

REGULARITY ON ABELIAN VARIETIES II: BASIC RESULTS ON LINEAR SERIES AND DEFINING EQUATIONS

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Abstract

We apply the theory of M -regularity developed in [PP] to the study of linear series given by multiples of ample line bundles on abelian varieties. We define an invariant of a line bundle, called M -regularity index, which is seen to govern the higher order properties and (partly conjecturally) the defining equations of such embeddings. We prove a general result on the behavior of the defining equations and higher syzygies in embeddings given by multiples of ample bundles whose base locus has no fixed components, extending a conjecture of Lazarsfeld proved in [Pa]. This approach also unifies essentially all the previously known results in this area, and is based on Fourier-Mukai techniques rather than representations of theta groups.

1. Introduction

This paper is mainly concerned with applying the theory of Mukai regularity (or M -regularity) introduced in [PP] to the study of linear series given by multiples of ample line bundles on abelian varieties. We show that this regularity notion allows one to define a new invariant of a line bundle, called M -regularity index, which will be seen to roughly measure how much better one can do, given a fixed line bundle, compared to the standard results of the theory. Based on the main result of [PP] (M -regularity criterion) and a related result proved here (W.I.T. regularity criterion), we show that all known results on such linear series can be recovered, and indeed generalized, under the same heading of M -regularity.

To make this precise, we start by recalling most of the basic results on ample line bundles existing in the literature. For simplicity we state them for powers of one line bundle, although most hold for suitable products of possibly distinct ones.

Theorem. *Let A be an ample line bundle on an abelian variety X . The following hold:*

- (1) A^2 is globally generated.
- (2) (Lefschetz Theorem) A^3 is very ample.
- (3) (Ohbuchi's Theorem [Oh1]) If A has no base divisor, then A^2 is very ample.
- (4) (Bauer-Szemberg Theorem [BSz]) A^{k+2} is k -jet ample, and the same holds for A^{k+1} if A has no base divisor (extending (1), (2) and (3)).
- (5) (Koizumi's Theorem [Ko]) A^3 gives a projectively normal embedding.

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(6)(Ohbuchi's Theorem [Oh2]) A^2 gives a projectively normal embedding if and only if 0_X does not belong to a finite union of translates of the base locus of A (cf. §5 for the concrete statement).

(7)(Mumford's Theorem [M2], [Ke1]) For $k \geq 4$, the ideal of X in the embedding given by A^k is generated by quadrics. In the embedding given by A^3 it is generated by quadrics and cubics.

(8)(Lazarsfeld's Conjecture [Pa], extending results of Kempf [Ke2]) A^{p+3} satisfies property N_p (extending (5) and (7)).

(9)(Khaled's Theorem [Kh]) If A is globally generated, then the ideal of X in the embedding given by A^2 is generated by quadrics and cubics.

These results turn out to be – some quick while others non-trivial – consequences of the general global generation criterion in [PP], called the M -regularity criterion. Together with a more technical extension (the W.I.T. regularity criterion), described below, this approach yields new results and extensions as well. To introduce them, we first need some terminology.

Let X be an abelian variety of dimension g over an algebraically closed field, with dual abelian variety \hat{X} , and let \mathcal{P} be a suitably normalized Poincaré line bundle on $X \times \hat{X}$. The Fourier-Mukai functor [Mu] is the derived functor associated to the functor $\hat{\mathcal{S}}(\mathcal{F}) = p_{\hat{X}*}(p_X^* \mathcal{F} \otimes \mathcal{P})$ from $\text{Mod}(X)$ to $\text{Mod}(\hat{X})$. A sheaf \mathcal{F} on X is said to satisfy the Weak Index Theorem (W.I.T.) with index $i(\mathcal{F}) = k$ if $R^i \hat{\mathcal{S}}(\mathcal{F}) = 0$ for all $i \neq k$, in which case $R^k \hat{\mathcal{S}}(\mathcal{F})$ is simply denoted $\hat{\mathcal{F}}$. A weaker condition, introduced in [PP], is the following: \mathcal{F} is called M -regular if $\text{codim}(\text{Supp } R^i \hat{\mathcal{S}}(\mathcal{F})) > i$ for all $i > 0$. Moreover, we will consider the *Fourier jump locus* of \mathcal{F} to be the locus of $\xi \in \hat{X}$ where $h^0(F \otimes P_\xi)$ is different from the generic value (where P_ξ is the line bundle on X classified by ξ).

Given an ample line bundle A on X , we define the M -regularity index of A to be

$$m(A) := \max\{l \mid A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p} \text{ is } M\text{-regular for all distinct } x_1, \dots, x_p \in X \text{ with } \sum k_i = l\}.$$

A first result is that this invariant governs the higher order properties of embeddings obtained from A .

