

GENERIC VANISHING FILTRATIONS AND PERVERSE OBJECTS IN DERIVED CATEGORIES OF COHERENT SHEAVES

MIHNEA POPA

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1. INTRODUCTION

This is a partly expository paper, intended to explain a very natural relationship between two a priori distinct notions appearing in the literature: *generic vanishing* in the context of vanishing theorems and birational geometry ([GL1], [GL2], [CIH], [Hac], [PP2], [PP4]), and *perverse coherent sheaves* in the context of derived categories ([Ka], [Be], [ABe], [YZ]). Criteria for checking either condition are provided and applied in geometric situations. A few new results, and especially new proofs, are included.

Let X and Y be noetherian schemes of finite type over a field, with $\mathbf{D}(X)$ and $\mathbf{D}(Y)$ denoting their bounded derived categories of coherent sheaves. If X is proper and P is a perfect object in $\mathbf{D}(X \times Y)$, we have an integral (or Fourier-Mukai-type) functor

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(Y), \quad \mathbf{R}\Phi_P(\cdot) := \mathbf{R}p_{Y*}(p_X^*(\cdot) \overset{\mathbf{L}}{\otimes} P).$$

Based on [PP2] and [PP4] one can introduce a Generic Vanishing (GV) filtration on the derived category $\mathbf{D}(X)$, with respect to this functor. Denote $\dim X = d$ and $\dim Y = g$.

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Definition. For every integer m , define the full subcategory of $\mathbf{D}(X)$

$$GV_m(X) := \{A \in \mathbf{D}(X) \mid \text{codim Supp } R^i \Phi_P A \geq i + m \text{ for all } i > 0\}.$$

We say that an object in $GV_m(X)$ *satisfies Generic Vanishing with index m* with respect to P . (The definition depends of course on P , and should rather be $GV_m^P(X)$, but since there will be no danger of confusion I will avoid this to simplify the notation.)

For dimension reasons we have

$$GV_g(X) = GV_{g+1}(X) = \dots = \mathbf{R}\Phi_P^{-1}(\mathbf{D}^{\leq 0}(Y)).$$

On the other hand, the negative indices can go indefinitely, as for $k \geq 0$ we have that $A \in GV_{-k}(X)$ is equivalent to $A[-1] \in GV_{-k-1}(X)$, i.e.

$$GV_{-k}(X) = GV_0(X)[-k], \text{ for all } k \geq 0.$$

(This is not the case for $k < 0$.) In conclusion the integral functor $\mathbf{R}\Phi_P$ induces a *Generic Vanishing filtration* on $\mathbf{D}(X)$ given by

$$\begin{aligned} \mathbf{R}\Phi_P^{-1}(\mathbf{D}^{\leq 0}(Y)) &= \dots = GV_g(X) \subset \dots \subset GV_1(X) \subset GV_0(X) =: GV(X) \subset \\ &\subset GV_{-1}(X) \subset \dots \subset GV_{-d}(X) \subset GV_{-d-1}(X) \subset \dots \end{aligned}$$

In §2 I compare this with a cohomological support loci filtration appearing in vanishing theorems.

By definition, the images of objects in the central component $GV(X)$ correspond via $\mathbf{R}\Phi_P$ to the negative component of a perverse t -structure on $\mathbf{D}(Y)$ introduced by Bezrukavnikov [Be] (following Deligne; cf. also [ABe]) and Kashiwara [Ka],¹ Grothendieck dual to the standard t -structure. I include in §3 a self-contained proof of the existence of this t -structure, following methods in [Ka] and [PP4], and identify the preimage of its positive part in $\mathbf{D}(X)$ via $\mathbf{R}\Phi_P$ as well. Perverse sheaves correspond via this functor to what will be called *geometric GV objects*, and furthermore via duality to objects satisfying the Weak Index Theorem with index d (WIT_d) with respect to a related kernel; see Theorem 3.8. (The standard example of such an object is the canonical bundle ω_X of a variety X with generically finite Albanese map, according to a fundamental theorem of Green-Lazarsfeld [GL1].) In case $\mathbf{R}\Phi_P$ is a Fourier-Mukai equivalence, this means that the subcategories of geometric GV objects and WIT_d objects in $\mathbf{D}(X)$ are hearts of natural t -structures.

The main point of the paper is that there are explicit cohomological criteria for detecting all of the individual components of the Generic Vanishing filtration, or the abelian category of perverse coherent sheaves and the components of a refined filtration on it. It is shown in §4 and §5 how $GV_m(X)$ can be characterized either in terms of the vanishing of certain hypercohomology (for $m \leq 0$), or in terms of commutative algebra (for $m > 0$). While quite basic theoretically, these criteria have concrete geometric applications which provided the initial motivation for undertaking this study. Below is a brief explanation.

¹This is one of many perverse t -structures, each depending on the choice of a perversity function. The main result of [Be] and [Ka] is to describe *all* functions that define such t -structures. Here the notion of perversity corresponds only to this particular t -structure (cf. Definition 3.4), which is very geometric in flavor, but only a sliver of the algebraic picture. It is an interesting question whether there exist natural geometric objects satisfying generic vanishing conditions associated to other perversity functions, especially since a few such functions have recently appeared in the study of Donaldson-Thomas invariants [Ba], [To1], [To2] or stability conditions [Me]. A different t -structure, roughly speaking associated to generic vanishing for \mathcal{D} -modules, will appear in [PS].

I characterize in §4 the negative part of the GV filtration, when Y is projective, by the vanishing of suitable hypercohomology groups. This is a slight generalization of the main technical results in [PP2]. These were in turn inspired by the approach to the Green-Lazarsfeld results for the canonical bundle in Hacon’s paper [Hac], where derived category techniques made their first appearance in the study of generic vanishing. The proof given here clarifies the treatment in [PP2]: after applying Grothendieck Duality to the integral functor, it emphasizes a distinct local part having to do with Kashiwara’s characterization of the perverse t -structure on $\mathbf{D}(Y)$, and a global one (for Y projective) which is a simple cohomological characterization of perverse sheaves.

Section 5 deals with the positive part of the filtration. It contains results that were announced (and proved for the correspondence between a smooth projective variety X and $\text{Pic}^0(X)$) in [PP4]. One introduces a filtration on the abelian category of perverse coherent sheaves given by a variant of Serre’s condition S_k . Due to a criterion of Auslander-Bridger [AB], this corresponds via $\mathbf{R}\Phi_P$ to the GV_m filtration with $m > 0$, and in the case of finite homological dimension with a filtration by syzygy conditions as well. In this last case, one can apply the Evans-Griffith Syzygy Theorem to bound the rank of the dual of a non-locally free perverse sheaf. Significantly, this rank is sometimes a standard invariant of X , like its holomorphic Euler characteristic $\chi(\omega_X)$ in the case of the Fourier-Mukai transform of ω_X . The homological commutative algebra needed here, as well as in §3, is reviewed in an Appendix in §7.

In §6 I present geometric applications of the two main criteria characterizing components of the GV filtration. The hypercohomology vanishing in Theorem 4.1 can often be checked in practice by reducing it to standard Kodaira-Nakano-type vanishing theorems. On projective varieties this accounts for essentially all known extensions of the generic vanishing theorems of [GL1] (cf. §6.1 and §6.2 – as briefly explained there, this whole circle of ideas will be generalized using M. Saito’s vanishing theorems for mixed Hodge modules in the upcoming [PS].) It can also be applied in connection with problems related to moduli spaces of vector bundles (cf. §6.5) and to cohomology classes on abelian varieties (cf. §6.4). On the other hand, the syzygy criterion in Theorem 5.3 is applied in conjunction with the Evans-Griffith theorem as explained in the previous paragraph, in situations when the cohomological support loci are known to be small, to the study of irregular varieties (cf. §6.3).

At the moment there is an obvious disconnect between the theoretical results in §3-5, which work in a general setting, and the applications in §6 and elsewhere, which are almost all in the context of the integral functor induced by a universal line bundle on $X \times \text{Pic}^0(X)$, for a smooth projective X . While this is not very restrictive from the point of view of birational geometry, it is natural to expect generic vanishing phenomena (or, equivalently, interesting perverse sheaves) on other moduli spaces, and also on some spaces with singularities. I give a few such examples in §6.5, but full results in this direction are still to be discovered, the main obstruction being the current poor understanding of moduli spaces of sheaves on varieties of dimension three or higher.² Hence overall this material has two rather distinct aspects: characterizing filtrations defined by support conditions in a formal setting on one hand, and applying this to the concrete geometric study of generic vanishing for the Picard variety (or other parameter spaces to a lesser extent) on the other. Work in

²More precisely, from the present perspective one needs a good understanding of the Fourier-Mukai transforms of their determinant line bundles.

the two directions has been done by somewhat disjoint groups of people. I hope this note indicates that the results and methods involved often overlap or are extremely similar, and will serve as an introduction to the geometric aspects for those more algebraically inclined and viceversa.

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2. PRELIMINARIES AND EXAMPLES

Integral functors. The technical results will be proved for noetherian schemes of finite type over a field.³ Given any such X , we denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves on X . For any other Y of this type, and P a perfect object in $\mathbf{D}(X \times Y)$ (or more generally of finite tor-dimension over X and Y), if X is proper we have an integral functor

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(Y), \quad \mathbf{R}\Phi_P(\cdot) := \mathbf{R}p_{Y*}(p_X^*(\cdot) \otimes^{\mathbf{L}} P).$$

If Y is also proper, we have the analogous

$$\mathbf{R}\Psi_P : \mathbf{D}(Y) \rightarrow \mathbf{D}(X), \quad \mathbf{R}\Psi_P(\cdot) := \mathbf{R}p_{X*}(p_Y^*(\cdot) \otimes^{\mathbf{L}} P).$$

According to standard terminology, an object A in $\mathbf{D}(X)$ is said to satisfy WIT_j , i.e. the Weak Index Theorem with index j , if $R^i\Phi_P A = 0$ for all $i \neq j$.

