

ON THE BEHAVIOR OF KODAIRA DIMENSION UNDER SMOOTH MORPHISMS

MIHNEA POPA AND CHRISTIAN SCHNELL

ABSTRACT. We prove several results on the additivity of Kodaira dimension under smooth morphisms of smooth projective varieties.

A. INTRODUCTION

The purpose of this paper is to prove some of the conjectures in [Pop21] about the additivity of the Kodaira dimension under smooth morphisms between smooth projective varieties. Our results are unconditional when varieties of general type are involved in some way, or when the base space is either of dimension ≤ 3 or a good minimal model of Kodaira dimension zero; for other varieties, we need a version of the non-vanishing conjecture due to Campana and Peternell [CP11]. We use the convention that an *algebraic fiber space* is a surjective morphism $f: X \rightarrow Y$ with connected fibers between two smooth projective varieties X and Y (over the field of complex numbers). We usually denote the general fiber by the letter F . We say that an algebraic fiber space is *smooth* if f is a smooth morphism.

A particular case of Conjectures 2.1 or 3.1 in [Pop21] predicts that if $f: X \rightarrow Y$ is a smooth algebraic fiber space, and X is of general type, then Y is of general type as well. Our first result is the following more general statement:

Theorem A. *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space whose general fiber F satisfies $\kappa(F) \geq 0$. Then Y is of general type if and only if $\kappa(X) = \kappa(F) + \dim Y$.*

By the Easy Addition formula [Mor87, Cor. 1.7], we always have the elementary inequality $\kappa(X) \leq \kappa(F) + \dim Y$. Also recall that Iitaka's conjecture, to the effect that $\kappa(X) \geq \kappa(F) + \kappa(Y)$, is known when Y is of general type [Vie83, Cor. IV]. One half of the theorem is therefore already known. Our contribution is to prove the other half: the identity $\kappa(X) = \kappa(F) + \dim Y$ implies that Y is of general type.

Note. Using the same techniques, one can show that if $f: X \rightarrow Y$ is an algebraic fiber space with Y not uniruled, $\kappa(F) \geq 0$, and $\kappa(X) = \kappa(F) + \dim Y$, and if $V \subseteq Y$ denotes the smooth locus of f , then V is of log general type. This answers positively another question suggested by [Pop21], when Y is not uniruled, but is weaker than [Pop21, Conj. 3.1]. See also Remark 1.

Our next result deals with smooth algebraic fiber spaces with canonically polarized fibers.

Theorem B. *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space such that each fiber F is canonically polarized. If $\kappa(Y) \geq 0$, or if Y is uniruled, then we have $\kappa(X) = \kappa(F) + \kappa(Y)$.*

For technical reasons, the case where $\kappa(Y) = -\infty$ and Y is not uniruled is missing; note however that a well-known conjecture in birational geometry, called the *non-vanishing conjecture*, predicts that $\kappa(Y) = -\infty$ is in fact equivalent to Y being uniruled. Thus up to this prediction, our theorem implies the additivity of the Kodaira dimension for any smooth

2020 *Mathematics Subject Classification.* 14D06, 14J10, 14D07.

Key words and phrases. Kodaira dimension; varieties of general type; weak positivity.

algebraic fiber space with fibers of this type. Unlike the other results in this paper, besides hyperbolicity-type techniques, Theorem B also relies on the existence and properties of moduli spaces of varieties of general type. We also observe that essentially the same proof works even when X and Y are only assumed to be quasi-projective, with the base of log Kodaira dimension $\kappa(Y) \geq 0$; see Theorem 2.2.¹

We also prove a more general result that depends on the following conjecture of Campana-Peternell [CP11, Conj. 2.4], which can be seen as a generalization of the non-vanishing conjecture:

Conjecture (Campana-Peternell). *Let Y be a smooth projective variety, and assume that $K_Y \sim_{\mathbb{Q}} A + B$, where A and B are \mathbb{Q} -divisors such that A is effective and B is pseudo-effective. Then $\kappa(Y) \geq \kappa(A)$.*

The non-vanishing conjecture is the case $A = 0$. Concretely, it says that if K_Y is pseudo-effective – which is equivalent to Y not being uniruled [BDPP13] – then $\kappa(Y) \geq 0$. It can be easily checked [CP11, p. 12–13] that the Campana-Peternell conjecture is implied by the existence of good minimal models; it therefore holds if $\dim Y \leq 3$, since this conjecture in the Minimal Model Program (MMP) is known to hold in this range. On the other hand, the Campana-Peternell conjecture is definitely weaker than the full abundance conjecture; in fact, we believe that it is basically equivalent to the non-vanishing conjecture [Sch22].

Theorem C. *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space, and assume that the Campana-Peternell conjecture holds. Then $\kappa(F) + \kappa(Y) \geq \kappa(X)$.*

Note. More precisely, we need the Campana-Peternell conjecture for the fibers of the Iitaka fibration of Y if $\kappa(Y) \geq 0$; or the non-vanishing conjecture for Y itself if $\kappa(Y) = -\infty$.

Iitaka’s conjecture predicts of course the opposite inequality $\kappa(X) \geq \kappa(F) + \kappa(Y)$, for an arbitrary morphism; it is known to be implied by the existence of a good minimal model for the geometric generic fiber of f by a result of Kawamata [Kaw85]. Therefore we obtain the following loosely formulated consequence.