Theorem. *If A is an ample line bundle on X and $k \geq m(A)$, then $A^{\otimes(k+2-m(A))}$ is k -jet ample.*

It is not hard to see that for example $m(A) \geq 1$ if and only if A has no base divisor. The theorem thus recovers and extends the results of Lefschetz, Ohbuchi and Bauer-Szemberg mentioned above. Most interestingly though, this shows that results with seemingly unrelated proofs are simply steps in a hierarchy of regularity conditions. It is interesting to note also that the M -regularity indices are quite intimately related to the Seshadri constants measuring local positivity (cf. [La2] for the case of abelian varieties); we will approach this in detail somewhere else.

By reversing the natural order in the body of the paper, the results presented in what follows suggest that it is quite natural to expect that a similar phenomenon governs the behavior of defining equations of X , and more generally higher syzygies, in embeddings of this kind.

Conjecture. *Let $p \geq m$ be non-negative integers. If A is ample and $m(A) \geq m$, then A^k satisfies N_p for any $k \geq p + 3 - m$.*

This extends Lazarsfeld's conjecture, which is the statement for $m = 0$, meaning no conditions on A . That case has already been proved in [Pa], by methods which are included in, and provide a basis for, the strategy adopted here.

The main result of this paper is a proof, and also a strengthening, of the conjecture above for $m = 1$, i.e. for line bundles whose base locus has no fixed components. We first recall the terminology introduced in [Gr]: property N_p for a very ample line bundle means that $I_{X,L}$, the homogeneous ideal of X in the corresponding embedding, is generated by quadratic forms, and also that – up to the p -th step – the higher syzygies between these forms are generated in the lowest possible degree, i.e. by linear ones. Thus, in this language, the property that $I_{X,L}$ be generated by quadrics is condition N_1 . Moreover, the property of "being off" by r from N_p was formalized in [Pa] into property N_p^r (cf. §6 for details). In a word, N_p^0 is equivalent to N_p , and N_p^1 means that $I_{X,L}$ is generated by quadrics and cubics.

Theorem. (char(k) does not divide $(p + 1)$ and $(p + 2)$.) *Let A be an ample line bundle on X , with no base divisor. Then:*

- (a) *If $k \geq p + 2$ then A^k satisfies property N_p .*
- (b) *More generally, if $(r + 1)(k - 1) \geq p + 1$ then A^k satisfies property N_p^r .*

The first instance of this theorem, worth emphasizing individually, is the following:

Corollary. (char(k) $\neq 2, 3$.) *Let A be an ample line bundle on X , with no base divisor. Then:*

- (a) *If $k \geq 3$ then I_{X,A^k} is generated by quadrics.*
- (b) *I_{X,A^2} is generated by quadrics and cubics.*

(Note in particular the improvement of Khaled's result above.)

For consistency reasons, we note that Ohbuchi's projective normality result (6) does not integrate in the discussion above, and does indeed suggest what happens in the cases left out by the above conjecture. However, it can still be obtained in a similar way, and in §5 we sketch its proof as a toy version of that of the syzygy theorem.

As previously mentioned, all the proofs of the statements above are based on the basic M -regularity theorem, which we recall below.

Theorem. (*M -regularity criterion, [PP] Theorem 2.4.*) *Let \mathcal{F} be a coherent sheaf and L an invertible sheaf supported on a subvariety Y of the abelian variety X (possibly X itself). If both \mathcal{F} and L are M -regular as sheaves on X , then $\mathcal{F} \otimes L$ is globally generated.*

This has to be combined with a refined study, in a relative setting, of the notion of skew-Pontrjagin product introduced in [Pa], and also with a different (but related) regularity criterion, which we prove here following a similar strategy. The new statement needs stronger hypotheses on the sheaf \mathcal{F} , but provides specific information about the loci where suitable tensor products are not globally generated.

Theorem. (W.I.T. regularity criterion.) *Let A be an ample line bundle on X . Let also F be a locally free sheaf on X such that*

- (1) *the Fourier-jump locus $J(F)$ is finite.*
- (2) *F^\vee satisfies the W.I.T. with index $i(F^\vee) = g$.*
- (3) *the torsion part of \hat{F}^\vee is a sum of (possibly zero) skyscraper sheaves on the points of $J(F)$.*

Then there is an inclusion of non-generation loci:

$$B(F \otimes A) \subset \bigcup_{\xi \in J(F)} B(A \otimes P_\xi^\vee).$$

(For a sheaf \mathcal{F} , we denote by $B(\mathcal{F})$ the locus where \mathcal{F} is not globally generated.)

It is interesting to note that the W.I.T. regularity criterion applies to some sheaves for which the M -regularity criterion does not apply and conversely.