The dual of an object A in $\mathbf{D}(X)$ is $A^\vee := \mathbf{R}\mathcal{H}om(A, \mathcal{O}_X)$. When X is Cohen-Macaulay, we also denote $\mathbf{R}\Delta A := \mathbf{R}\mathcal{H}om(A, \omega_X)$, where ω_X is the dualizing sheaf of X . When X is in addition projective, Grothendieck duality takes the following form (cf. [PP2] Lemma 2.2):

Lemma 2.1. *Assume that X is Cohen-Macaulay and projective, of dimension d . Then for any A in $\mathbf{D}(X)$, the Fourier-Mukai and duality functors satisfy the exchange formula*

$$(\mathbf{R}\Phi_P A)^\vee \cong \mathbf{R}\Phi_{P^\vee}(\mathbf{R}\Delta A)[d].$$

t -structures. Recall the following well-known

Definition 2.2. Let \mathbf{D} be a triangulated category. A t -structure on \mathbf{D} is a pair of full subcategories $(\mathbf{D}^{\leq 0}, \mathbf{D}^{\geq 0})$, with

$$\mathbf{D}^{\leq n} := \mathbf{D}^{\leq 0}[-n] \text{ and } \mathbf{D}^{\geq n} := \mathbf{D}^{\geq 0}[-n], \text{ satisfying}$$

(a) $\mathbf{D}^{\leq 0} \subset \mathbf{D}^{\leq 1}$ and $\mathbf{D}^{\geq 1} \subset \mathbf{D}^{\geq 0}$.

³ Note that all of the results in §3–6, except for those in §6 involving the Evans-Griffith syzygy theorem, hold more generally for schemes of finite type over the spectrum of a ring with dualizing complex. The reader will also note that many of the results hold, with appropriate formulations, for complex analytic spaces.

(b) $\mathrm{Hom}_{\mathbf{D}}(A, B) = 0$, for all $A \in \mathbf{D}^{\leq 0}$ and $B \in \mathbf{D}^{\geq 1}$.

(c) For any $X \in \mathbf{D}$ there exists a triangle

$$A \rightarrow X \rightarrow B \rightarrow A[1]$$

with $A \in \mathbf{D}^{\leq 0}$ and $B \in \mathbf{D}^{\geq 1}$.

The category $\mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$ is called the *heart* (or *core*) of the t -structure; it is an abelian category. The main example is of course, for a scheme X as above, the *standard t -structure* on $\mathbf{D}(X)$ given by $(\mathbf{D}^{\leq 0}(X), \mathbf{D}^{\geq 0}(X))$. Its heart is $\mathrm{Coh}(X)$. Another important t -structure will appear in §3.

Cohomological support loci. Generic vanishing conditions were originally given in terms of cohomological support loci, so it is natural to compare the definition of a GV_m -object A in the Introduction with the condition that the i -th cohomological support locus of A has codimension $\geq i + m$. For any $y \in Y$ we denote $P_y = \mathbf{L}i_y^*P$ in $\mathbf{D}(X_y) = \mathbf{D}(X)$, where $i_y : X_y = X \times \{y\} \hookrightarrow X \times Y$ is the inclusion.

Definition 2.3. Given an object A in $\mathbf{D}(X)$, the i -th cohomological support locus of A with respect to P is

$$V_P^i(A) := \{y \in Y \mid H^i(X_y, A \otimes^{\mathbf{L}} P_y) \neq 0\}.$$

Although $\mathrm{Supp} R^i\Phi_P A$ and $V_P^i(A)$ are in general different, they carry the same numerical information in the following sense:

Lemma 2.4. *For every $m \in \mathbb{Z}$, the following conditions are equivalent:*

- (1) A is a GV_m -object with respect to P .
- (2) $\mathrm{codim} V_P^i(A) \geq i + m$ for all i .

Proof. I reproduce the proof in [PP2] Lemma 3.6, for the sake of completeness. Since by cohomology and base change (for the hypercohomology of bounded complexes – cf. [EGA III] 7.7, especially 7.7.4, and Remarque 7.7.12(ii)) we have that $\mathrm{Supp} R^i\Phi_P A \subseteq V_P^i(A)$, it is enough to prove that (1) implies (2). The proof is by descending induction on i . There certainly exists an integer s such that $H^j(X, A \otimes^{\mathbf{L}} P_y) = 0$ for any $j > s$ and for any $y \in Y$. Then, by base change, $\mathrm{Supp} R^s\Phi_P A = V_P^s(A)$. The induction step is as follows: assume that there is a component \bar{V} of $V_P^i(A)$ of codimension less than $i + m$. Since (1) holds, the generic point of \bar{V} cannot be contained in $\mathrm{Supp} R^i\Phi_P A$ and so, again by base change, we have that $\bar{V} \subset V_P^{i+1}(A)$. This implies that $\mathrm{codim} V_P^{i+1}(A) < i + m$, which contradicts the inductive hypothesis. \square

Examples. Here are some basic examples related to the Generic Vanishing filtration. Many more examples will appear in §6.

- (1) The pioneering result on generic vanishing is the following theorem of Green-Lazarsfeld, [GL1] Theorem 1: if X is a smooth projective variety and the general fiber of its Albanese map has dimension k , then $\omega_X \in GV_{-k}(X)$. This is of course with respect to the integral functor $\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(\mathrm{Pic}^0(X))$ given by a normalized Poincaré bundle on $X \times \mathrm{Pic}^0(X)$. If $k = 0$ we will see below criteria for when $\omega_X \in GV_m(X)$ for some $m > 0$.

(2) Let C be a smooth projective curve of genus $g \geq 2$, and P a Poincaré bundle on $C \times \text{Pic}^0(C)$. It is a simple exercise to check that if L is a line bundle of degree d on C , then $V^i(L) = \emptyset$ for $i \geq 2$ and

$$V^1(L) \cong W_{2g-2-d}(C) \subset \text{Pic}^{2g-2-d}(C),$$

the image of the Abel-Jacobi map of the $(2g - 2 - d)$ -th symmetric product of C . This has dimension $2g - 2 - d$ for $d \geq g - 1$. Hence the restriction of the Generic Vanishing filtration to $\text{Pic}(C)$ looks as follows

$$GV_g^*(C) \subset \dots \subset GV_1^*(C) \subset GV_0^*(C) \subset GV_{-1}^*(C)$$

where $GV_m^*(C) := GV_m(C) \cap \text{Pic}(C)$ is equal to the whole of $\text{Pic}(C)$ for $m = -1$ and to $\text{Pic}^{\geq g-1+m}(C)$ for $m \geq 0$.

As a side remark, note that one can similarly introduce more refined loci

$$V_p^1(L) = \{\alpha \in \text{Pic}^0(C) \mid h^1(C, L \otimes \alpha) \geq p\}.$$

As we vary d and p , these correspond to all Brill-Noether loci $W_d^r(C)$ (see e.g. [ACGH] Ch.IV, §3), which can be quite complicated. An understanding of the pieces of the Generic Vanishing filtration on $\mathbf{D}(C)$, and of suitable refinements, could then be seen as a broad generalization of Brill-Noether theory to arbitrary coherent sheaves, or even objects in the derived category. The same goes for $\mathbf{D}(X)$ for higher dimensional X , where even the line bundle picture is quite mysterious.

3. COMPARISON WITH A BEZRUKAVNIKOV-KASHIWARA PERVERSE t -STRUCTURE

Let Y be a Cohen-Macaulay scheme of finite type over a field. Bezrukavnikov [Be] and Kashiwara [Ka] have introduced a t -structure on $\mathbf{D}(Y)$ which corresponds to the standard t -structure via taking derived duals, as part of a general procedure of defining perverse t -structures on the bounded derived category of coherent sheaves.⁴ The exposition here is closer to that of [Ka]. Explicitly, define

$${}^p\mathbf{D}^{\leq 0}(Y) := \{A \in \mathbf{D}(Y) \mid \text{codim Supp } \mathcal{H}^i A \geq i \text{ for all } i \geq 0\} \text{ and}$$

$${}^p\mathbf{D}^{\geq 0}(Y) := \{A \in \mathbf{D}(Y) \mid \mathcal{H}_Z^i(A) = 0, \forall Z \subset X \text{ closed with } \text{codim } Z > i\}.$$

Here $\mathcal{H}_Z^i(\cdot)$ denotes local hypercohomology with support in Z . The following was proved in [Ka] Proposition 4.3 in the smooth case (cf. also [Be] Lemma 5, in a more general setting):

Proposition 3.1. (1) *The pair $({}^p\mathbf{D}^{\leq 0}(Y), {}^p\mathbf{D}^{\geq 0}(Y))$ is a t -structure on $\mathbf{D}(Y)$.*

(2) *We have $\mathbf{R}\Delta(\mathbf{D}^{\geq 0}(Y)) = {}^p\mathbf{D}^{\leq 0}(Y)$ and $\mathbf{R}\Delta(\mathbf{D}^{\leq 0}(Y)) = {}^p\mathbf{D}^{\geq 0}(Y)$.*

Regarding (2), the proof in [Ka] shows the inclusions from left to right. Together with the fact that the pair $({}^p\mathbf{D}^{\leq 0}(Y), {}^p\mathbf{D}^{\geq 0}(Y))$ is a t -structure (cf. [Ka] Theorem 3.5), this formally implies that they are both equalities. To keep the exposition self-contained, I give below direct proofs for both equalities in (2) for Y as above, using arguments from [PP4] §2 and [Ka] §4 plus some consequences of duality for local cohomology listed in Corollary 7.2. It implies that (1) also holds, by transfer of the standard t -structure via duality. The

⁴In fact Bezrukavnikov works under slightly more general, and Kashiwara under slightly more restrictive hypotheses.

two equalities are treated separately in the following two Propositions, stated slightly more generally.

Proposition 3.2. *For every $k \geq 0$ we have $\mathbf{R}\Delta(\mathbf{D}^{\geq -k}(Y)) = {}^p\mathbf{D}^{\leq k}(Y)$, i.e. for $A \in \mathbf{D}(Y)$*

$$\text{codim Supp } \mathcal{H}^i A \geq i - k \text{ for all } i > 0 \iff R^i \Delta A = 0 \text{ for all } i < -k.$$

Proof. Since the condition $\text{codim Supp}(\mathcal{H}^i A) \geq i - k$ is non-trivial only for $i \geq k$, by replacing A by $A[k]$ we can reduce to the case $k = 0$.