Corollary D. *The conjectures of the MMP imply [Pop21, Conj. 2.5], stating that if $f: X \rightarrow Y$ is a smooth algebraic fiber space, then $\kappa(X) = \kappa(F) + \kappa(Y)$.*

In any event, we obtain unconditional results – and hence the solution to [Pop21, Conj. 2.1] – for algebraic fiber spaces over surfaces and threefolds.

Corollary E. *If $f: X \rightarrow S$ is a smooth algebraic fiber space, with S a smooth projective surface, then $\kappa(X) = \kappa(F) + \kappa(S)$.*

Corollary F. *If $f: X \rightarrow Y$ is a smooth algebraic fiber space, with Y a smooth projective threefold, then $\kappa(F) + \kappa(Y) \geq \kappa(X)$.*

In Corollary E we have equality due to the fact that the Iitaka conjecture is known over surfaces; see [Cao18], and the references therein. This is not yet known when Y is an arbitrary threefold.

Theorem C can be easily reduced to the two cases $\kappa(Y) = -\infty$ and $\kappa(Y) = 0$, and is therefore a consequence of the following two results.

Proposition G. *Let $f: X \rightarrow Y$ be a smooth morphism between smooth projective varieties, with Y uniruled. Then $\kappa(X) = -\infty$.*

¹Note that since this paper was written, Campana [Cam22] has extended this result to morphisms whose fibers have semiample canonical bundle, with no conditions on the base.

Thus the additivity of Kodaira dimension for smooth morphisms over varieties Y with $\kappa(Y) = -\infty$ holds if one assumes the non-vanishing conjecture. The proof of Proposition G is the first step in the proof of Theorem B. It works by reduction to the case $Y = \mathbb{P}^1$, where a stronger version is a deep result of Viehweg-Zuo [VZ01]. Therefore the main new input is the following:

Theorem H. *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space with $\kappa(Y) = 0$.*

- (i) *If the Campana-Peternell conjecture holds on Y , so for instance if Y has a good minimal model, then $\kappa(F) \geq \kappa(X)$.*
- (ii) *If Y is actually a good minimal model, in other words if $K_Y \sim_{\mathbb{Q}} 0$, then $\kappa(X) = \kappa(F)$. Moreover, we have $P_m(F) \geq P_m(X)$ for all $m \geq 1$, with equality if Y is simply connected (e.g. Calabi-Yau).*

Thus [Pop21, Conj. 2.5] holds unconditionally when Y is a good minimal model with $\kappa(Y) = 0$. When Y is an abelian variety, this is a special case of [MP21, Thm. A(2)]. Note for completeness that the conjecture also holds when X is a good minimal model with $\kappa(X) = 0$, since then so is Y by [TZ20].

The proofs of the results in this paper rely on a Hodge module construction from [PS14, PS17], which was in turn heavily inspired by techniques of Viehweg-Zuo in their work on hyperbolicity. Because of the smoothness assumption, semistable reduction and the technical passage to logarithmic Higgs bundles are however not needed. The proof of Theorem H is somewhat more delicate (than for instance that of Theorem A), needing both an additional technical ingredient and important analytic results from [PT18, CP17] (see also [HPS18]).

Remark. The disadvantage of this method is the lack of symmetry between the assumptions and the conclusion. Say $f: X \rightarrow Y$ is a smooth algebraic fiber space, and D a divisor on Y . Roughly speaking, we start from the assumption that $mK_X - f^*D$ is *effective* for some $m \geq 1$; and the conclusion is that $m'K_Y - D$ is *pseudo-effective* for $m' \gg m$. The Campana-Peternell conjecture removes this asymmetry, and allows us to put “pseudo-effective” or “effective” in both places; but it would of course be desirable to have unconditional results.

We conclude by observing that the Campana-Peternell conjecture, together with the techniques above, also leads to the following characterization of $\kappa(Y)$ that can be seen as a generalization of Theorem A.

Corollary I. *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space, with general fiber F satisfying $\kappa(F) \geq 0$. If the Campana-Peternell conjecture holds on X and Y , then*

$$\kappa(Y) = \max\{\kappa(L) \mid mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1\},$$

where the maximum is taken over all line bundles on Y .

Acknowledgements. We thank Dori Bejleri, Gavril Farkas, Fanjun Meng, Sung Gi Park, Valentino Tosatti and Ziquan Zhuang for their comments and suggestions. In particular, Zhuang’s help was crucial for the proof of Theorem B. M.P. is partially supported by NSF grant DMS-2040378. Ch.S. is partially supported by NSF grant DMS-1551677 and by a Simons Fellowship from the Simons Foundation. He thanks the Max-Planck-Institute for providing him with excellent working conditions during his stay in Bonn.

B. PROOFS

1. Proof of Theorem A. We start by considering a general construction. Let A be an ample line bundle on Y . According to [Mor87, Prop. 1.14], the condition that $\kappa(X) = \kappa(F) + \dim Y$ is equivalent to having

$$H^0(X, \omega_X^{\otimes m} \otimes f^*A^{-1}) \neq 0$$

for some $m \geq 1$. Using the projection formula, this is equivalent to the existence of a nontrivial (hence injective) morphism

$$(1.1) \quad L := A \otimes (\omega_Y^{-1})^{\otimes m} \hookrightarrow f_* \omega_{X/Y}^{\otimes m}.$$

Note that by the invariance of plurigena, the sheaf $f_* \omega_{X/Y}^{\otimes m}$ is locally free of rank $P_m(F)$ on all of Y .