The underlying principle in this article is the use of vanishing theorems and Fourier-Mukai methods for vector bundles, or even arbitrary coherent sheaves, in the study of linear series. This completely bypasses methods based on theta-functions and representations of theta-groups originating in [M2] (and employed in the original proofs of most of the previously known results) as it was somewhat hinted that it could be possible by earlier work of Kempf. Better still, the main advantage of the present methods is that they apply to a much wider spectrum of problems on abelian varieties, as it is described in [PP].

The paper is organized as follows: in Section 2 we recall the main terminology and results from [PP], and we introduce further notions of generation of sheaves. Section 3 contains the definition of the M -regularity index and the corresponding result on higher order properties of embeddings. Sections 4 and 5 are devoted to a rather long list of technical results needed in the study of defining equations. In the former we prove the W.I.T. regularity criterion, while in the latter we introduce the notion of relative skew Pontrjagin product and study its properties under various operations. Finally, Section 6 is devoted to the main results of this paper, on defining equations and syzygies of abelian varieties embedded with powers of line bundles whose base locus has no fixed components.

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2. Background and preliminary results

In what follows X will be an abelian variety over an algebraically closed ground field k . Restrictions on $\text{char}(k)$ will be specified along the paper. We denote by \hat{X} the dual of X , which we identify with $\text{Pic}^0(X)$. Given $\xi \in \hat{X}$, P_ξ will denote the line bundle on X classified by ξ . For a positive integer n , X_n will denote the group of n -torsion points of X . When it appears in the text, we will always be in a situation where $\text{char}(k)$ does not divide n .

Various notions of generation of sheaves. Let \mathcal{F} be an arbitrary coherent sheaf on X . The support of the cokernel of the evaluation map $H^0(\mathcal{F}) \otimes \mathcal{O}_X \rightarrow \mathcal{F}$ will be referred to as the *non-generation locus* of \mathcal{F} and denoted $B(\mathcal{F})$. The usual notions of global generation and generic global generation mean that $B(\mathcal{F})$ is empty or a proper subset respectively. On abelian varieties it is useful to consider weaker notions of generation, which can be in fact defined on any irregular variety: \mathcal{F} is said to be *continuously globally generated* (cf. [PP] §2) if the map

$$\bigoplus_{\xi \in U} H^0(\mathcal{F} \otimes P_\xi) \otimes P_\xi^\vee \rightarrow \mathcal{F}$$

is surjective for *any non-empty Zariski-open set* $U \subset \hat{X}$. For a line bundle A this just means that $\bigcap_{\xi \in U} B(A \otimes P_\xi)$ is empty. In what follows we introduce an even weaker variant, needed in the sequel.

Definition 2.1. Given a sheaf \mathcal{F} , we define its *Fourier jump locus* as the locus $J(\mathcal{F}) \subset \hat{X}$ consisting of $\xi \in \hat{X}$ where $h^0(\mathcal{F} \otimes P_\xi)$ jumps, i.e. it is different from its minimal value over $\text{Pic}^0(X)$. \mathcal{F} is said to be *weakly continuously generated* if the map

$$\bigoplus_{\xi \in U} H^0(\mathcal{F} \otimes P_\xi) \otimes P_\xi^\vee \rightarrow \mathcal{F}$$

is surjective for *any non-empty Zariski-open set* $U \subset \hat{X}$ containing $J(\mathcal{F})$. Continuous global generation obviously implies weak continuous generation and the two notions are equivalent if the Fourier jump locus of \mathcal{F} is empty.

Remark 2.2. The following facts are easy to check (cf. also [PP] Remark 2.11):

- (a) If \mathcal{F} is weakly continuously generated, then there exist $\xi_1, \dots, \xi_k \in \hat{X}$ such that the map $\bigoplus_{i=1}^k H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \rightarrow \mathcal{F}$ is surjective.
- (b) If \mathcal{F} is continuously globally generated then there is a positive integer N such that for *general* $\xi_1, \dots, \xi_N \in \hat{X}$, the map $\bigoplus_{i=1}^N H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \rightarrow \mathcal{F}$ is surjective.
- (c) If \mathcal{F} is weakly continuously generated and $J(\mathcal{F})$ is finite, say $J(\mathcal{F}) = \{\xi_1, \dots, \xi_n\}$, then there is a positive integer N such that for *general* $\xi_{n+1}, \dots, \xi_{n+N} \in \hat{X}$ the map $\bigoplus_{i=1}^{n+N} H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes P_{\xi_i}^\vee \rightarrow \mathcal{F}$ is surjective.

The following lemma, proved in [PP] Proposition 2.12, shows how to produce global generation from continuous global generation.