Start first with A such that $\text{codim Supp}(\mathcal{H}^i A) \geq i$ for all i . We have a spectral sequence

$$E_2^{pq} := \mathcal{E}xt^p(\mathcal{H}^q A, \omega_Y) \Rightarrow R^{p-q} \Delta A.$$

In other words, to converge to $R^i \Delta A$, we start with the E_2 -terms $\mathcal{E}xt^{i+j}(\mathcal{H}^j A, \omega_Y)$. Since $\text{codim Supp } \mathcal{H}^j A \geq j$ for all j , by Corollary 7.2(a) we get $\mathcal{E}xt^{i+j}(\mathcal{H}^j A, \omega_Y) = 0$ for all $i < 0$, which by the spectral sequence implies that $R^i \Delta A = 0$ for $i < 0$.

Now start with A such that $R^i \Delta A = 0$ for $i < 0$. Since $\mathbf{R}\Delta$ is an involution on $\mathbf{D}(Y)$, we have a spectral sequence

$$E_2^{pq} := \mathcal{E}xt^p(R^q \Delta A, \omega_Y) \Rightarrow \mathcal{H}^{p-q} A.$$

In other words to converge to $\mathcal{H}^i A$, we start with the E_2 -terms $\mathcal{E}xt^{i+j}(R^j \Delta A, \omega_Y)$. But Corollary 7.2(b) implies that for all j we have

$$\text{codim Supp } \mathcal{E}xt^{i+j}(R^j \Delta A, \omega_Y) \geq i + j.$$

Since by hypothesis we must have $j \geq 0$, this implies that the codimension of the support of all E_2 -terms is at least i . But then chasing through the spectral sequence this immediately implies that $\text{codim Supp } \mathcal{H}^i A \geq i$. \square

Proposition 3.3. *For every $k \geq 0$ we have $\mathbf{R}\Delta(\mathbf{D}^{\leq -k}(Y)) = {}^p\mathbf{D}^{\geq k}(Y)$, i.e. for $A \in \mathbf{D}(Y)$*

$$\mathcal{H}_Z^i(A) = 0, \text{ for all } Z \subset X \text{ closed with } \text{codim } Z > i - k \iff R^i \Delta A = 0 \text{ for all } i > -k.$$

Proof. One can again reduce to the case $k = 0$ by shifting by k . Let A such that $R^i \Delta A = 0$ for $i > 0$. Denote $B = \mathbf{R}\Delta A$. For any closed Z , we use the derived local duality isomorphism

$$\mathbf{R}\Gamma_Z(A) \cong \mathbf{R}\mathcal{H}om(B, \mathbf{R}\Gamma_Z(\omega_Y)).$$

There is a (double) spectral sequence computing the right hand side, namely

$$\mathcal{E}xt^p(H^i B, \mathcal{H}_Z^j(\omega_Y)) \Rightarrow R^{p-i+j} \mathcal{H}om(B, \mathbf{R}\Gamma_Z(\omega_Y)).$$

Now $H^i B = 0$ for $i > 0$, while $\mathcal{H}_Z^j(\omega_Y) = 0$ for $j < \text{codim } Z$ by Corollary 7.2(a). This implies that $\mathcal{H}_Z^i(A) = 0$ for $i < \text{codim } Z$ as well.

Let now A such that $\mathcal{H}_Z^j(A) = 0$ for any closed Z such that $j < \text{codim } Z$. By [Ka] Proposition 4.6, this is equivalent to the fact that A can be represented by a bounded complex F^\bullet of locally free \mathcal{O}_Y -modules in non-negative degrees. But then we have a spectral sequence with E_2 -terms $\mathcal{E}xt^{i+j}(F^j, \omega_Y)$ converging to $R^i \Delta A$. Now $j \geq 0$, hence $\mathcal{E}xt^{i+j}(F^j, \omega_Y) = 0$ for $i > 0$, since the F^j are flat. This implies that $R^i \Delta A = 0$ for $i > 0$. \square

Definition 3.4. Coherent perverse t -structures were defined and studied in general in [Be] Theorem 1 and [Ka] Theorem 5.9, by means of perversity (or support) functions. The particular t -structure in Proposition 3.1 corresponds to the perversity function

$$p : \{0, \dots, g\} \rightarrow \mathbb{Z}, p(m) = g - m,$$

where $g = \dim Y$ (or equivalently $p' : Y \rightarrow \mathbb{Z}, p'(y) = \dim \mathcal{O}_{Y,y}$). It is further studied via rigid dualizing complexes in a more general context in [YZ], where it is called the *rigid perverse t -structure*. A *perverse coherent sheaf* on Y is an object in the heart of the t -structure $({}^p\mathbf{D}^{\leq 0}(Y), {}^p\mathbf{D}^{\geq 0}(Y))$. We denote these by

$$\text{Per}(Y) = {}^p\mathbf{D}^{\leq 0}(Y) \cap {}^p\mathbf{D}^{\geq 0}(Y).$$

By Proposition 3.1 we have simply that $\text{Per}(Y) = \mathbf{R}\Delta(\text{Coh}(Y))$.

Comparison via integral functors. Let now in addition X be a Cohen-Macaulay scheme, projective over k , and let P be a perfect object in $\mathbf{D}(X \times Y)$. The goal is to use the above discussion after applying the functor $\mathbf{R}\Phi_P$ to objects in $\mathbf{D}(X)$. When Y is projective, the next result is the equivalence of (1) and (3) in [PP2] Theorem 3.7. In general it was stated (in the smooth case) in [PP4] Theorem 5.2 – the proof was given there only in a special case, but the method is similar. Denote

$$Q := P^\vee \otimes p_Y^* \omega_Y.$$

Corollary 3.5. *Let $A \in \mathbf{D}(X)$ and $k \geq 0$. Then*

$$A \in GV_{-k}(X) \iff \mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \in \mathbf{D}^{\geq d-k}(Y).$$

Proof. First, it is clear by definition that for $k \geq 0$ we have

$$A \in GV_{-k}(X) \iff \mathbf{R}\Phi_P A \in {}^p\mathbf{D}^{\leq k}(Y).$$

This result is then simply a reinterpretation of the equivalence in Proposition 3.2 applied to $\mathbf{R}\Phi_P A$, via duality. Indeed, by Lemma 2.1 we have that $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \in \mathbf{D}^{\geq d-k}(Y)$ if and only if $(\mathbf{R}\Phi_{Q^\vee} A)^\vee \in \mathbf{D}^{\geq -k}(Y)$. But the projection formula implies

$$(\mathbf{R}\Phi_{Q^\vee} A)^\vee \cong \mathbf{R}\Delta(\mathbf{R}\Phi_P A).$$

□

Example 3.6 (The Fourier transform of the canonical bundle). In the situation of Example (1) in §2, we saw that $\omega_X \in GV_{-k}(X)$ (with respect to a Poincaré bundle on $X \times \text{Pic}^0(X)$). On the other hand, by dimension reasons we have $R^i \Phi_{P^\vee} \mathcal{O}_X = 0$ for $i > d$. This means that when X is of maximal Albanese dimension (i.e. $k = 0$), $\omega_X \in GV(X)$ and equivalently by Corollary 3.5, $\mathbf{R}\Phi_{P^\vee} \mathcal{O}_X$ is a sheaf supported in degree d (see Theorem 6.1 below for a more general statement). In other words, $\mathbf{R}\Phi_P \omega_X$ is a perverse sheaf on $\text{Pic}^0(X)$.

In concrete geometric applications it is often the case as in the Example above that P is a sheaf, flat over Y , and $\mathbf{R}\Delta A \in \mathbf{D}^{\leq 0}(X)$ (usually a sheaf). By basic properties of push-forwards we then have $R^i \Phi_Q(\mathbf{R}\Delta A) = 0$ for all $i > d$, as d is the dimension of the fibers of the second projection. The GV such objects deserve a special name, as they are intimately related to perverse sheaves in $\mathbf{D}(Y)$ via $\mathbf{R}\Phi_P$.

Definition 3.7. A *geometric GV -object* on X is an object $A \in GV(X)$ such that

$$R^i \Phi_Q(\mathbf{R}\Delta A) = 0 \text{ for all } i > d.$$

The conclusion of the discussion in this section can be summarized in the following result. With respect to [PP2] I am also adding the new observation that under reasonable assumptions (for example when X and Y are smooth and P lives in non-negative degrees) the GV -objects in non-negative degrees are always geometric.⁵ The picture will be expanded in the next two sections.

Theorem 3.8. *Let X, Y and P be as above. For $A \in \mathbf{D}(X)$, the following are equivalent:*

- (1) A is a geometric GV -object.
- (2) $\mathbf{R}\Phi_P A$ is a perverse coherent sheaf.
- (3) $\mathbf{R}\Delta A$ satisfies WIT_d with respect to $Q = P^\vee \otimes p_Y^* \omega_Y$.

Assuming in addition that $P \otimes p_X^* \omega_X \in {}^p\mathbf{D}^{\geq 0}(X \times Y)$ (i.e. it can be represented by a complex of locally free sheaves in non-negative degrees),

$$\mathbf{R}\Phi_P(GV(X) \cap \mathbf{D}^{\geq 0}(X)) \subset \text{Per}(Y)$$

i.e. GV -objects in non-negative degrees are geometric.