We now use Viehweg's well-known fiber product trick, in the following form: let $X^{(m)}$ denote the m -fold fiber product $X \times_Y \cdots \times_Y X$, with its induced morphism $f^{(m)} : X^{(m)} \rightarrow Y$. Note that this is still a smooth algebraic fiber space, with general fiber isomorphic to the m -fold product $F \times \cdots \times F$. Moreover,

$$(1.2) \quad f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m} \simeq (f_* \omega_{X/Y}^{\otimes m})^{\otimes m}$$

for example by [Mor87, Cor. 4.11]. This gives us an inclusion

$$L^{\otimes m} \hookrightarrow f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m}.$$

By replacing X by $X^{(m)}$, and the fiber space f by $f^{(m)}$, we are therefore allowed to assume from the beginning that we are working with a fiber space f such that

$$(1.3) \quad H^0(X, B^{\otimes m}) \neq 0 \quad \text{where} \quad B = \omega_{X/Y} \otimes f^* L^{-1}.$$

Under this assumption, one can bring into play the machinery used in [PS14, PS17], strongly inspired in turn by the celebrated method developed by Viehweg-Zuo [VZ01, VZ02], based on the existence of so-called Viehweg-Zuo sheaves. Note however that, unlike in the works above, the line bundle L we consider here is usually not positive any more.

First, using [PS17, Thm. 2.2] and denoting $\mathcal{A}_Y = \text{Sym } T_Y$, (1.3) implies that there exists a graded \mathcal{A}_Y -module \mathcal{G}_\bullet that is coherent over \mathcal{A}_Y , and has the following among its special properties:

- (i) One has $\mathcal{G}_0 \simeq L$.
- (ii) Each \mathcal{G}_k is torsion-free.
- (iii) There exists a regular holonomic \mathcal{D}_Y -module \mathcal{M} with good filtration $F_\bullet \mathcal{M}$, and an inclusion of graded \mathcal{A}_Y -modules $\mathcal{G}_\bullet \subseteq \text{gr}_\bullet^F \mathcal{M}$.
- (iv) The filtered \mathcal{D} -module $(\mathcal{M}, F_\bullet \mathcal{M})$ underlies a polarizable Hodge module M on Y , with strict support Y , and $F_k \mathcal{M} = 0$ for $k < 0$.

We next follow closely [PS17, §3.2]. The argument is based on considering the chain of \mathcal{O}_Y -module homomorphisms

$$0 \longrightarrow \mathcal{G}_0 \xrightarrow{\theta_0} \mathcal{G}_1 \otimes \Omega_Y^1 \xrightarrow{\theta_1 \circ \text{id}} \mathcal{G}_2 \otimes (\Omega_Y^1)^{\otimes 2} \longrightarrow \cdots$$

obtained from the graded \mathcal{A}_Y -module structure on \mathcal{G}_\bullet and the generalized Kodaira-Spencer maps

$$\theta_k : \mathcal{G}_k \longrightarrow \mathcal{G}_{k+1} \otimes \Omega_Y^1,$$

and originated in work of Kovács and Viehweg-Zuo. Using the identification $\mathcal{G}_0 \simeq L$ and the fact that $\text{Ker}(\theta_k)^\vee$ is a weakly positive sheaf for each k ,² the exact same argument as in the proof of [PS17, Thm. 3.5] shows that at least one of the following two cases must happen:

- (i) The line bundle $L^{-1} = A^{-1} \otimes \omega_Y^{\otimes m}$ is pseudo-effective.
- (ii) There exists a weakly positive sheaf \mathcal{W} on Y and an inclusion

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer $N > 0$.

²This requires the negativity of the kernels of generalized Kodaira-Spencer maps for pure Hodge modules with strict support Y , proved in [PW16] using results by Zuo and Brunebarbe; see the references in *loc. cit.*

In the first case, we immediately get that ω_Y is big, and hence that Y is of general type. In the second case, we argue as follows. We rewrite the above inclusion as

$$(1.4) \quad \mathcal{W} \otimes A \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m},$$

and observe that $\mathcal{W} \otimes A$ is big in the sense of Viehweg [PS17, Lem. 3.2]. Now $\det \Omega_Y^1 = \omega_Y$, hence there exists also a (split) inclusion

$$\omega_Y^{\otimes m} \hookrightarrow (\Omega_Y^1)^{\otimes m \cdot \dim Y}.$$

Putting everything together we deduce the existence of an inclusion

$$\mathcal{H} \hookrightarrow (\Omega_Y^1)^{\otimes M}$$

for some $M > 0$, where \mathcal{H} is a big torsion-free sheaf. The result of Campana-Păun [CP19, Thm. 7.11] implies then that Y is of general type. (Campana and Păun only state the result with \mathcal{H} being a line bundle, but their proof still works when \mathcal{H} has higher rank; see also [Sch17] for a streamlined proof of this important result.)