Lemma 2.3. *If \mathcal{F} is a continuously globally generated sheaf on X and L is a continuously globally generated sheaf on X which is everywhere of rank 1 on its support, then $\mathcal{F} \otimes L$ is globally generated.*

The proposition below is a variation of this result, relating the notions of weak continuous generation and generic global generation. At least if the Fourier jump locus of \mathcal{F} is *finite*, one can describe the non-generation locus of $\mathcal{F} \otimes L$ in terms of $J(\mathcal{F})$.

Proposition 2.4. *Let \mathcal{F} be a weakly continuously generated sheaf on X .*

- (a) *If \mathcal{E} is a sheaf such that $\mathcal{E} \otimes P_\xi$ is generically globally generated for any $\xi \in \hat{X}$, then $\mathcal{F} \otimes \mathcal{E}$ is generically globally generated.*

- (b) If $\mathcal{E} \otimes P_\xi$ is globally generated for any $\xi \in \hat{X}$, then $\mathcal{F} \otimes \mathcal{E}$ is globally generated.
(c) Assume that the Fourier jump locus $J(\mathcal{F})$ is finite, and let L be a continuously globally generated line bundle on X . Then $B(\mathcal{F} \otimes L) \subset \bigcup_{\xi \in J(\mathcal{F})} B(L \otimes P_\xi^\vee)$.

Proof. (a) By Remark 2.2(a), the map $\bigoplus_{i=1}^k H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes \mathcal{E} \otimes P_{\xi_i}^\vee \rightarrow \mathcal{F} \otimes \mathcal{E}$ is surjective. Therefore we have the inclusion of non-generation loci $B(\mathcal{F} \otimes \mathcal{E}) \subset \bigcup_{i=1}^k B(\mathcal{E} \otimes P_{\xi_i}^\vee)$. This proves the assertion since, by hypothesis, $B(\mathcal{E} \otimes P_{\xi_i}^\vee)$ are proper subvarieties. This same argument also proves (b).

(c) If $J(\mathcal{F}) = \{\xi_1, \dots, \xi_N\}$, then the map $\bigoplus_{i=1}^{n+N} H^0(\mathcal{F} \otimes P_{\xi_i}) \otimes L \otimes P_{\xi_i}^\vee \rightarrow \mathcal{F} \otimes L$ is surjective for general $\xi_{n+1}, \dots, \xi_N \in \hat{X}$ (cf. Remark 2.2(c)). Therefore $B(\mathcal{F} \otimes L)$ is contained in the union of $\bigcup_{i=1}^n B(L \otimes P_{\xi_i})$ with the intersection – for all $\xi_{n+1}, \dots, \xi_{n+N}$ general in \hat{X} – of $\bigcup_{i=n+1}^{n+N} B(L \otimes P_{\xi_i}^\vee)$. Since L is a continuously globally generated line bundle, this intersection is empty. This implies that $B(\mathcal{F} \otimes L) \subset \bigcup_{i=1}^n B(L \otimes P_{\xi_i}^\vee)$. \square

Fourier-Mukai functor, index theorems and M -regularity. According to Mukai [Mu], one considers the left-exact functor $\hat{\mathcal{S}}$ from the category of \mathcal{O}_X -modules to the category of $\mathcal{O}_{\hat{X}}$ -modules defined as $\hat{\mathcal{S}}(\mathcal{F}) = p_{\hat{X}*}(p_X^* \mathcal{F} \otimes \mathcal{P})$. Mukai's main result [Mu] Theorem 2.2 is that the derived functor $\mathbf{R}\hat{\mathcal{S}}$ establishes an equivalence of categories between $\mathbf{D}(X)$ and $\mathbf{D}(\hat{X})$. A sheaf \mathcal{F} on X is said to satisfy the *Index Theorem (I.T.) with index $i(\mathcal{F}) = k$* if $H^j(\mathcal{F} \otimes P_\xi) = 0$ for any $\xi \in \hat{X}$ and any $j \neq k$. More generally, \mathcal{F} is said to satisfy the *Weak Index Theorem (W.I.T.) with index $i(\mathcal{F}) = k$* if $R^j \hat{\mathcal{S}}(\mathcal{F}) = 0$ for $j \neq k$. In this case $R^{i(\mathcal{F})} \hat{\mathcal{S}}(\mathcal{F})$ is simply denoted $\hat{\mathcal{F}}$. A useful consequence of Mukai's theory is the following lemma ([Mu], Cor.2.5):

Lemma 2.5. *If \mathcal{F} and \mathcal{G} both satisfy W.I.T., then there is a natural isomorphism*

$$\phi : \text{Ext}^j(\mathcal{F}, \mathcal{G}) \cong \text{Ext}^{j+i(\mathcal{F})-i(\mathcal{G})}(\hat{\mathcal{F}}, \hat{\mathcal{G}}).$$

We recall from [PP] the following weakening of the W.I.T. condition with index 0, and the main regularity result proved there.

