorem 3.4 below, a result of Campana.

Part (11) is [PS22, Corollary D]; a more careful explanation of what exactly is needed is given in *loc. cit.* □

We note that via standard methods one can perform a reduction on the base, which in particular generates further examples (and was already used in Theorem 2.8(4)). We restrict the discussion to smooth morphisms for simplicity.

Lemma 2.10. *Any smooth algebraic fiber space over Y satisfies Conjecture 2.1 if there is a birational morphism $Y' \to Y$, and a nontrivial fibration $g\colon Y' \to Z$ of smooth projective varieties with general fiber G, such that $\kappa(Y) = \dim Z + \kappa(G)$*[3] *and any smooth fiber space over G satisfies Conjecture 2.1.*

Proof. Let $f\colon X \to Y$ be a smooth algebraic fiber space. After a birational base change on Y, we may assume that there is a fiber space $g\colon Y \to Z$ to a smooth projective variety, such that its fiber G satisfies Conjecture 2.1. (Doing the corresponding birational base change on X is ok, since the fiber product is still a smooth fibration.) Let's denote by F the general fiber of f, and by H the general fiber of $h = g \circ f$. We then have that $H \to G$ is a smooth fibration with fiber F, so by assumption

$$\kappa(F) + \kappa(G) \geq \kappa(H).$$

Adding $\dim Z$ to both sides of the inequality and using the hypothesis, we obtain

$$\kappa(F) + \kappa(Y) \geq \dim Z + \kappa(H) \geq \kappa(X),$$

where the last inequality is the Easy Addition formula applied to h. □

Remark 2.11. One consequence is that in order to establish Conjecture 2.1 for smooth morphisms, it would be enough to show it over all bases Y with $\kappa(Y) = -\infty$ or $\kappa(Y) = 0$. Indeed, when $\kappa(Y) > 0$, we can simply consider a smooth model $g\colon Y' \to Z$ of the Iitaka fibration of Y. Then either Y is of general type and the statement is clear, or the fiber G of g is positive dimensional with $\kappa(G) = 0$, and the lemma applies.

3. The most general conjecture. The strongest proposal about projective morphisms that I would like to make, easily seen to imply all the other conjectures in this note, is the following:

Conjecture 3.1. *If $f : U \to V$ is smooth projective algebraic fiber space between smooth quasi-projective varieties, with general fiber F, then*

$$\kappa(U) = \kappa(F) + \kappa(V).$$

[3] This happens for instance if g is a model of the Iitaka fibration of Y, or if Z is of general type.

In other words, in the presence of smooth morphisms, subadditivity in the log version of Iitaka's conjecture should become *additivity*. As noted in the previous section, this is quite open even when U and V are projective.

There is a mounting body of evidence in favor of this conjecture. To begin with, it is known to hold when V is of log general type, i.e. $\kappa(V) = \dim V$, without any smoothness hypothesis on f; see Remark 3.10 below. Here are some sample recent results on other cases. The first regards base spaces that compactify to abelian varieties:

Theorem 3.2 ([MP21, Theorem A])**.** *Let $f\colon X \to A$ be an algebraic fiber space, with X a smooth projective variety and A an abelian variety. Assume that f is smooth over an open set $V \subseteq A$, and denote $U = f^{-1}(V)$ and the general fiber of f by F. Then $\kappa(U) = \kappa(V) + \kappa(F)$.*

Another is that the conjecture holds when U is of log general type; this is a recent theorem of Park. Extending a result shown in [PS22] in the projective case, he proves the following more general fact:

Theorem 3.3 ([Par22, Theorem 1.5])**.** *In the situation of Conjecture 3.1, assume that $\kappa(F) \geq 0$. Then*

$$V \text{ is of log general type} \iff \kappa(U) = \kappa(F) + \dim V.$$

In particular, if U is of log general type, then V is of log general type.

In [Par22, Theorem 1.10], Park also completes the proof of the conjecture when V is a curve; due to results in [VZ01] and [MP21], the cases that he needs to establish are when V is $\mathbf{P}^1$ minus one or two points.