Proof. By definition we have $\mathbf{R}\Phi_P^{-1}({}^p\mathbf{D}^{\leq 0}(Y)) = GV(X)$. On the other hand, by Corollary 3.5, this is equivalent to $R^i\Phi_Q(\mathbf{R}\Delta A) = 0$ for all $i < d$. For the other half, note that the identity in Proposition 3.3 applied to $\mathbf{R}\Phi_P A$ implies after shifting by k that

$$\mathbf{R}\Phi_P^{-1}({}^p\mathbf{D}^{\geq 0}(Y)) = \{A \in \mathbf{D}(X) \mid \mathbf{R}\Delta(\mathbf{R}\Phi_P A) \in \mathbf{D}^{\leq 0}(Y)\}.$$

But note that the formula at the end of the proof of Corollary 3.5, combined with Lemma 2.1, implies

$$\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \cong \mathbf{R}\Delta(\mathbf{R}\Phi_P A)[-d]$$

which gives

$$\mathbf{R}\Phi_P^{-1}({}^p\mathbf{D}^{\geq 0}(Y)) = \{A \in \mathbf{D}(X) \mid R^i\Phi_Q(\mathbf{R}\Delta A) = 0 \text{ for } i > d\}.$$

Finally, the last statement in the Theorem uses part (2) of Proposition 3.9 below. \square

Proposition 3.9. (1) *If $P \in \mathbf{D}^{\leq 0}(X \times Y)$, then $\mathbf{R}\Phi_P({}^p\mathbf{D}^{\leq 0}(X)) \subset \mathbf{D}^{\leq d}(Y)$.*

(2) *If $P \otimes p_X^* \omega_X \in {}^p\mathbf{D}^{\geq 0}(X \times Y)$, then $\mathbf{R}\Phi_P(\mathbf{D}^{\geq 0}(X)) \subset {}^p\mathbf{D}^{\geq 0}(Y)$.*

Proof. (1) Let $A \in {}^p\mathbf{D}^{\leq 0}(X)$. We have a spectral sequence

$$E_2^{pq} := R^p\Phi_P(\mathcal{H}^q A) \Rightarrow R^{p+q}\Phi_P A.$$

By definition we have $\dim \text{Supp } \mathcal{H}^q A \leq d - q$ for all q . Since P is supported only in negative degrees, we get by basic properties of push-forwards that $R^p\Phi_P(\mathcal{H}^q A) = 0$ for all $p > d - q$, which implies the result.

(2) Let $A \in \mathbf{D}^{\geq 0}(X)$. By Proposition 3.2 we have $\mathbf{R}\Delta A \in {}^p\mathbf{D}^{\leq 0}(X)$. Since $\omega_Y \otimes p_Y^\vee \cong \mathcal{O}_Y$, by Proposition 3.3 we have $Q \in \mathbf{D}^{\leq 0}(X \times Y)$. By (1) we then have $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \in \mathbf{D}^{\leq d}(Y)$. But we noted in the proof of the Theorem that $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \cong \mathbf{R}\Delta(\mathbf{R}\Phi_P A)[-d]$. We deduce $\mathbf{R}\Delta(\mathbf{R}\Phi_P A) \in \mathbf{D}^{\leq 0}(Y)$, which by Proposition 3.3 is equivalent to $\mathbf{R}\Phi_P A \in {}^p\mathbf{D}^{\geq 0}(Y)$. \square

⁵I thank C. Schnell for help with this issue.

Remark 3.10 (Gorenstein assumption). When Y is Gorenstein, the right hand side condition in Corollary 3.5, the definition of a geometric GV -object, and condition (3) in Theorem 3.8 can all be phrased in terms of the simpler looking $\mathbf{R}\Phi_{P^\vee}(\mathbf{R}\Delta A)$ instead of $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A)$. Since this will always be the case in applications, we will use it freely in what follows.

Example 3.11. With respect to the last statement in Theorem 3.8, it is not the case that the preimage in $\mathbf{D}(X)$ of the category of perverse sheaves is equal to $GV(X) \cap \mathbf{D}^{\geq 0}(X)$. This does not happen even in the most favorable case: let X be an abelian variety, $Y = \widehat{X}$ its dual, and P a Poincaré bundle on $X \times \widehat{X}$. Consider L a nondegenerate line bundle on X (i.e. with $\chi(L) \neq 0$) satisfying WIT_1 , so that L^{-1} satisfies WIT_{g-1} . Consider $A = L[1] \notin \mathbf{D}(X)^{\geq 0}$. Then $\mathbf{R}\Phi_P A$ is a vector bundle supported in degree 0, so $A \in GV(X)$. On the other hand $\mathbf{R}\Phi_{P^\vee}(\mathbf{R}\Delta A) \cong R^{g-1}\Phi_{P^\vee} L^{-1}[-g]$ is supported in degree g , so $\mathbf{R}\Phi_P A$ is perverse.

An immediate consequence of Proposition 3.1 and its avatar Theorem 3.8 is extra structure on the category of geometric objects satisfying Generic Vanishing with respect to Fourier-Mukai equivalences.

Corollary 3.12. *If $\mathbf{R}\Phi_P$ is an equivalence, the geometric GV -objects with respect to P (or dually the WIT_d -objects with respect to Q) form the heart of a t -structure on $\mathbf{D}(X)$, whose negative half consists of the category of GV -objects.*

It is tempting to wonder whether this still holds under weaker assumptions on $\mathbf{R}\Phi_P$.

4. HYPERCOHOMOLOGY VANISHING CHARACTERIZATION OF $\text{Per}(Y)$ AND OF $GV_m(X)$ WITH $m \leq 0$

The main result of this section is a characterization of the negative part of the Generic Vanishing filtration. It is a slightly more general version of [PP2] Theorem 3.7 and Theorem A, in turn following Hacon's approach to generic vanishing for the canonical bundle in [Hac]. The proof given below follows a strategy different from that in the papers above, emphasizing the connection with Proposition 3.1 and the fact that the projectivity of Y is needed only for a simple cohomological characterization of perverse sheaves. By a *sufficiently positive* ample line bundle I always mean a sufficiently high power of an ample line bundle, and use the notation $L \gg 0$.

Theorem 4.1. *Let X and Y be Cohen-Macaulay schemes of finite type over k , of dimension d and g respectively with X projective. Let $P \in \mathbf{D}(X \times Y)$ be a perfect object. For $A \in \mathbf{D}(X)$ and $k \geq 0$, the following are equivalent:*

- (1) $A \in GV_{-k}(X)$.
- (2) $\text{codim}_Y V_P^i(A) \geq i - k$ for all i .
- (3) $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \in \mathbf{D}^{\geq d-k}(Y)$.

If in addition Y is also projective, they are equivalent to the hypercohomology vanishing

- (4) $H^i(X, A \otimes^{\mathbf{L}} \mathbf{R}\Psi_{P[g]}(L^{-1})) = 0$ for all $i > k$ and any $L \gg 0$ on Y .

Proof. The equivalence between (1) and (2) is Lemma 2.4. The equivalence between (1) and (3) is Corollary 3.5, based on Proposition 3.2. So we only need to prove the equivalence between (1) and (4) when Y is projective. We have by definition that $A \in GV_{-k}(X)$ if and only if $(\mathbf{R}\Phi_P A)[k] \in {}^p\mathbf{D}^{\leq 0}(Y)$. It is a standard consequence of the derived Projection Formula and Leray isomorphism that

$$H^j(Y, \mathbf{R}\Phi_P A \otimes L^{-1}) \cong H^j(X, A \otimes^{\mathbf{L}} \mathbf{R}\Psi_P(L^{-1})) \cong H^{j-g}(X, A \otimes^{\mathbf{L}} \mathbf{R}\Psi_{P[g]}(L^{-1}))$$

(cf. [PP2] Lemma 2.1). Hence all we need to show is the vanishing

$$H^j(Y, \mathbf{R}\Phi_P A \otimes L^{-1}) = 0 \text{ for all } j > g + k \text{ and any } L \gg 0 \text{ on } Y,$$

which is a consequence of Corollary 3.5, and of Lemma 4.2 which describes perverse sheaves cohomologically via a standard Serre Vanishing argument. \square

Lemma 4.2. *On a Cohen-Macaulay scheme Y , projective over k and of dimension g*

$$\text{Per}(Y) = \{A \in \mathbf{D}(Y) \mid H^j(Y, A \otimes L^{-1}) = 0 \text{ for all } j \neq g \text{ and any } L \gg 0 \text{ on } Y\}.$$

More precisely:

- $A \in {}^p\mathbf{D}^{\leq 0}(Y)$ if and only if $H^j(Y, A \otimes L^{-1}) = 0$ for all $j > g$ and any $L \gg 0$ on Y
- $A \in {}^p\mathbf{D}^{\geq 0}(Y)$ if and only if $H^j(Y, A \otimes L^{-1}) = 0$ for all $j < g$ and any $L \gg 0$ on Y .

Proof. It is enough to prove the two statements at the end. Note first that $A \in {}^p\mathbf{D}^{\leq 0}(Y)$ is equivalent to $\mathbf{R}\Delta A \in \mathbf{D}^{\geq 0}(Y)$ by Proposition 3.2. Now by Grothendieck-Serre duality we have

$$H^j(Y, A \otimes L^{-1}) \cong H^{g-j}(Y, \mathbf{R}\Delta A \otimes L)^*.$$

We have a spectral sequence

$$E_2^{pq} := H^p(Y, R^q \Delta A \otimes L) \Rightarrow H^{p+q}(Y, \mathbf{R}\Delta A \otimes L).$$

By Serre Vanishing, for $L \gg 0$ we have $E_2^{pq} = 0$ for all $p > 0$ and all q , so the spectral sequence degenerates and

$$H^{g-j}(Y, \mathbf{R}\Delta A \otimes L) \cong H^0(Y, R^{g-j} \Delta A \otimes L).$$

Again by Serre's theorem, this is nonzero for $L \gg 0$ if and only if $R^{g-j} \Delta A \neq 0$, hence the assertion. The proof of the assertion for $A \in {}^p\mathbf{D}^{\geq 0}(Y)$ is completely analogous, using Proposition 3.3. \square

The same argument shows that $\mathbf{R}\Phi_Q(\mathbf{R}\Delta A) \in \mathbf{D}^{\leq d-k}(Y)$ (i.e. A is geometric) if and only if

$$H^i(X, A \otimes^{\mathbf{L}} \mathbf{R}\Psi_{P[g]}(L^{-1})) = 0$$

for all $i < k$ and all L sufficiently positive on Y . In particular one can check the perversity of the Fourier-Mukai transform via vanishing on X , which is crucial for the applications in §7.