Definition 2.6. A sheaf \mathcal{F} on X is said to be *Mukai-regular* (or simply *M -regular*) if $R^i \hat{\mathcal{S}}(\mathcal{F})$ is supported in codimension $> i$ for any $i > 0$, where for $i = g$ this means that the support $R^g \hat{\mathcal{S}}(\mathcal{F})$ is empty. This happens in particular if the cohomological support loci

$$V^i(\mathcal{F}) := \{\xi \mid h^i(\mathcal{F} \otimes P_\xi) \neq 0\} \subset \text{Pic}^0(X)$$

have codimension $> i$ for all i .

Theorem 2.7. (M -regularity criterion, [PP] Theorem 2.4 and Proposition 2.13.) *Let \mathcal{F} be an M -regular sheaf on X , possibly supported on a subvariety Y of X . Then the following hold:*

- (a) \mathcal{F} is continuously globally generated.
(b) Let also A be a line bundle on Y , continuously globally generated as a sheaf on X . Then $\mathcal{F} \otimes A$ is globally generated.

3. The M -regularity index and properties of embeddings

In this section we show that the concept of M -regularity is well-adapted to the study of linear series, and provides a uniform point of view on the study of (higher order) properties of embeddings. This will serve as an introduction to the deeper facts on defining equations treated in the subsequent sections via stronger regularity techniques. We first need to recall the notion of k -jet ampleness (cf. e.g. [BS] in general and [BSz] in the context of abelian varieties).

Definition 3.1. A line bundle A is called k -jet ample, $k \geq 0$, if the restriction map

$$H^0(A) \longrightarrow H^0(A \otimes \mathcal{O}_X/m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p})$$

is surjective for any distinct points x_1, \dots, x_p on X such that $\sum k_i = k + 1$.

Remark 3.2. In particular 0-jet ample means globally generated, 1-jet ample means very ample. The notion of k -jet ampleness is stronger than a related notion of k -very ampleness, which takes into account 0-dimensional subschemes of length equal to $k + 1$.

Lemma 3.3. For an ample line bundle A on the abelian variety X , the following are equivalent:

- (i) A is k -jet ample.
- (ii) $A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}$ satisfies I.T. with index 0 for all x_1, \dots, x_p such that $\sum k_i = k + 1$.
- (iii) $A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_l}^{k_l}$ is globally generated for all x_1, \dots, x_l such that $\sum k_i = k$.

Proof. This is based on the immediate fact that, since $h^1(A) = 0$ as we are on an abelian variety, k -jet ampleness is equivalent to the vanishing

$$H^1(A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}) = 0$$

for all x_1, \dots, x_p such that $\sum k_i = k + 1$. If A is k -jet ample, then so is any translate, thus (i) is equivalent to (ii) by the very definition. The equivalence with (iii) also follows quickly, since the required global generation is equivalent to the surjectivity of

$$H^0(A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_l}^{k_l}) \longrightarrow H^0(A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_l}^{k_l} \otimes \mathcal{O}_X/m_x)$$

for every $x \in X$. □

The key definition is given below. We note that it is suggested naturally by Theorem 2.7 and Lemma 3.3, and as a result the theorem which follows is almost tautological.

Definition 3.4. The M -regularity index of A is defined as

$$m(A) := \max\{l \mid A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p} \text{ is } M\text{-regular for all distinct } x_1, \dots, x_p \in X \text{ with } \sum k_i = l\}.$$

The following description provides more intuition for this definition.

Proposition 3.5. We say that a line bundle A is k -jet ample in codimension r if the set of points x for which there exist x_2, \dots, x_p and k_1, \dots, k_p with $\sum k_i = k + 1$ such that $h^1(A \otimes m_x^{k_1} \otimes m_{x_2}^{k_2} \otimes \dots \otimes m_{x_p}^{k_p}) > 0$ has codimension r in X . We have that $m(A) \geq k + 1$ if A is k -jet ample in codimension ≥ 2 .