Moreover, the methods of [PS22] show that the conjecture holds when F is canonically polarized, assuming that $\kappa(V) \geq 0$. However, more generally and quite importantly, Campana has shown that this last assumption can be removed, and furthermore:

Theorem 3.4 ([Cam22, Theorem 1])**.** *Conjecture 3.1 holds when F has semiample canonical bundle.*

Remark 3.5. Campana's proof works for any type of fiber F for which the so-called isotriviality conjecture for fibrations over a special base is known to hold; see *loc. cit.*

We summarize this discussion:

Theorem 3.6. *Conjecture 3.1 holds when:*

(1) V is of log general type [Kaw81], [Mae86].
(2) V compactifies to an abelian variety [MP21].
(3) V is a curve [VZ01],[MP21], [Par22].
(4) U is of log general type [Par22].
(5) F has semiample canonical bundle [Cam22].

Remark 3.7 (**Variation, $C^+_{n,m}$, and the Kebekus-Kovács conjecture**)**.** Recall that Viehweg's $C^+_{n,m}$ conjecture states that, when $\kappa(Y) \geq 0$, one has

$$\kappa(X) \geq \kappa(F) + \max\{\mathrm{Var}(f), \kappa(Y)\}.$$

(See [Mor87, §7] for a survey.) In fact, just as with Iitaka's conjecture, it seems to make sense to ask for a bit more:

Conjecture 3.8. *Let $f : U \to V$ be a projective algebraic fiber space, with U and V smooth quasi-projective varieties and $\kappa(V) \geq 0$. If F is the generic fiber of f, then*

$$\kappa(U) \geq \kappa(F) + \max\{\kappa(V), \mathrm{Var}(f)\}.$$

It is interesting to note that, when combined with this generalized version of $C^+_{n,m}$, Conjecture 2.1 implies the Kebekus-Kovács conjecture [KK08, Conjecture 1.6], i.e the inequality

$$\kappa(V) \geq \mathrm{Var}(f)$$

for a smooth projective fiber space $f \colon U \to V$ with $\kappa(V) \geq 0$.[4]

The log version. Even though the logarithmic Kodaira dimension necessarily made an appearance, all the statements above should be seen as being about the standard case of varieties. Iitaka's conjecture however has more general log analogues; see e.g. [Fuj20] for a general overview. In this direction, a generalization of Conjecture 3.1 is as follows:

Conjecture 3.9. *Let $f \colon X \to Y$ be an algebraic fiber space between smooth projective varieties, and let E be an SNC divisor on X and D an SNC divisor on Y such that* $\mathrm{Supp}(f^*D) \subseteq E$. *Assume that f is log-smooth over $V = Y \smallsetminus D$,*[5] *and let F be a general fiber over a point of V. Then*

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, K_F + E_F).$$

The reader can specialize this statement to various weaker versions, including those discussed earlier in the standard case. For instance, the case of log smooth morphisms to smooth projective varieties corresponds to $D = 0$.

Remark 3.10 (Base of log general type). Due to a result of Kawamata [Kaw81, Theorem 30] for $\kappa(X, K_X + E) \geq 0$, and Maehara [Mae86, Corollary 2] in general, Conjecture 3.9 holds when the base is of log general type, i.e. $\kappa(Y, K_Y + D) = \dim Y$. This is of course a result about Iitaka's conjecture (since the opposite inequality follows from Easy Addition), so it does not require the log-smoothness hypothesis.

4. A conjecture on the Kodaira codimension. A weaker but quite interesting consequence of the conjectures in the previous sections can be phrased in terms of the *Kodaira codimension* of a smooth quasi-projective variety U, defined in [Mor87] as

$$\kappa c(U) := \dim U - \kappa(U).$$

Concretely, at least for smooth fiber spaces, the inequality

$$\kappa(F) + \kappa(V) \geq \kappa(U)$$

predicted as part of Conjecture 3.1 implies the following:

Conjecture 4.1. *If $f : U \to V$ is smooth projective morphism of smooth quasi-projective varieties, then*

$$\kappa c(U) \geq \kappa c(V).$$

[4]When f is the restriction of $f \colon X \to Y$ with $\kappa(Y) \geq 0$, the usual $C^+_{n,m}$ conjecture suffices.

[5]The log-smooth condition means that each stratum of the pair (X, E), including X itself, is smooth over V via f.

I am including this statement because I like its symmetric form, but also because, although weaker, it can still be very useful. For instance, this is what was originally proved in [PS14] when V is an abelian variety. It may also possibly be more approachable in some instances. Note though that while writing the first version of this note, in many significant cases Conjecture 4.1 was known to hold, while the stronger conjectures in the earlier sections were not; however in view of recent work this is not the case anymore.