Remark 1 (The non-smooth case). As mentioned in the introduction, when Y is not uniruled (which we recall implies that ω_Y is pseudo-effective by [BDPP13]) the proof can be extended without much effort to the general case, when f is only smooth over an open set $V \subseteq Y$, to deduce that V is of log general type. By blowing up it is immediate to reduce to the case where $Y \setminus V = D$, a simple normal crossing divisor. The only difference in the argument is that (1.2) is now an isomorphism only away from D , which in practice means that in the proof above we have to replace L by $L(-rD)$ for some integer $r > 0$. The same steps now lead to the bigness of a line bundle of the form

$$\omega_Y^{\otimes a} \otimes \mathcal{O}_Y(bD)$$

for some $a, b > 0$, and the pseudo-effectivity of ω_Y is then needed to deduce the bigness of $\omega_Y(D)$ by multiplying by a suitable multiple of either ω_Y or $\mathcal{O}_Y(D)$. (Compare also with the proof of [PS17, Thm. 4.1].)

Note also that if $\kappa(Y) \geq 0$ and the complement of V in Y has codimension at least 2, it is not hard to check that $\kappa(V) = \kappa(Y)$; see e.g. [MP21, Lem. 2.6]. Thus under this assumption the conclusion is still that Y is of general type.

2. Proof of Theorem B. We divide the proof into a few steps.

Step 1. In this step we deal with the case when Y is uniruled,³ so in particular $\kappa(Y) = -\infty$, by easy reduction to the case $Y = \mathbb{P}^1$, in which case a theorem of Viehweg-Zuo applies. We are in fact proving Proposition G; no assumption on the fibers of f is necessary. See also Remark 2 below for an alternative proof based on the techniques used in Theorem A.

The hypothesis means that there exists a variety Z , which can be assumed to be smooth and projective, and a dominant rational map

$$Z \times \mathbb{P}^1 \dashrightarrow Y.$$

By resolving the indeterminacies of this map, we obtain a generically finite surjective morphism $\varphi: W \rightarrow Y$, such that W admits a morphism $g: W \rightarrow Z$ with general fiber \mathbb{P}^1 . We consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\psi} & X \\ \downarrow \tilde{f} & & \downarrow f \\ W & \xrightarrow{\varphi} & Y \\ \downarrow g & & \\ Z & & \end{array}$$

h (curved arrow from \tilde{X} to Z)

³As mentioned in the introduction, conjecturally this is equivalent to $\kappa(Y) = -\infty$.

where $h = g \circ \tilde{f}$. Since ψ is generically finite we have $\kappa(\tilde{X}) \geq \kappa(X)$ (see [Mor87, Cor. 2.3(i)]), and therefore it suffices to show that $\kappa(\tilde{X}) = -\infty$. Note now that the general fiber H of h admits a smooth morphism (with fiber F) to the general fiber of g , i.e. to \mathbb{P}^1 . By [VZ01, Thm. 2] it follows that $\kappa(H) = -\infty$, and therefore by Easy Addition (see [Mor87, Cor. 1.7]) we also have $\kappa(\tilde{X}) = -\infty$. As promised, this proves Proposition G.

Step 2. In this step we assume that $\kappa(Y) \geq 0$, and show that we can reduce to the case $\kappa(Y) = 0$. First, since the fibers of f are of general type, the subadditivity

$$\kappa(X) \geq \kappa(F) + \kappa(Y)$$

conjectured by Iitaka holds by [Kol87]. To prove Theorem B, it therefore suffices to show

$$(2.1) \quad \kappa(F) + \kappa(Y) \geq \kappa(X).$$

We show the reduction for this inequality.

To this end, we consider the Iitaka fibration $g: Y \rightarrow Z$, which after base change can be assumed to be a morphism, with Z smooth and projective. We again denote $h = g \circ f$, with general fiber H , and we also denote the general fiber of g by G . Thus we have a smooth morphism $H \rightarrow G$, with fiber F , and since $\kappa(G) = 0$, we may assume that $\kappa(F) \geq \kappa(H)$. On the other hand, by definition we have $\dim Z = \kappa(Y)$, and Easy Addition applied to h gives

$$\kappa(H) + \dim Z \geq \kappa(X).$$

Putting the two inequalities together, we obtain (2.1).

Step 3. Under the assumption that the fibers of f are canonically polarized, by Taji [Taj16, Thm. 1.5] we also know the validity of Campana's isotriviality conjecture, and hence the Kebekus-Kovács conjecture, stating that if $\kappa(Y) \geq 0$, then $\kappa(Y) \geq \text{Var}(f)$. Using this in combination with Step 2, we may therefore assume that $\text{Var}(f) = 0$, i.e. that f is birationally isotrivial.⁴

Step 4. In this final step, we show the assertion in the theorem under the assumption that $\text{Var}(f) = 0$ (with no assumptions on Y). This is a consequence of the existence and properties of the moduli space of canonically polarized varieties, as suggested to us by Ziquan Zhuang.

Concretely, the family f induces a morphism

$$\varphi: Y \rightarrow \mathbf{M}$$

to the appropriate coarse moduli space of canonically polarized varieties [Vie95, Kol21]. Since f is birationally isotrivial, it follows that φ is constant on an open set, and therefore constant everywhere; in other words the family f is isotrivial. Since varieties of general type have finite automorphism group (\mathbf{M} is the coarse moduli space of a Deligne-Mumford stack), it is therefore well-known that there is a finite étale base change $\tilde{Y} \rightarrow Y$ such that

$$\tilde{X} := X \times_Y \tilde{Y} \simeq F \times \tilde{Y},$$

where F is any fiber of f . We thus have

$$\kappa(X) = \kappa(\tilde{X}) = \kappa(F) + \kappa(\tilde{Y}) = \kappa(F) + \kappa(Y).$$

This concludes the proof.