Proof. If we assume that $m(A) < k + 1$, then there exist x_1, \dots, x_p and k_1, \dots, k_p with $\sum k_i = k + 1$ such that the set

$$\{y \in X \mid h^i(t_y^* A \otimes m_{x_1}^{k_1} \otimes m_{x_2}^{k_2} \otimes \dots \otimes m_{x_p}^{k_p}) > 0\}$$

has codimension ≤ 1 . Since this is the same as the set

$$\{y \in X \mid h^i(A \otimes m_{x_1-y}^{k_1} \otimes m_{x_2-y}^{k_2} \otimes \dots \otimes m_{x_p-y}^{k_p}) > 0\},$$

the assertion follows immediately. \square

Example 3.6. (Small values of $m(A)$.) If A is an ample line bundle, then $m(A) \geq 1$ if and only if A does not have a base divisor. Also, if A gives a birational map which is an isomorphism outside a codimension 2 subset, then $m(A) \geq 2$. Both assertions follow immediately from the proposition above.

Example 3.7. (Abelian surfaces.) (i) Let A be a polarization of type $(1, 2)$ on an abelian surface. It is an immediate consequence of the Decomposition Theorem (cf. [LB] 4.3.1) that A has a base divisor if and only if X is a product of elliptic curves E and F and $A = \mathcal{O}_X(E + 2F)$. On the other hand, it is not hard to see that we always have $m(A) \leq 1$ (for example $A \otimes m_x^2$ is not M -regular, where x is any point on X). Thus $m(A) = 1$ exactly when the pair (X, A) is not of the above form, while otherwise it is 0. Polarizations of type $(1, 3)$ are globally generated, and so $m(A) \geq 1$, but again an argument similar to Proposition 3.5 shows that $m(A) = 1$.

(ii) Let A be a polarization of type $(1, 4)$. If X is general (cf. [LB] Ch.10 §5, or the original [BLvS], for more precise conditions), then A gives a birational morphism to \mathbf{P}^3 which is not an embedding, and whose exceptional set is a curve (we thank the referee for pointing this out to us). However from the properties of this curve and the fact that the map separates tangent vectors outside a codimension 2 subset, it follows easily that $m(A) \geq 2$, although one cannot directly use Proposition 3.5. On the other hand, the special such abelian variety is a cover of a product of elliptic curves, and in that case $m(A) = 1$.

(iii) On a general abelian surface (more precisely, by Reider's Theorem [LB] 10.4.1, one on which there are no elliptic curves C such that $C \cdot A = 2$), a polarization A of type $(1, d)$ with $d \geq 5$ is very ample, and so $m(A) \geq 2$.

Based on this definition we obtain the following theorem, which extends and places in a natural setting the basic results on embeddings given by multiples of ample line bundles existing in the literature.

Theorem 3.8. *If A and $M_1, \dots, M_{k+1-m(A)}$ are ample line bundles on X , $k \geq m(A)$, then $A \otimes M_1 \otimes \dots \otimes M_{k+1-m(A)}$ is k -jet ample. In particular $A^{\otimes(k+2-m(A))}$ is k -jet ample.*

Proof. By definition, $A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}$ is M -regular for any $x_1, \dots, x_p \in X$ as long as $\sum k_i = m(A)$. This in turn implies that $M_1 \otimes A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_p}^{k_p}$ is globally generated, by Theorem 2.7. Now, by Lemma 3.3, this is the same as saying that $M_1 \otimes A \otimes m_{x_1}^{k_1} \otimes \dots \otimes m_{x_l}^{k_l}$ satisfies I.T. with index 0 for all x_1, \dots, x_l and $\sum k_i = m(A) + 1$. As this is a strong form of M -regularity, it allows us to continue the same procedure inductively to obtain the desired conclusion. \square

In particular, for small values of $m(A)$ (namely 0 and 1), this recovers as particular cases the theorems of Lefschetz, Ohbuchi, and more generally Bauer-Szemberg, mentioned in the introduction (cf. also Example 3.6).

We conclude by noting that a deeper reason for considering these invariants is that they seem to link in a natural way the geometry of the abelian variety in the embedding given by a line bundle with the equations, and more generally the syzygies, of that embedding. This will be explained in detail at the end of §6.

4. Cohomological criteria for weak continuous generation

In this section we provide a criterion, based on the weak index theorem, for the weak continuous generation of locally free sheaves on abelian varieties. To put the result into perspective, we recall that a key step in the proof of Lazarsfeld's conjecture [Pa] was based on the fact that if F is a locally free sheaf on X satisfying I.T. with index 0, and A is an ample line bundle, then $F \otimes A$ is globally generated. Theorem 2.7 above, proved in [PP], provides a generalization of that criterion widely extending its range of applicability. For the purposes of this paper, we also need the following different generalization of the result mentioned above, based on even weaker hypotheses (but note the locally freeness assumption):

Theorem 4.1. (W.I.T. regularity criterion) *Let F be a locally free sheaf on X such that F^\vee satisfies W.I.T. with index g and the torsion part of $\widehat{F^\vee}$ is a torsion-free sheaf on a reduced subscheme of X . Then the following hold:*