Remark 4.2. (1) The conjecture holds when f is étale, since then $\kappa(U) = \kappa(V)$. When the dimension of the fibers is positive, one may also assume that f is an algebraic fiber space, as then it is not hard to see that in the Stein factorization the finite morphism is in fact étale.

(2) The conjecture is not true when f is not smooth. For instance, it is well known that there exist smooth projective surfaces S of general type with $q(S) = 1$, so with a surjective morphism to an elliptic curve; such a morphism cannot be smooth. Or consider any base point free pencil on a variety with $\kappa(X) \geq 0$. This induces a morphism $f \colon X \to \mathbf{P}^1$, but such a morphism is never smooth by [VZ01, Theorem 0.2]. There are many other examples.

References

[Cam22] F. Campana, *Kodaira additivity, birational isotriviality and specialness*, preprint arXiv:2207.05412, v.2 (2022).

[CP19] F. Campana and M. Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49.

[Cao18] J. Cao, *Kodaira dimension of algebraic fiber spaces over surfaces*, Algebr. Geom. **5** (2018), no. 6, 728–741.

[CP17] J. Cao and M. Păun, *Kodaira dimension of algebraic fiber spaces over abelian varieties*, Invent. Math. **207** (2017), no. 1, 345–387.

[Fuj20] O. Fujino, *Iitaka conjecture – an introduction*, SpringerBriefs in Mathematics. Springer, Singapore (2020), 128 pp.

[HPS18] C. Hacon, M. Popa, and C. Schnell, *Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun*, Contemporary Mathematics **712** (2018), 143–195.

[Kaw81] Y. Kawamata, *Characterization of abelian varieties*, Compositio Math. **43** (1981), no. 2, 253–276.

[Kaw82] Y. Kawamata, *Kodaira dimension of algebraic fiber spaces over curves*, Invent. Math. **66** (1982), no. 1, 51–71.

[Kaw85] Y. Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.

[KK08] S. Kebekus and S. Kovács, *Families of canonically polarized varieties over surfaces*, Invent. Math. **172** (2008), no. 3, 657–682.

[Kol87] J. Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., North-Holland, Amsterdam **10** (1987), 361–398.

[LPS20] L. Lombardi, M. Popa, and C. Schnell, *Pushforwards of pluricanonical bundles under morphisms to abelian varieties*, J. Eur. Math. Soc. **22** (2020), no. 8, 2511–2536.

[Mae86] K. Maehara, *The weak 1-positivity of direct image sheaves*, J. Reine Angew. Math. **364** (1986), 112–129.

[MP21] F. Meng and M. Popa, *Kodaira dimension of fibrations over abelian varieties*, preprint arXiv:2111.14165 (2021).

[Mor87] S. Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math. **46** (1987), 269–331.

[NU73] I. Nakamura and K. Ueno, *An addition formula for Kodaira dimensions of analytic fibre bundles whose fibre are Moisezon manifolds*, J. Math. Soc. Japan **25** (1973), 363–371.

[Par22] S. G. Park, *Logarithmic base change theorem and smooth descent of positivity of log canonical divisor*, preprint arXiv:2210.02825 (2022).

[PS14] M. Popa and C. Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. **179** (2014), no. 3, 1109–1120.

[PS17] M. Popa and C. Schnell, *Viehweg's hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713.

[PS22] M. Popa and C. Schnell, *On the behavior of Kodaira dimension under smooth morphisms*, preprint arXiv:2202.02825 (2022).

[Taj20] B. Taji, *Birational geometry of smooth families of varieties admitting good minimal models*, preprint arXiv:2005.01025 (2020).

[TZ20] V. Tosatti and Y. Zhang, *Collapsing Hyperkähler manifolds*, Ann. Sci. Éc. Norm. Supér. **53** (2020), no. 3, 751–786.

[Vie83] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension of certain fiber spaces*, Adv. Studies Pure Math. **1** (1983), 329–353.

[VZ01] E. Viehweg and K. Zuo, *On the isotriviality of families of projective manifolds over curves*, J. Algebraic Geom. **10** (2001), 781–799.

[VZ02] E. Viehweg and K. Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin (2002), 279–328.

[WW20] C. Wei and L. Wu, *Isotriviality of smooth families of varieties of general type*, preprint arXiv:2001:08360, to appear in Manuscripta Math. (2020).

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