Remark 2. Proposition G can be rephrased as saying that if $f: X \rightarrow Y$ is smooth and $\kappa(X) \geq 0$, then K_Y is pseudo-effective. Here is an alternative proof of this fact, resembling that of Theorem A. We consider the construction in the proof of that theorem, only now we

⁴It is worth noting that in this proof, the hyperbolicity-type techniques present in the other results in this paper are hidden in the results from [VZ01] and [Taj16].

take $A = \mathcal{O}_Y$, which is possible because $\kappa(X) \geq 0$. The exact same proof shows then that either we have directly that K_Y is pseudo-effective (case (i)), or that there is an inclusion

$$\mathcal{W} \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m},$$

where \mathcal{W} has pseudo-effective determinant (case (ii)). But [CP19, Thm. 7.6] says precisely that in the latter case K_Y is again pseudo-effective.

Remark 3. In fact, recent work in symplectic geometry can be used to prove the following strengthening of Proposition G: *Let $f: X \rightarrow Y$ be a smooth algebraic fiber space. If K_X is pseudo-effective, then K_Y is also pseudo-effective.* Here is a sketch of the proof. If K_Y is not pseudo-effective, then Y is covered by rational curves. After restricting to a smooth rational curve whose normal bundle has nonnegative degree, we obtain a smooth algebraic fiber space $g: Z \rightarrow \mathbb{P}^1$ such that K_Z is pseudo-effective. But a very recent theorem by Pieloch [Pie21, Thm. 1.1] – answering a question by Starr [Sta15] – says that Z is covered by sections of g , and therefore uniruled. This is a contradiction.

Remark 4 (The quasi-projective case). A very similar proof to that of Theorem B leads in fact to a statement for smooth morphisms between quasi-projective varieties, addressing an analogous case of the more general [Pop21, Conj. 3.1]. We thank Sung Gi Park for pointing this out.

Theorem 2.2. *Let $f: U \rightarrow V$ be a smooth projective algebraic fiber space with canonically polarized fibers. If $\kappa(V) \geq 0$, then we have $\kappa(U) = \kappa(F) + \kappa(V)$.*

Proof. This is the analogue of Steps 2.-4. in the proof of Theorem B.

Since the fibers are of general type, the analogue of Iitaka’s conjecture, namely

$$\kappa(U) \geq \kappa(F) + \kappa(V)$$

holds by [KP17, Thm. 1.3]. Therefore we only need to focus on the opposite inequality. We can then reduce to the case $\kappa(V) = 0$ by using the analogue of the Iitaka fibration and the Easy Addition theorem in the case of open varieties; see [Iit77, Thm. 4 and 5].

Now again by [Taj16, Thm. 1.5], which is also valid in the quasi-projective case, we may therefore assume that $\text{Var}(f) = 0$. Then the exact same argument as in Step 4 of the proof of Theorem B, based on the properties of the moduli space of varieties of general type, shows that the relative canonical model U' of U over V has an étale cover \tilde{U} which is birational to $F \times V$. The statement then follows, since on one hand $\kappa(U) = \kappa(U')$ as U and U' admit proper birational morphisms from a fixed smooth quasi-projective variety W (see [Iit77, §3]), and on the other hand the log Kodaira dimension is preserved under étale covers (see [Iit77, Thm. 3]). \square

Note however that a more general statement, removing the condition on the base, and allowing for fibers with semiample canonical bundle, appeared very recently in [Cam22].

Remark 5. If confirmed, the results of [WW20] would lead to validity of the statement of Theorem 2.2 in the case when the fibers of f are only assumed to be of general type. One would only need to replace f by its relative canonical model, and apply the same argument.

3. Proof of Theorem C. If $\kappa(Y) = -\infty$ the result follows from the non-vanishing conjecture together with Proposition G, while if $\kappa(Y) = 0$, it follows from Theorem H(i). If $\kappa(Y) > 0$, we can reduce to the case $\kappa = 0$ using the standard argument involving the Iitaka fibration: we may assume that this is a morphism $g: Y \rightarrow Z$, with general fiber G satisfying $\kappa(G) = 0$. Therefore the general fiber H of $h = g \circ f$ has a smooth morphism $H \rightarrow G$ with fiber F , and consequently $\kappa(F) \geq \kappa(H)$. On the other hand, Easy Addition for h gives

$$\kappa(H) + \kappa(Y) = \kappa(H) + \dim Z \geq \kappa(X).$$

Hence the remaining point is to prove Theorem H.

4. Proof of Theorem H. If $\kappa(F) = -\infty$, then we also have $\kappa(X) = -\infty$ by Easy Addition. We may therefore assume throughout that $\kappa(F) \geq 0$. The key players are again the vector bundles

$$\mathcal{F}_m := f_*\omega_{X/Y}^{\otimes m}, \quad \text{for } m \geq 1,$$

We also consider their twists

$$\mathcal{E}_m := \mathcal{F}_m \otimes \omega_Y^{\otimes m} \simeq f_*\omega_X^{\otimes m} \quad \text{for } m \geq 1,$$

for which we have

$$r_m := \text{rk}(\mathcal{E}_m) = \text{rk}(\mathcal{F}_m) = P_m(F) \quad \text{and} \quad h^0(Y, \mathcal{E}_m) = P_m(X).$$