(a) *F is weakly continuously generated.*
 (b) *Let moreover A be a continuously globally generated line bundle on a subvariety of X . Then*

- (i) *$F \otimes A$ is generically globally generated.*
- (ii) *If the Fourier-jump locus $J(F)$ is finite then $B(F \otimes A) \subset \bigcup_{\xi \in J(F)} B(A \otimes P_\xi)$.*

Corollary 4.2. *Let F and A be a locally free sheaf, respectively an invertible sheaf on X . If F satisfies the hypotheses of Theorem 4.1 and A is globally generated, then $F \otimes A$ is globally generated.*

The corollary follows immediately from Theorem 4.1(a) and Proposition 2.4(b). Turning to the proof of Theorem 4.1, note that, in view of Proposition 2.4(c), the only thing to prove is part(a). This in turn follows in a standard way (cf. [Pa] §2(b) or [PP] §2) from the following corresponding generalization of a result of Mumford-Kempf-Lazarsfeld type on multiplication maps (cf. [PP] Theorem 2.5).

Lemma 4.3. *Let F be a locally free sheaf on X satisfying the hypotheses of Theorem 4.1 and let \mathcal{H} be a coherent sheaf on X satisfying I.T. with index 0. Then the sum of multiplication maps*

$$\mathcal{M}_U : \bigoplus_{\xi \in U} H^0(F \otimes P_\xi^\vee) \otimes H^0(\mathcal{H} \otimes P_\xi) \xrightarrow{\oplus m_\xi} H^0(F \otimes \mathcal{H})$$

is surjective for any non-empty Zariski-open set $U \subset \widehat{X}$ containing $J(F)$.

Proof. The argument follows the proof of [PP] Theorem 2.5 (although in fact the hypotheses allow us to avoid the use of derived categories). The statement is equivalent to proving the injectivity of the dual map (note that F is locally free):

$$\mathcal{M}_U^\vee : \text{Ext}^g(\mathcal{H}, F^\vee) \xrightarrow{\prod m_\xi^\vee} \prod_{\xi \in U} \text{Hom}(H^0(\mathcal{H} \otimes P_\xi), H^g(F^\vee \otimes P_\xi)),$$

where the maps m_ξ^\vee are the co-multiplication maps taking an extension class $e \in \text{Ext}^g(\mathcal{H}, F^\vee)$ to its connecting map $H^0(\mathcal{H} \otimes P_\xi) \rightarrow H^g(F^\vee \otimes P_\xi)$. The index hypotheses allows us to write \mathcal{M}_U^\vee as the composition of the map on global sections $\phi : \text{Ext}^g(H, F^\vee) \rightarrow \text{Hom}(\widehat{\mathcal{H}}, \widehat{F^\vee})$, followed by the evaluation map

$$ev_U : H^0(\mathcal{H}om(\widehat{\mathcal{H}}, \widehat{F^\vee})) \xrightarrow{\prod ev_\xi} \prod_{\xi \in U} \mathcal{H}om(\widehat{\mathcal{H}}, \widehat{F^\vee})(\xi),$$

where for a sheaf \mathcal{E} , we denote $\mathcal{E}(\xi) := \mathcal{E} \otimes k(\xi)$. In addition the hypotheses imply, by Lemma 2.5 above, that the map ϕ is an isomorphism.

On the other hand, if U is a Zariski-open set containing $J(F)$, the map ev_U is injective: note that by Nakayama's Lemma, given a non-zero global section s of a sheaf \mathcal{E} on X , we have that $s(x) \in \mathcal{E}(x)$ vanishes identically on a Zariski-open set U only if either U does not meet a component of the support of the torsion part of \mathcal{E} , or if the torsion part of \mathcal{E} is a sheaf on a non-reduced subscheme of X . Taking $\mathcal{E} = \mathcal{H}om(\widehat{H}, \widehat{F^\vee}) \cong (\widehat{H})^\vee \otimes \widehat{F^\vee}$ in our case, we see that the torsion part $\tau(\mathcal{H}om(\widehat{H}, \widehat{F^\vee}))$ is isomorphic to $(\widehat{H})^\vee \otimes \tau(\widehat{F^\vee})$. The hypothesis that U contains the Fourier jump locus of F excludes the first possibility (since the support of the torsion part of $\widehat{F^\vee}$ is certainly contained in $J(F)$). The second possibility is excluded by hypothesis. \square

Remark 4.4. More generally, Lemma 4.3 and, consequently, Theorem 4.1 continue to hold (with the same proof) under the hypotheses that F is a locally free sheaf on an m -dimensional Cohen-Macaulay subvariety Y of X , $\mathcal{H}om(F, \omega_Y) = \text{Ext}^{g-m}(F, \mathcal{O}_X)$ satisfies W.I.T. with index m , and the torsion part of the Fourier transform $\mathcal{H}om(F, \omega_Y)^\widehat{}$ is a torsion-free sheaf on a reduced subvariety of X .