Corollary 4.3. *For $A \in \mathbf{D}(X)$ we have*

$$\mathbf{R}\Phi_P A \in \text{Per}(Y) \iff H^i(X, A \otimes^{\mathbf{L}} \mathbf{R}\Psi_{P[g]}(L^{-1})) = 0, \text{ for all } i \neq 0 \text{ and all } L \gg 0.$$

Remark 4.4. (1) The path pursued in [Hac] and [PP2] is to prove instead the equivalence between (3) and (4) in Theorem 4.1, still based on Serre Vanishing. This equivalence is the natural extension to integral functors of a basic degeneration of the Leray spectral sequence used in the proof of Grauert-Riemenschneider-type theorems. Concretely, let $f : X \rightarrow Y$ be a morphism of smooth projective varieties, and consider $P := \mathcal{O}_\Gamma$ as a sheaf on $X \times Y$, where $\Gamma \subset X \times Y$ is the graph of f . Hence P induces the integral functor $\mathbf{R}\Phi_P = \mathbf{R}f_*$, and $\mathbf{R}\Psi_P$ is the adjoint $\mathbf{L}f^*$. Consider A and $B = \mathbf{R}\Delta A$ objects in $\mathbf{D}(X)$. A routine calculation shows that the equivalence of (3) and (4) in Theorem 4.1 applied to A is the same as the well-known statement (individually for each i):

$$R^i f_* B = 0 \iff H^i(X, B \otimes f^* L) = 0 \text{ for all } L \gg 0 \text{ on } Y.$$

For instance, say $B = \omega_X$ and f is generically finite. Then for any ample L on Y , $f^* L$ is big and nef, so $H^i(X, \omega_X \otimes f^* L) = 0$ by Kawamata-Viehweg vanishing. We get that $R^i f_* \omega_X = 0$ for all $i > 0$, which is of course Grauert-Riemenschneider vanishing (in the projective case).

(2) It is interesting to note that the vanishing condition in (4) is of a different nature from standard vanishing theorems. For instance, when X and Y are dual abelian varieties and $\mathbf{R}\Phi_P$ is the standard Fourier-Mukai functor, Mukai showed that $\phi_L^* \mathbf{R}\Psi_{P[g]}(L^{-1}) \cong \oplus L$, where $\phi_L : Y \rightarrow X$ is the standard isogeny associated to L (see the proof of Theorem 6.1 below for more details). This suggests that, at least when P is a locally free sheaf, $\mathbf{R}\Psi_{P[g]}(L^{-1})$ should be interpreted as a positive vector bundle, but which is less and less so as L becomes more positive.

5. COMMUTATIVE ALGEBRA FILTRATION ON $\text{Per}(Y)$, DESCRIBING $GV_m(X)$ WITH $m > 0$

This section is concerned with a characterization of the positive part of the Generic Vanishing filtration, extending the results in [PP4] §3. Note to begin with that the subcategories $GV_m(X)$ with $m > 0$ are not obtained simply by shifting $GV_0(X)$, as in the negative case. The main result, Theorem 5.3, is at this stage merely a matter of notation and of navigating through results in other sections and in the Appendix.

Let Y be a Cohen-Macaulay scheme of finite type over a field, not necessarily projective. The characterization of perverse coherent sheaves in Proposition 3.1 gives

$$\text{Per}(Y) = \{A \in \mathbf{D}(Y) \mid \mathbf{R}\Delta A \text{ is a sheaf in degree } 0\}.$$

We can naturally consider a filtration on $\text{Per}(Y)$ according to the singularities of the sheaf $\mathbf{R}\Delta A$. To this end, define for $m \geq 0$ (see Definition 7.4):

$$\text{Per}_m(Y) := \{A \in \text{Per}(Y) \mid \mathbf{R}\Delta A \text{ satisfies } S'_m\}.$$

We get a filtration

$$\{\text{locally free sheaves}\} \subset \text{Per}_\infty(Y) = \dots = \text{Per}_g(Y) \subset \dots \subset \text{Per}_1(Y) \subset \text{Per}_0(Y) = \text{Per}(Y)$$

If we restrict to the subcategory $\text{Per}^{\text{fhd}}(Y) \subset \text{Per}(Y)$ consisting of objects such that $\mathbf{R}\Delta A$ has finite homological dimension (so all of $\text{Per}(Y)$ if Y is smooth), then we have

$$\{\text{maximal Cohen - Macaulay sheaves}\} = \text{Per}_\infty^{\text{fhd}}(Y) = \dots = \text{Per}_g^{\text{fhd}}(Y).$$

(Note that in general \mathcal{F} is a maximal Cohen-Macaulay sheaf if and only if $\mathbf{R}\Delta\mathcal{F}$ is a maximal Cohen-Macaulay sheaf; see [BH] Theorem 3.3.10.) On the other hand, by the Auslander-Bridger criterion in Proposition 7.5

$$\mathrm{Per}_m^{\mathrm{fhd}}(Y) = \{A \in \mathrm{Per}^{\mathrm{fhd}}(Y) \mid \mathbf{R}\Delta A \text{ is an } m\text{-th syzygy sheaf}\}.$$

Note that this last thing holds for the entire $\mathrm{Per}_m(Y)$ if Y is Gorenstein in codimension less than or equal to one. The Evans-Griffith Syzygy Theorem 7.6 can be rephrased as follows.

Corollary 5.1. *Let $m > 0$ be an integer, and let A be a perverse sheaf in $\mathrm{Per}_m^{\mathrm{fhd}}(Y)$ which is not a locally free sheaf. Then $\mathrm{rank}(\mathbf{R}\Delta A) \geq m$.*

The equivalence of (b) and (c) in Proposition 7.5 gives a characterization of objects in $\mathrm{Per}_m(Y)$.

Lemma 5.2. *For $A \in \mathrm{Per}(Y)$ and $m > 0$*

$$A \in \mathrm{Per}_m(Y) \iff \mathrm{codim} \mathrm{Supp} \mathcal{H}^i A \geq i + m \text{ for all } i > 0.$$

This last condition corresponds to the GV_m -piece of the Generic Vanishing filtration, in the case of an integral functor $\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$. Recall that if A is a geometric GV -object in $\mathbf{D}(X)$, then $\mathbf{R}\Phi_P A$ is perverse in $\mathbf{D}(Y)$, so $\widehat{\mathbf{R}\Delta A} := \mathbf{R}\Phi_Q(\mathbf{R}\Delta A)[d]$ is a sheaf.

Theorem 5.3. *Let X and Y be Cohen-Macaulay schemes of finite type over k , with X projective. Fix a kernel $P \in \mathbf{D}(X \times Y)$, and let A be a geometric GV -object in $\mathbf{D}(X)$, with respect to P . Let $m > 0$ be an integer. Then the following are equivalent:*

- (1) $A \in GV_m(X)$.
- (2) $\mathbf{R}\Phi_P A \in \mathrm{Per}_m(Y)$.
- (3) $\widehat{\mathbf{R}\Delta A}$ satisfies S'_m .

If these conditions are satisfied and in addition $\mathbf{R}\Phi_P A \in \mathrm{Per}_m^{\mathrm{fhd}}(Y)$ or Y is Gorenstein in codimension less than or equal to one, then they are also equivalent to

- (4) $\widehat{\mathbf{R}\Delta A}$ is an m -th syzygy sheaf.

Proof. The equivalence of (1) and (2) is the content of Lemma 5.2. The equivalence of (2) and (3) follows by definition and Lemma 2.1. In the Gorenstein or finite homological dimension case, the equivalence of (3) and (4) is the Auslander-Bridger criterion quoted in Proposition 7.5. \square

Definition 5.4 (Generic Vanishing Index). Let A be an object in $\mathbf{D}(X)$. The *Generic Vanishing index* of A (with respect to P) is the integer

$$gv(A) := \min_{i>0} \{\mathrm{codim} \mathrm{Supp} R^i \Phi_P A - i\} = \min_{i>0} \{\mathrm{codim} V_P^i(A) - i\}.$$

(The last equality holds due to Lemma 2.4.) If $\mathrm{Supp} R^i \Phi_P A = \emptyset$ for all $i > 0$, we declare $gv(A) = \infty$. By definition $A \in GV_m(X)$ if and only if $gv(A) \geq m$.

Theorem 5.3 and Corollary 5.1 imply then the following useful

Corollary 5.5. *If A is a geometric GV -object with $gv(A) < \infty$, then $\text{rank}(\widehat{\mathbf{R}\Delta A}) \geq gv(A)$.*

6. GEOMETRIC APPLICATIONS

The characterizations of GV_m -objects (or of perverse objects and the syzygy filtration on them) given in §4 and §5 can often be checked in practice. From a derived category point of view, one obtains nontrivial concrete examples of perverse coherent sheaves. This produces a number of different geometric applications, some described in what follows. The general literature on applications of generic vanishing theorems to birational geometry is very extensive, a small sampling being given by [GL1], [EL], [CH1], [CH2], [CH3], [CH4], [HP], [PP2], [PP4]. For instance, some of the most striking applications include results of Ein-Lazarsfeld [EL] on singularities of theta divisors, and of Chen-Hacon [CH1] on a conjecture of Kollár on characterizing abelian varieties (cf. also [Pa2] for a proof using the interpretation in §6.2 as well) and [CH4] on Ueno’s conjecture on the Albanese map of varieties with Kodaira dimension equal to 0.

6.1. Generic Vanishing Theorems. The derived category approach to generic vanishing theorems was pioneered by Hacon [Hac]. The work described here is mostly taken from [PP2], and grew out of trying to extend Hacon’s approach and the Green-Lazarsfeld results [GL1].

In what follows let X be a smooth projective complex variety of dimension d , with Albanese map $a : X \rightarrow A$. Let P be a Poincaré line bundle on $X \times \text{Pic}^0(X)$, and consider as usual

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0(X)).$$

Every GV condition will be considered with respect to this functor. For a \mathbb{Q} -divisor L on X , we define κ_L to be $\kappa(L|_F)$, the Iitaka dimension along the generic fiber F of a , if $\kappa(L) \geq 0$, and 0 if $\kappa(L) = -\infty$.

Theorem 6.1 ([PP2] Theorem B). *Let L be a line bundle and D an effective \mathbb{Q} -divisor on X such that $L - D$ is nef. If the dimension of the Albanese image $a(X)$ is $d - k$, then $\omega_X \otimes L \otimes \mathcal{J}(D)$ belongs to $GV_{-(k-\kappa_{(L-D)})}(X)$, where $\mathcal{J}(D)$ is the multiplier ideal sheaf associated to D .⁶ In particular, if L is a nef line bundle, then $\omega_X \otimes L$ belongs to $GV_{-(k-\kappa_L)}(X)$.*

The simplest instance of this (explaining also the terminology “generic vanishing”) is the following:

Corollary 6.2. *Let X be a smooth projective variety, and L a nef line bundle on X . Assume that either one of the following holds:*

- (1) X is of maximal Albanese dimension (i.e. $k = 0$).
- (2) $\kappa(L) \geq 0$ and $L|_F$ is big, where F is the generic fiber of a .