We have for each m a (split) inclusion

$$\det \mathcal{F}_m \hookrightarrow \mathcal{F}_m^{\otimes r_m},$$

and therefore using the Viehweg fiber product trick just as in the proof of Theorem A, we obtain an inclusion

$$(\det \mathcal{F}_m)^{\otimes m} \hookrightarrow f_*^{(mr_m)} \omega_{X^{(mr_m)}/Y}^{\otimes m}.$$

Proceeding precisely as in the proof of that theorem (where the role of L there is now played by $\det \mathcal{F}_m$), the arguments from [PS17] lead then to one of the following two possibilities:

- (1) $(\det \mathcal{F}_m)^{-1}$ is pseudo-effective.
- (2) There exists a weakly positive sheaf \mathcal{W} on Y , and an inclusion

$$\mathcal{W} \otimes \det \mathcal{F}_m \hookrightarrow (\Omega_Y^1)^{\otimes N}$$

for some positive integer $N > 0$.

Recall on the other hand that \mathcal{F}_m is a weakly positive sheaf by [Vie83, Thm. III], and therefore $\det \mathcal{F}_m$ is a pseudo-effective line bundle, e.g. by [Vie83, Lem. 1.6 (6)]. If (1) holds, then both $\det \mathcal{F}_m$ and $(\det \mathcal{F}_m)^{-1}$ are pseudo-effective, and so $\det \mathcal{F}_m \equiv 0$ (i.e. it is numerically trivial).

On the other hand, if (2) holds, by taking the saturation of the left hand side inside $(\Omega_Y^1)^{\otimes N}$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{W}' \otimes \det \mathcal{F}_m \longrightarrow (\Omega_Y^1)^{\otimes N} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where \mathcal{W}' is a weakly positive sheaf, and \mathcal{Q} is torsion-free. We also know that $\det \mathcal{Q}$ is pseudo-effective; this follows from [CP19, Thm. 1.3], which says that if K_Y is pseudo-effective, then any quotient of any tensor power of Ω_Y^1 has pseudo-effective determinant. Passing to determinants, we then obtain an isomorphism of the form

$$(4.1) \quad \omega_Y^{\otimes a} \simeq (\det \mathcal{F}_m)^{\otimes b} \otimes P,$$

where $a, b > 0$ and P is a pseudo-effective line bundle.

Remark 6. Up to here we only used that $\kappa(F) \geq 0$ and K_Y is pseudo-effective.

If we assume that $K_Y \sim_{\mathbb{Q}} 0$, from (4.1) we again deduce that $(\det \mathcal{F}_m)^{-1}$ is pseudo-effective, and therefore as above $\det \mathcal{F}_m \equiv 0$. It follows from [CP17, Thm. 5.2] (see also [HPS18, Cor. 27.2]) that \mathcal{F}_m is a hermitian flat vector bundle, and therefore comes from a unitary representation of the fundamental group. If we assume in addition that Y is simply connected, we then obtain $\mathcal{F}_m \simeq \mathcal{O}_Y^{\oplus r_m}$, or equivalently

$$\mathcal{E}_m \simeq (\omega_Y^{\otimes m})^{\oplus r_m}.$$

Note that by hypothesis $\omega_Y^{\otimes m} \simeq \mathcal{O}_X$ when m is sufficiently divisible, and otherwise vanishes. Consequently $P_m(X) = P_m(F)$ when $P_m(Y) = 1$, and $P_m(X) = 0$ otherwise. In general we

only have $r_m \geq h^0(Y, \mathcal{E}_m)$ (because unitary representations are semisimple), and therefore by the same argument we obtain

$$P_m(F) \geq P_m(X) \quad \text{for all } m \geq 1,$$

hence also $\kappa(F) \geq \kappa(X)$. On the other hand, a result of Cao-Păun [CP17, Thm. 5.6] states, for any algebraic fiber space f , that if $c_1(\det \mathcal{F}_m) = 0 \in H^2(X, \mathbb{R})$ for some $m \geq 2$, then Iitaka's conjecture holds for f . Cao and Păun prove this result under the assumption that $c_1(\det \mathcal{F}_m) = 0 \in H^2(X, \mathbb{Z})$, but because of [HPS18, Thm. 27.2], the weaker assumption is sufficient to conclude that \mathcal{F}_m is a hermitian flat bundle; the rest of the argument then proceeds as in [CP17]. In our case, this gives the opposite inequality $\kappa(X) \geq \kappa(F)$, and concludes the proof of (ii).

We are thus left with proving (i). We need the following lemma.

Lemma 4.2. *Let E be a nonzero vector bundle on a projective variety X . If $h^0(X, E) \geq \text{rk } E + 1$, then there is a line bundle L with $h^0(X, L) \geq 2$ and an inclusion $L \hookrightarrow E^{\otimes r}$ for some $1 \leq r \leq \text{rk } E$.*

Proof. Let $\mathcal{F} \subseteq E$ be the torsion-free subsheaf generated by the global sections of E , and let $1 \leq r \leq \text{rk } E$ be the generic rank of \mathcal{F} . Choose r global sections $s_1, \dots, s_r \in H^0(X, E)$ that generate the stalk of \mathcal{F} at a general point of X . The resulting morphism $\mathcal{O}_X^{\oplus r} \rightarrow \mathcal{F}$ is injective; it cannot be surjective, for otherwise it would be an isomorphism, and then $h^0(X, E) = h^0(X, \mathcal{F}) = r \leq \text{rk } E$, contradicting the assumption that $h^0(X, E) \geq \text{rk } E + 1$. Let $x \in X$ be one of the points where the morphism fails to be surjective on stalks. Since \mathcal{F} is globally generated, depending on the situation we can do one of the following two things:

- If s_1, \dots, s_r remain linearly independent at x , we can find an additional global section $s \in H^0(X, E)$ such that $s(x)$ does not lie in the linear span of $s_1(x), \dots, s_r(x)$ inside the stalk E_x . It follows that, among the $r + 1$ sections

$$s_1 \wedge \cdots \wedge s_r, \quad s_1 \wedge \cdots \wedge s_{i-1} \wedge s \wedge s_{i+1} \wedge \cdots \wedge s_r \quad \text{for } i = 1, \dots, r$$

of the bundle $\bigwedge^r E$, at least 2 are linearly independent.