⁶Multiplier ideals are treated for example in [La] Ch.9. Here are some easily understood examples: when D is integral one has $\mathcal{J}(D) = \mathcal{O}_X(-D)$, while when D is a simple normal crossings \mathbb{Q} -divisor one has $\mathcal{J}(D) = \mathcal{O}_X(-[D])$.

Then $\omega_X \otimes L$ belongs to $GV(X)$. In particular

$$H^i(X, \omega_X \otimes L \otimes P) = 0 \text{ for all } i > 0 \text{ and } P \in \text{Pic}^0(X) \text{ general}$$

and consequently $\chi(\omega_X \otimes L) \geq 0$.

Corollary 6.3. *Under the hypotheses of Corollary 6.2, $\mathbf{R}\Phi_P(\omega_X \otimes L)$ is a perverse coherent sheaf on $\text{Pic}^0(X)$.*

A more precise statement for ω_X is given in Corollary 6.11.

I only sketch the proof of Corollary 6.2, under the hypothesis (1), as an example of the use of Theorem 4.1. This contains all the key ideas needed for Theorem 6.1, the rest involving a standard extension to multiplier ideal sheaves, the additivity of the Iitaka dimension, and extensions of the Kawamata-Viehweg vanishing theorem (for full details cf. [PP2] §5). The main idea of the proof is due to Hacon [Hac], with a refinement from [PP2] that allows for bypassing Hodge-theoretic results, hence the extension to twists by arbitrary nef line bundles.

Let L be a nef line bundle on X . It is enough to show that $\omega_X \otimes L$ satisfies condition (4) in Theorem 4.1. Let M be an ample line bundle on $\widehat{A} \cong \text{Pic}^0(X)$, and assume for simplicity that it is symmetric, i.e. $(-1_{\widehat{A}})^*M \cong M$. We consider the two different Fourier transforms $\mathbf{R}SM = R^0SM = p_{A*}(p_{\widehat{A}}^*M \otimes \mathcal{P})$ (on A), where \mathcal{P} is a Poincaré bundle on $A \times \widehat{A}$ so that $P \cong (a \times \text{id}_{\widehat{A}})^*\mathcal{P}$, and $\mathbf{R}\Psi_{P[g]}(M^{-1}) = R^g\Psi_P(M^{-1}) =: \widehat{M}^{-1}$ (on X). These are both locally free sheaves. One can check with a little care that

$$\widehat{M}^{-1} \cong a^*R^gS(M^{-1}) \cong a^*(R^0SM)^\vee.$$

On the other hand, by [Muk] 3.11, the vector bundle R^0SM has the property:

$$\phi_M^*(R^0SM) \cong H^0(M) \otimes M^{-1}.$$

Here $\phi_M : \widehat{A} \rightarrow A$ is the standard isogeny induced by M . We consider then the fiber product $X' := X \times_A \widehat{A}$ induced by a and ϕ_M :

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ b \downarrow & & \downarrow a \\ \widehat{A} & \xrightarrow{\phi_M} & A \end{array}$$

It follows that

$$(1) \quad \psi^*\widehat{M}^{-1} \cong \psi^*a^*(R^0SM)^\vee \cong b^*(H^0(M) \otimes M) \cong H^0(M) \otimes b^*M.$$

What we want is the vanishing

$$H^i(X, \omega_X \otimes L \otimes \widehat{M}^{-1}) = 0 \text{ for all } i > 0.$$

(Note that this will work for *any* ample line bundle M , the condition $M \gg 0$ being required only for the equivalence in Theorem 4.1 to hold.) Since ψ , like ϕ_M , is étale, it is enough to prove this after pull-back to X' , so for $H^i(X', \omega_{X'} \otimes \psi^*L \otimes \psi^*\widehat{M}^{-1})$. But by (1) we see that this amounts to the vanishing

$$H^i(X', \omega_{X'} \otimes \psi^*L \otimes b^*M) = 0 \text{ for all } i > 0.$$

Now ψ^*L is nef and b^*M is big and nef (as the pull-back of an ample line bundle by a generically finite map), so this follows from Kawamata-Viehweg vanishing.

A completely similar approach, replacing at the end Kawamata-Viehweg by other standard vanishing theorems, proves the following results for higher direct images of canonical bundles and for bundles of holomorphic forms.

Theorem 6.4. *Let $f : Y \rightarrow X$ be a morphism, with X, Y smooth projective varieties. Let L be a nef line bundle on $f(Y)$ (reduced image of f). If the dimension of $f(Y)$ is d and that of its image via the Albanese map of X is $d - k$, then $R^j f_* \omega_Y \otimes L$ is a $GV_{-(k-\kappa_L)}$ -sheaf on X for any j .*

Theorem 6.5. *Let X be a smooth projective variety of dimension d , with Albanese image of dimension $d - k$. Denote by f the maximal dimension of a fiber of a , and consider $l := \max\{k, f - 1\}$. Then:*

- (1) Ω_X^j belongs to $GV_{-(d-j+l)}(X)$ for all j .
- (2) $\text{codim}_{\hat{A}} V^i(\Omega_X^j) \geq \max\{i + j - d - l, d - i - j - l\}$, for all i and all j .
- (3) If L is a nef line bundle on X and a is finite, then $\Omega_X^j \otimes L$ belongs to $GV_{-(d-j)}(X)$ for all j .

Theorem 6.4 is again due to Hacon [Hac] in the case $L = \mathcal{O}_X$. Another statement of a slightly different flavor for Ω_X^j , not covered by these methods, can be found in [GL1]. For the first statement one uses Kollár's vanishing and torsion-freeness theorems for higher direct images of canonical bundles, while for the second Nakano and Bogomolov-Sommese vanishing. Details and further results along these lines can be found in [PP2] §5. Note that Theorem 6.4 produces other natural examples of perverse sheaves on Picard varieties.

Corollary 6.6. *Let X be a smooth projective variety and $a : X \rightarrow A$ its Albanese map, and let L be any nef line bundle on $a(X)$. Then $\mathbf{R}\Phi_P(R^i a_* \omega_X \otimes L)$ are perverse sheaves on $\text{Pic}^0(X)$ for all i , where $\mathbf{R}\Phi_P$ is the standard Fourier-Mukai functor on $\mathbf{D}(A)$.*

Remark 6.7. All of these results, suitably interpreted, hold more generally for morphisms $X \rightarrow A$, with A an abelian variety and \hat{A} replacing $\text{Pic}^0(X)$. This allows in particular for a more general statement of Theorem 6.4, where X can be singular. It is also interesting to note that the methods presented here give an algebraic proof of generic vanishing in characteristic 0 (as well as in positive characteristic in some cases), which had traditionally been approached via transcendental methods. (For details cf. [PP2], end of §5 and Remark 6.7.)

Generic vanishing and perverse coherent sheaves via mixed Hodge modules. The upcoming [PS] generalizes the results above, when $L = \mathcal{O}_X$, using M. Saito's theory of mixed Hodge modules (see e.g. [Sa]). The main point is that every mixed Hodge module on an abelian variety gives rise to a collection of perverse coherent sheaves on \hat{A} , namely the graded pieces of the filtration on the underlying \mathcal{D} -module. This is again an application of the criterion in §4, together with the vanishing theorem [Sa] Proposition 2.33, for the graded pieces of the de Rham complex associated to a \mathcal{D} -module of this type.

Theorem 6.8 ([PS]). *Let A be a complex abelian variety, and M a mixed Hodge module on A with underlying filtered \mathcal{D} -module (\mathcal{M}, F) . Then for each integer k , the transform $\mathbf{R}\Phi_P(\text{Gr}_k^F \mathcal{M})$ is a perverse coherent sheaf on \hat{A} .*

This provides new information, via the Decomposition Theorem, even when applied to the push forward to $A = \text{Alb}(X)$ of the Hodge module $\mathbb{Q}_X^H[d]$ associated to the trivial

variation of Hodge structures. For $s \in \mathbb{N}$, let $A_s = \{y \in A \mid \dim a^{-1}(y) \geq s\}$, and define the *defect* of a to be

$$\delta(a) = \max_s (2s - \dim X + \dim A_s),$$

where the maximum is over all values of s up to the maximal dimension of a fiber of a .

Theorem 6.9 ([PS]). *Let X be a smooth complex projective variety of dimension d . Then Ω_X^j is a $GV_{-(d-j+\delta(a))}$ -sheaf for each j .*

For instance, if the Albanese map is semi-small (so that in particular that X is of maximal Albanese dimension) then $\delta(a) = 0$, and one obtains that X satisfies the generic Nakano vanishing theorem: for every $i, j \in \mathbb{N}$ with $i + j > d$,

$$\text{codim } V^i(\Omega_X^j) \geq i + j - d.$$

In [PS] we explain how Theorem 6.9, or its proof, imply all of the standard generic vanishing theorems mentioned above for $L = \mathcal{O}_X$. We provide further applications to cohomological support loci for local systems, as well as to constructing perverse coherent sheaves with respect to a Bezrukavnikov-Kashiwara t -structure different from the one appearing in this paper.

6.2. Vanishing of higher direct images. The equivalence between (1) and (3) in Theorem 4.1 says that once the (geometric) GV -condition is established for an object A , the corresponding integral transform of the Grothendieck dual object is, up to shift, a sheaf. Again in the setting of $\mathbf{R}\Phi_P : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0(X))$, here is the main instance of this:

Corollary 6.10. *Let X be a smooth projective variety of dimension d and Albanese dimension $d - k$. Then*

$$R^i \Phi_{P^\vee} \mathcal{O}_X (\cong R^i p_{2*}(P^\vee)) = 0 \text{ for } i \notin [d - k, d].$$

In particular, if X is of maximal Albanese dimension, then \mathcal{O}_X satisfies WIT_d , so that its Fourier-Mukai transform is a sheaf

$$\widehat{\mathcal{O}_X} \cong \mathbf{R}\Phi_{P^\vee} \mathcal{O}_X[d].$$

Although with the methods presented in this paper this statement is an immediate consequence of generic vanishing, its history follows somewhat the opposite direction: a more precise version was first proved when $X = A$ is an abelian variety by Mumford [Mum] §13. It was then conjectured to be true in general by Green-Lazarsfeld [GL2] Problem 6.2. This was proved by Hacon [Hac] and Pareschi [Pa1], both showing that it further implies generic vanishing. (A similar statement holds for compact Kähler manifolds, cf. [PP4].) The existence of the sheaf $\widehat{\mathcal{O}_X}$ (or equivalently the perversity of $\mathbf{R}\Phi_{P^\vee} \mathcal{O}_X$) has significant consequences via the results in §5, as we will see in §6.3.

In the reverse direction, the non-vanishing of higher direct images implies a more precise version of Corollary 6.3 in the case of the canonical bundle.

Corollary 6.11. *The object $\mathbf{R}\Phi_{P^\vee} \mathcal{O}_X$ is a perverse sheaf if and only if X is of maximal Albanese dimension.*

Proof. The if part follows from Corollary 6.3. Assume now that $\mathbf{R}\Phi_{P^\vee} \mathcal{O}_X$ is perverse. We have seen that this is equivalent to the fact that $R^i \Phi_{P^\vee} \mathcal{O}_X = 0$ for $i \neq d$. We claim on the other hand that a result of Kollár implies that if the Albanese map a of X has generic

fiber of dimension k , then $R^{d-k}\Phi_{P\nu}\mathcal{O}_X \neq 0$. Indeed, by [Ko] Theorem 3.1 one has a decomposition in the derived category

$$\mathbf{R}a_*\omega_X \cong \bigoplus_{i=0}^k R^i a_*\omega_X[-i].$$

This decomposition implies in particular that $H^d(X, \omega_X) \cong H^{d-k}(A, R^k a_*\omega_X)$, as the support of all $R^i a_*\omega_X$ is at most $d - k$ dimensional. Hence we must have $R^k a_*\omega_X \neq 0$. But then the claim follows by a standard application of Grothendieck duality Lemma 2.1. \square

Finally, the extra results listed in §6.1 produce other examples of WIT_d -objects, so via Fourier-Mukai duals of perverse sheaves. For instance:

- The same statements as in Corollary 6.10 hold replacing \mathcal{O}_X by L^{-1} for any nef line bundle L on X .
- Under the hypotheses of Theorem 6.4, $A_j := \mathbf{R}\Delta(R^j f_*\omega_X \otimes L)$ satisfies WIT_d for each j , so $\widehat{A}_j = \mathbf{R}\Phi_P A_j[d]$ are all sheaves.

6.3. Bounding the holomorphic Euler characteristic and applications to irregular varieties. The syzygy filtration in §5, combined with the Evans-Griffith theorem, can be applied to bound the holomorphic Euler characteristic of irregular varieties (i.e. those with $h^1(\mathcal{O}_X) \neq 0$, or equivalently $b_1(X) \neq 0$) of maximal Albanese dimension. Starting with such a variety X , we consider the sheaf $\widehat{\mathcal{O}}_X$ as in Corollary 6.10, and find the suitable $\text{Per}_m(Y)$ to which its dual $\mathbf{R}\Phi_P \omega_X$ belongs according to the existence of nontrivial irregular fibrations of X .

Corollary 6.12. *If X is a smooth projective complex variety of maximal Albanese dimension:*

- (i) $\mathbf{R}\Phi_P \omega_X \in \text{Per}_{gv(\omega_X)}(\text{Pic}^0(X))$.
- (ii) $\chi(\omega_X) \geq gv(\omega_X)$.

Proof. This follows immediately from Theorem 5.3 and Corollary 5.5 applied to ω_X . The only thing that needs mention is that $gv(\omega_X) < \infty$, as $0 \in V_{\beta}^d(\omega_X)$ with $d = \dim X$. \square

The key point is that according to [GL2], $gv(\omega_X)$ is bounded in terms of the fibrations of X over lower dimensional varieties of maximal Albanese dimension (those that cannot be fibered further should be considered the building blocks in the study of irregular varieties).

Corollary 6.13. *Let X be of maximal Albanese dimension, with $q = h^1(X, \mathcal{O}_X)$ and $d = \dim X$. If X is not fibered over any nontrivial normal projective variety of maximal Albanese dimension satisfying $q(\tilde{Y}) - \dim Y \geq q - d - m + 1$ for any smooth model \tilde{Y} , then*

$$\mathbf{R}\Phi_P \omega_X \in \text{Per}_m(\text{Pic}^0(X)) \text{ and } \chi(\omega_X) \geq m.$$

Proof. One applies Corollary 6.12, noting that the hypothesis implies $gv(\omega_X) \geq m$. Indeed, an argument based on [GL2] Theorem 0.1 says that X is fibered over a normal

projective variety Y of lower dimension such that any smooth model \tilde{Y} is of maximal Albanese dimension, and

$$q(X) - \dim X - (q(\tilde{Y}) - \dim Y) \leq gv(\omega_X).$$

The argument is described in the proof of [PP4] Theorem B. \square

The most significant instance of this result is an extension to arbitrary dimension of a classical result on surfaces due to Castelnuovo and de Franchis. I only state a slightly weaker version here for simplicity. Note that it holds in the Kähler case as well.

Theorem 6.14 ([PP4] Theorem A). *Let X be an irregular smooth projective complex variety. If X does not admit any surjective morphism with connected fibers onto a normal projective variety Y with $0 < \dim Y < \dim X$ and with any smooth model \tilde{Y} of maximal Albanese dimension, then*

$$\chi(\omega_X) \geq q(X) - \dim X.$$

Along similar lines, arguments based on Theorem 5.3 lead to classification results. For instance, they are crucial in another recent extension to arbitrary dimension due to Barja-Lahoz-Naranjo-Pareschi of a well-known fact on the bicanonical map of surfaces of general type.

Theorem 6.15 ([BLNP] Corollary B). *Under the same assumptions as in Theorem 6.14, if in addition X is of general type and $\dim X < q(X)$, the linear system $|2K_X|$ induces a birational map unless X is birational to a theta divisor in a principally polarized abelian variety.*

6.4. Perverse sheaves coming from special subvarieties of abelian varieties. Let (A, Θ) be an indecomposable principally polarized abelian variety (ppav) of dimension g , and let P be a Poincaré bundle on $A \times \hat{A}$. For subvarieties $X \subset A$, the question whether $\mathbf{R}\Phi_P(\mathcal{I}_X(\Theta))$ is a perverse sheaf on \hat{A} is related to a beautiful geometric problem. First, note that some of the most widely studied special subvarieties of ppav's satisfy this property:

- If $(J(C), \Theta)$ is the polarized Jacobian of a curve of genus $g \geq 2$, and for any $1 \leq d \leq g - 1$ we denote by W_d the image of the d -th symmetric product of C in $J(C)$ via an Abel-Jacobi map, then $\mathcal{I}_{W_d}(\Theta)$ is a geometric GV -sheaf, i.e. $\mathbf{R}\Phi_P(\mathcal{I}_X(\Theta))$ is perverse (cf. [PP1] Proposition 4.4).
- If $X \subset \mathbf{P}^4$ is a smooth cubic hypersurface, $(J(X), \Theta)$ is its intermediate Jacobian (a five dimensional ppav which is not the Jacobian of a curve), and $F \subset J(X)$ is the Fano surface parametrizing lines on X , then $\mathcal{I}_F(\Theta)$ is a geometric GV -sheaf, i.e. $\mathbf{R}\Phi_P(\mathcal{I}_F(\Theta))$ is perverse (cf. [Hö] Theorem 1.2).

Note also that further results along these lines in the case of Prym varieties can be found in [CLV].

What the W_d 's and F have in common is that they are subvarieties whose cohomology classes are *minimal* (i.e. not divisible in $H^*(A, \mathbb{Z})$). Indeed, $[W_d] = \frac{\theta^{g-d}}{(g-d)!}$ and $[F] = \frac{\theta^3}{3!}$ by results of Poincaré and Clemens-Griffiths respectively. An problem stemming from work of Beauville and Ran in low dimensions, and suggested in general by Debarre [De], is to show that these are in fact the only subvarieties of ppav's representing

minimal cohomology classes. In [PP3] Conjecture A, it is proposed that this should also be equivalent to the fact that $\mathbf{R}\Phi_P(\mathcal{I}_X(\Theta))$ is a perverse sheaf. One direction is known: in [PP3] Theorem B it is shown that if X is a nondegenerate closed reduced subscheme of pure dimension d of a ppav (A, Θ) of dimension g and $\mathcal{I}_X(\Theta)$ is a GV -sheaf, then X is Cohen-Macaulay and $[X] = \frac{\theta^{g-d}}{(g-d)!}$. The main ingredient is the criterion provided by Theorem 4.1 above.

6.5. Moduli spaces of vector bundles. The most natural higher rank analogue of generic vanishing involving $\text{Pic}^0(X)$ is to consider the (singular) moduli space $M_X(r)$ of semistable rank r vector bundles with trivial Chern classes on a smooth projective X . Some bounds on the dimension of cohomological support loci in $M_X(r)$ are given by Arapura [Ar] §7, but a more thorough understanding remains a very interesting problem. (Note however that the *structure* of these loci is very well understood in [Ar], by means of non-abelian Hodge theory.)

On the other hand, generic vanishing statements can be considered whenever M is a fine moduli space of objects on X , with a universal object \mathcal{E} on $X \times M$ inducing the functor

$$\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \longrightarrow \mathbf{D}(M).$$

In other words, the problem is to study the variation of the cohomology of the objects parametrized by M . In practice, in order to apply Theorem 4.1 the main difficulty to be overcome is a good understanding of the transform $\widehat{A}^{-1} = \mathbf{R}\Psi_{\mathcal{E}}(A^{-1})[\dim M]$, with A a very positive line bundle on M . Few concrete examples seem to be known: besides the case of curves, there are only sparse examples of generic vanishing phenomena on moduli spaces in higher dimension, coming from constructions of Yoshioka on $K3$ and abelian surfaces and of Bridgeland-Maciocia on threefolds with fibrations by such surfaces. I finish with a very brief discussion of these.