- If s_1, \dots, s_r do not remain linearly independent at x , we can in any case find other global sections $t_1, \dots, t_r \in H^0(X, E)$ such that $t_1(x), \dots, t_r(x) \in E_x$ form part of a minimal system of generators of \mathcal{F}_x . Then

$$s_1 \wedge \cdots \wedge s_r \quad \text{and} \quad t_1 \wedge \cdots \wedge t_r$$

are linearly independent global sections of the bundle $\bigwedge^r E$.

Either way, by construction all these sections come from global sections of the sheaf $\bigwedge^r \mathcal{F}$. Let L be the reflexive hull of $\bigwedge^r \mathcal{F}$; then L is a line bundle with $h^0(X, L) \geq 2$. We also have inclusions

$$L \hookrightarrow \bigwedge^r E \hookrightarrow E^{\otimes r}$$

because E is locally free. □

Let us now suppose for the sake of contradiction that $P_m(X) \geq P_m(F) + 1$. The lemma, applied to the vector bundle \mathcal{E}_m , then gives us an inclusion

$$L \hookrightarrow \mathcal{E}_m^{\otimes r} \simeq \mathcal{F}_m^{\otimes r} \otimes \omega_Y^{\otimes mr}$$

for some integer $1 \leq r \leq r_m$, where L is a line bundle with $h^0(X, L) \geq 2$, and so $\kappa(L) \geq 1$. Arguing as above, using the fiber product trick, we obtain one of the following two possibilities:

(1) $L^{-1} \otimes \omega_Y^{\otimes mr}$ is pseudo-effective.

(2) There exists a weakly positive sheaf \mathcal{W} on Y , and a short exact sequence

$$0 \rightarrow \mathcal{W} \otimes L \rightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes mr} \rightarrow \mathcal{Q} \rightarrow 0$$

for some positive integer $N > 0$.

In case (1), since $\kappa(L) \geq 1$, the Campana-Peternell conjecture implies that $\kappa(Y) \geq 1$, contradicting our assumption that $\kappa(Y) = 0$.

In case (2), as above $\det \mathcal{Q}$ must be a pseudo-effective line bundle by [CP19, Thm. 1.3]. Passing to determinants in the short exact sequence in (2), we now obtain an isomorphism

$$\omega_Y^{\otimes a} \simeq L^{\otimes b} \otimes P$$

where $a, b > 0$ and P is a pseudo-effective line bundle. We then conclude just as in case (1).

5. Proof of Corollary I. Let $f: X \rightarrow Y$ be a smooth algebraic fiber space whose general fiber F satisfies $\kappa(F) \geq 0$. We use the notation

$$\kappa_f(Y) := \max\{\kappa(Y, L) \mid mK_X - f^*L \text{ is pseudo-effective for some } m \geq 1\},$$

and show that $\kappa(Y) = \kappa_f(Y)$ provided that $\kappa(F) \geq 0$. One inequality is easy. Our assumption that $\kappa(F) \geq 0$ implies that the relative canonical divisor $K_{X/Y}$ is pseudo-effective; the reason is that $\omega_{X/Y}$ has a canonical singular hermitian metric with semi-positive curvature [PT18, Thm. 4.2.2] (see also [HPS18, Thm. 27.1]). Since $K_X - f^*K_Y \equiv K_{X/Y}$, we get $\kappa_f(Y) \geq \kappa(Y)$.

To prove the other inequality, let L be a line bundle on Y such that $mK_X - f^*L$ is pseudo-effective, and such that $\kappa_f(Y) = \kappa(Y, L)$. We may assume that $\kappa(Y, L) \geq 0$, and consider an integer $n \geq 1$ such that the linear system $|nL|$ gives the Iitaka fibration of L . After replacing Y by a log resolution of the base locus of $|nL|$, and $f: X \rightarrow Y$ by the fiber product, we can assume that $nL \sim B + E$ with B semi-ample and E effective; consequently, $nmK_X - f^*B \sim n(mK_X - f^*L) + E$ is still pseudo-effective. Moreover, because of the assumption on n we have that $\kappa(Y, B) = \kappa(Y, L)$, hence after replacing L by B we may assume from the beginning that L is semi-ample. The Campana-Peternell conjecture (on X) implies that $mK_X - f^*L$ is effective for $m \gg 0$; this is proved in [Sch22]. We can now apply exactly the same argument as in the proof of Theorem A, up to the line containing (1.4), and obtain a short exact sequence

$$0 \rightarrow \mathcal{W} \otimes L \rightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m} \rightarrow \mathcal{Q} \rightarrow 0$$

for some $N \geq 1$, with \mathcal{W} weakly positive. (Concretely, \mathcal{W} is a torsion-free sheaf with a semi-positively curved singular hermitian metric.) Since $m \geq 1$, both K_Y and $\det \mathcal{Q}$ must therefore be pseudo-effective: this follows by combining [CP19, Thm. 7.6] (to conclude that K_Y is pseudo-effective) and [CP19, Thm. 1.3] (to conclude that any quotient of any tensor power of Ω_Y^1 has pseudo-effective determinant). Passing to determinants in the short exact sequence above, we see that there are two integers $a, b \geq 1$ such that

$$aK_Y \sim bL + \det \mathcal{Q} + \det \mathcal{W}.$$

As both $\det \mathcal{Q}$ and $\det \mathcal{W}$ are pseudo-effective, the Campana-Peternell conjecture (on Y) now gives us the desired inequality $\kappa(Y) \geq \kappa(Y, L) = \kappa_f(Y)$.

REFERENCES

- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.
- [Cam22] Frédéric Campana, *Kodaira additivity, birational isotriviality and specialness*, 2022, preprint arXiv:2207.05412.
- [Cao18] Junyan Cao, *Kodaira dimension of algebraic fiber spaces over surfaces*, Algebr. Geom. **5** (2018), no. 6, 728–741.
- [CP11] Frédéric Campana and Thomas Peternell, *Geometric stability of the cotangent bundle and the universal cover of a projective manifold*, Bull. Soc. Math. France **139** (2011), no. 1, 41–74, With an appendix by Matei Toma.

- [CP17] Junyan Cao and Mihai Păun, *Kodaira dimension of algebraic fiber spaces over abelian varieties*, Invent. Math. **207** (2017), no. 1, 345–387.
- [CP19] Frédéric Campana and Mihai Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, Publ. Math. Inst. Hautes Études Sci. **129** (2019), 1–49.
- [HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell, *Algebraic fiber spaces over abelian varieties: Around a recent theorem by Cao and Păun*, Local and global methods in algebraic geometry, Contemp. Math., vol. 712, Amer. Math. Soc., Providence, RI, 2018, pp. 143–195.
- [Iit77] S. Iitaka, *On logarithmic Kodaira dimension of algebraic varieties*, Complex analysis and algebraic geometry, 1977, pp. 175–189.
- [Kaw85] Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46.
- [Kol87] János Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 361–398.
- [Kol21] ———, *Families of varieties of general type*, book in preparation, 2021.
- [KP17] Sándor J. Kovács and Zolt Patakfalvi, *Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension*, J. Amer. Math. Soc. **30** (2017), no. 4, 959–1021.
- [Mor87] Shigefumi Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331.
- [MP21] Fanjun Meng and Mihnea Popa, *Kodaira dimension of fibrations over abelian varieties*, 2021, preprint arXiv:2111.14165.
- [Pie21] Alex Pieloch, *Sections and unirulings of families over \mathbb{P}^1* , 2021, preprint arXiv:2110.09494v1.
- [Pop21] Mihnea Popa, *Conjectures on the Kodaira dimension*, 2021, preprint arXiv:2111.10900, to appear in the Shokurov 70th birthday volume.
- [PS14] Mihnea Popa and Christian Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. (2) **179** (2014), 1–12.
- [PS17] ———, *Viehweg’s hyperbolicity conjecture for families with maximal variation*, Invent. Math. **208** (2017), no. 3, 677–713.
- [PT18] Mihai Păun and Shigeharu Takayama, *Positivity of twisted relative pluricanonical bundles and their direct images*, J. Algebraic Geom. **27** (2018), no. 2, 211–272.
- [PW16] Mihnea Popa and Lei Wu, *Weak positivity for Hodge modules*, Math. Res. Lett. **23** (2016), no. 4, 1139–1155.
- [Sch17] Christian Schnell, *On a theorem of Campana and Păun*, Épijournal Géom. Algébrique **1** (2017), Art. 8, 9.
- [Sch22] ———, *Singular metrics and a conjecture by Campana and Peternell*, 2022, preprint arXiv:2202.01295.
- [Sta15] Jason Starr, *Does every smooth, projective morphism to $\mathbb{C}P^1$ admit a section?*, MathOverflow, 2015, <https://mathoverflow.net/q/185171> (version: 2017-12-20).
- [Taj16] Behrouz Taji, *The isotriviality of families of canonically-polarized manifolds over a special quasi-projective base*, Compos. Math. **152** (2016), no. 7, 1421–1434.
- [TZ20] Valentino Tosatti and Yuguang Zhang, *Collapsing Hyperkähler manifolds*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 3, 751–786.
- [Vie83] Eckart Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353.
- [Vie95] ———, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 30, Springer-Verlag, Berlin, 1995.
- [VZ01] Eckart Viehweg and Kang Zuo, *On the isotriviality of families of projective manifolds over curves*, J. Algebraic Geom. **10** (2001), no. 4, 781–799.
- [VZ02] ———, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328.
- [WW20] Chuanhao Wei and Lei Wu, *Isotriviality of smooth families of varieties of general type*, 2020, preprint arXiv:2001.08360.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, 1 OXFORD ST., CAMBRIDGE, MA 02138

Email address: `mpopa@math.harvard.edu`

DEPARTMENT OF MATHEMATICS, STONY BROOK UNIVERSITY, STONY BROOK, NY 11794-3651

Email address: `christian.schnell@stonybrook.edu`