Curves. The issue whether $\mathbf{R}\Phi_{\mathcal{E}}\omega_X$ is a perverse sheaf on moduli spaces of higher rank vector bundles on a curve X is not so interesting, as it is easy to check by hand. However, Theorem 4.1 is useful when applied to bundles other than ω_X . For instance, when applied to higher rank stable bundles, it links the entire indeterminacy loci of determinant line bundles on these moduli spaces to a well-known construction of Raynaud, as first considered by Hein [He]. This is of a somewhat different flavor from the main direction of this note, so I will refrain from including details. The interested reader can consult [PP2] §7.

Some surfaces and threefolds. Consider first X to be a complex abelian or $K3$ surface. For a coherent sheaf E on X , the Mukai vector of E is

$$v(E) := \text{rk}(E) + c_1(E) + (\chi(E) - \epsilon \cdot \text{rk}(E))[X] \in H^{ev}(X, \mathbb{Z}),$$

where ϵ is 0 if X is abelian and 1 if X is $K3$. Given a polarization H on X and a vector $v \in H^{ev}(X, \mathbb{Z})$, we consider the moduli space $M_H(v)$ of sheaves E with $v(E) = v$, stable with respect to H . If the Mukai vector v is primitive and isotropic, and H is general, the moduli space is $M = M_H(v)$ is smooth, projective and fine, and it is in fact again an abelian or $K3$ surface (cf. e.g. [Yo1]). The universal object \mathcal{E} on $X \times M$ induces an equivalence of derived categories $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(M)$. Yoshioka gives many examples that amount to satisfying:

(*) The Mukai vector v is primitive and isotropic, and the structure sheaf $\mathcal{O}_X(\cong \omega_X)$ satisfies WIT_2 with respect to $\mathbf{R}\Phi_{\mathcal{E}^\vee}$ (or equivalently is GV with respect to $\mathbf{R}\Phi_{\mathcal{E}}$ by Theorem 4.1).

- Let (X, H) be a polarized $K3$ surface such that $\text{Pic}(X) = \mathbb{Z} \cdot H$, with $H^2 = 2n$. Let $k > 0$ be an integer such that kH is very ample. Consider $v = k^2n + kH + [X]$. It is shown in [Yo3] Lemma 2.4 that under these assumptions (*) is satisfied.⁷

- Let (X, H) be a polarized abelian surface with $\text{Pic}(X) = \mathbb{Z} \cdot H$. Write $H^2 = 2r_0k$, with $(r_0, k) = 1$. Consider the Mukai vector $v_0 := r_0 + c_1(H) + k[X]$. By [Yo2], Theorem 2.3 and the preceding remark, in this situation (*) is again satisfied by \mathcal{O}_X (among many other examples).

A Calabi-Yau fibration is a morphism $\pi : X \rightarrow S$ of smooth projective varieties, with connected fibers, such that $K_X \cdot C = 0$ for all curves C contained in fibers of π . Assuming that X is a threefold, it is an elliptic, abelian surface, or $K3$ -fibration (in the sense that the nonsingular fibers are of this type). Say π is flat, and consider a polarization H on X , and Y an irreducible component of the relative moduli space $M^{H,P}(X/S)$ of sheaves on X (over S), semistable with respect to H , and with fixed Hilbert polynomial P . The choice of P induces on every smooth fiber X_s invariants which are equivalent to the choice of a Mukai vector $v \in H^{ev}(X_s, \mathbb{Z})$ as above. Assuming that Y is also a threefold, and fine, Bridgeland and Maciocia (cf. [BM], Theorem 1.2) proved that it is smooth, and the induced morphism $\hat{\pi} : Y \rightarrow S$ is a Calabi-Yau fibration of the same type as π . If \mathcal{E} is a universal sheaf on $X \times Y$, then $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \rightarrow \mathbf{D}(Y)$ is an equivalence. Now if the first half of (*) is satisfied for each smooth fiber of π , it is proved in [BM] §7 that the moduli space $M^{H,P}(X/S)$ does have a fine component Y which is a threefold, so the above applies. The same method based on Theorem 4.1 leads to the following:

Proposition 6.16 ([PP2], Proposition 7.7). *Let X be a smooth projective threefold with a smooth Calabi-Yau fibration $\pi : X \rightarrow S$ of relative dimension two. Let H be a polarization on X and P a Hilbert polynomial, and assume that condition (*) is satisfied for each fiber of π . Consider a fine three-dimensional moduli space component $Y \subset M^{H,P}(X/S)$, and let \mathcal{E} be a universal sheaf on $X \times Y$. Then ω_X is a GV_{-1} -sheaf with respect to \mathcal{E} . In particular*

$$H^i(X, \omega_X \otimes E) = 0, \text{ for all } i > 1 \text{ and all } E \in Y \text{ general.}$$

This can be extended to the case of singular fibers, involving a slightly technical condition which should nevertheless be often satisfied; see [PP2] Remark 7.8. A similar statement holds for threefold elliptic fibrations.

7. APPENDIX: SOME HOMOLOGICAL COMMUTATIVE ALGEBRA

A useful technical point is that over Cohen-Macaulay rings one can avoid finite homological dimension hypotheses in statements of Auslander-Buchsbaum-type, by involving the canonical module. The key fact is the following consequence of Grothendieck duality for local cohomology.

⁷It is also shown in *loc. cit.* §2 that in fact $X \cong M$.

Lemma 7.1 ([BH], Corollary 3.5.11). *Let (R, m) be a local Cohen-Macaulay ring of dimension n with canonical module ω_R , and let M be a finitely generated module over R of depth t and dimension d . Then:*

(a) $\text{Ext}_R^i(M, \omega_R) = 0$ for $i < n - d$ and $i > n - t$.

(b) $\text{Ext}_R^i(M, \omega_R) \neq 0$ for $i = n - d$ and $i = n - t$.

(c) $\dim \text{Ext}_R^i(M, \omega_R) \leq n - i$ for all $i \geq 0$.

Corollary 7.2. *Let \mathcal{F} be a coherent sheaf on a Cohen-Macaulay scheme of finite type over a regular local ring, or a complex analytic space X . Then:*

(a) $\mathcal{E}xt^i(\mathcal{F}, \omega_X) = 0$ for all $i < \text{codim Supp } \mathcal{F}$ and $\mathcal{E}xt^i(\mathcal{F}, \omega_X) \neq 0$ for $i = \text{codim Supp } \mathcal{F}$.

(b) $\text{codim Supp } \mathcal{E}xt^i(\mathcal{F}, \omega_X) \geq i$ for all i .

(c) If \mathcal{F} is locally free, then $\mathcal{E}xt^i(\mathcal{F}, \omega_X) = 0$ for all $i > 0$.

Let now X be a noetherian scheme of finite type over a field, or a complex analytic space.

Definition 7.3. Let \mathcal{F} be a coherent sheaf on X . Then \mathcal{F} is called a k -th syzygy sheaf if locally there exists an exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_k \longrightarrow \dots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G} \longrightarrow 0$$

with \mathcal{E}_j locally free for all j . It is well known for example that if X is normal, then 1-st syzygy sheaf is equivalent to torsion-free, and 2-nd syzygy sheaf is equivalent to reflexive. Every coherent sheaf is declared to be a 0-th syzygy sheaf. A locally free one is declared to be an ∞ -syzygy sheaf.

Following [AB] and [EG1], we consider a variant of Serre's condition S_k .

Definition 7.4. A coherent sheaf \mathcal{F} on X satisfies property S'_k if for all x in the support of \mathcal{F} we have:

$$\text{depth } \mathcal{F}_x \geq \min\{k, \dim \mathcal{O}_{X,x}\}.$$

Proposition 7.5. *Let X be Cohen-Macaulay, and let \mathcal{F} be a coherent sheaf on X . Consider the following conditions:*

(a) \mathcal{F} is a k -th syzygy sheaf.

(b) $\text{codim Supp } \mathcal{E}xt^i(\mathcal{F}, \omega_X) \geq i + k$, for all $i > 0$.

(c) \mathcal{F} satisfies S'_k .

Then (b) is equivalent to (c), and if in addition \mathcal{F} has finite homological dimension or X is Gorenstein in codimension less than or equal to one, they are both equivalent to (a).

Proof. When \mathcal{F} is of finite homological dimension or X is Gorenstein in codimension less than or equal to one, the equivalence between (a) and (c) is a well-known result of Auslander-Bridger, [AB] Theorem 4.25. The equivalence between (b) and (c) is also standard in the case of finite homological dimension, when it follows from the Auslander-Buchsbaum formula (cf. [HL] Proposition 1.1.6(ii)). We note here that a

variation of the usual argument, involving Lemma 7.1, proves it in general. Consider first $x \in \text{Supp } \mathcal{E}xt^i(\mathcal{F}, \omega_X)$ for some $i > 0$. Then by Lemma 7.1(a) we get that $i \leq \dim \mathcal{O}_{X,x} - \text{depth } \mathcal{F}_x$. Combined with S'_k , this implies $\dim \mathcal{O}_{X,x} \geq i + k$. On the other hand, consider $x \in \text{Supp } \mathcal{F}$. We need to show that $\dim \mathcal{O}_{X,x} - \text{depth } \mathcal{F}_x \leq \max\{\dim \mathcal{O}_{X,x} - k, 0\}$. But by Lemma 7.1 (a) and (b), we have

$$\dim \mathcal{O}_{X,x} - \text{depth } \mathcal{F}_x = \max \{p \mid \text{Ext}_{\mathcal{O}_{X,x}}^p(\mathcal{F}_x, \omega_{X,x}) \neq 0\}.$$

But for any such p we have $x \in \text{Supp } \mathcal{E}xt^p(\mathcal{F}, \omega_X)$. Hence either the maximum is 0 and we're done, or it is positive and for all such $p > 0$ we have $\dim \mathcal{O}_{X,x} \geq p + k$. \square

Finally, a key result here is the Syzygy Theorem of Evans-Griffith.

Theorem 7.6 ([EG1] Corollary 1.7). *Let X be a Cohen-Macaulay scheme over a field, and \mathcal{F} a k -th syzygy sheaf of finite homological dimension on X which is not locally free. Then $\text{rank}(\mathcal{F}) \geq k$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, 851 S. MORGAN STREET,
CHICAGO, IL 60607, USA

E-mail address: mpopa@math.uic.edu