5. Relative Pontrjagin products

Pontrjagin products, multiplication maps and relative Pontrjagin products.

One of the key points emphasized in [Pa] is the relation between multiplication maps of sheaves on abelian varieties (which are in turn involved in the study of linear series) and (skew) Pontrjagin products (cf. Proposition 5.2 below). Here we develop an analogue relative setting for skew Pontrjagin products, required for our applications.

Terminology/Notation 5.1. (Skew Pontrjagin product, P.I.T. and P.W.I.T.)

Let us recall first that, given two sheaves \mathcal{E} and \mathcal{G} on X , their *skew Pontrjagin product* (see [Pa] §1) is defined as

$$\mathcal{E} \hat{*} \mathcal{G} := p_{1*}((p_1 + p_2)^*(\mathcal{E}) \otimes p_2^*(\mathcal{G})).$$

We will see, in the spirit of [Mu] §3, the skew Pontrjagin product as a bifunctor from $\text{Mod}(X) \times \text{Mod}(X)$ to $\text{Mod}(X)$, and we denote by $\hat{*}^{\mathbf{R}}$ its derived functor. Moreover we

adopt the following terminology: the pair $(\mathcal{E}, \mathcal{G})$ satisfies the *Pontrjagin Index Theorem (P.I.T.)* with index $k = p(\mathcal{E}, \mathcal{G})$ if $h^i((T_x^* \mathcal{E}) \otimes \mathcal{G}) = 0$ for any $i \neq k$ and for any $x \in X$. If $R^i p_{1*}((p_1 + p_2)^*(\mathcal{E}) \otimes p_2^*(\mathcal{G})) = 0$ for $i \neq k$ we will say that $(\mathcal{E}, \mathcal{G})$ satisfies the *Weak Pontrjagin Index Theorem (P.W.I.T.)* with index $k = p(\mathcal{E}, \mathcal{G})$ and in this case (by abuse of notation) we will denote again

$$\mathcal{E} \hat{*} \mathcal{G} = R^k p_{1*}((p_1 + p_2)^*(\mathcal{E}) \otimes p_2^*(\mathcal{G})).$$

We will also use the following notation: given two sheaves \mathcal{E} and \mathcal{G} on X , we denote by $\mathcal{M}(E, G)$ the locus of $x \in X$ where the multiplication map

$$m_x : H^0(T_x^* \mathcal{E}) \otimes H^0(\mathcal{G}) \rightarrow H^0((T_x^* \mathcal{E}) \otimes \mathcal{G})$$

is not surjective. The relationship between skew Pontrjagin products and multiplication maps is provided by the following:

Proposition 5.2. ([Pa] Proposition 1.1) *Let \mathcal{E} and \mathcal{G} be sheaves on X such that $(\mathcal{E}, \mathcal{G})$ satisfies P.I.T. with $p(E, G) = 0$. Then*

$$\mathcal{M}(\mathcal{E}, \mathcal{G}) = B(\mathcal{E} \hat{*} \mathcal{G}).$$

Remark 5.3. If E and G are locally free and (E, G) satisfies P.I.T. with $p(E, G) = 0$, then (E^\vee, G^\vee) satisfies P.I.T. with $p(E^\vee, G^\vee) = g$ and, by relative Serre duality, $(E \hat{*} G)^\vee \cong E^\vee \hat{*} G^\vee$. In other words the dual of $E \hat{*} F$ is also a skew Pontrjagin product.

In view of Theorems 2.7 and 4.1, given a pair (E, G) satisfying P.I.T. with $p(E, G) = 0$ as in the remark above, in order to study the surjectivity of the multiplication map $m_0 : H^0(F) \otimes H^0(G) \rightarrow H^0(F \otimes G)$ it is then natural to investigate whether there exists an ample line bundle A on X such that the "mixed" product $((E \hat{*} G) \otimes A^\vee)^\vee \cong (E^\vee \hat{*} F^\vee) \otimes A$ satisfies W.I.T. with $i((E^\vee \hat{*} F^\vee) \otimes A) = g$. Following [Pa], an appropriate strategy turns out to be the following: first one establishes a suitable result of "exchange of Pontrjagin and tensor product under cohomology". Then, to prove the required vanishing, one uses the fact that, when the sheaves involved are line bundles algebraically equivalent to powers of a given one, say A , there is a suitable positive integer n such that, pulling back via multiplication by n , the skew Pontrjagin product is a trivial bundle tensored by a suitable power of A . Here we generalize this technique to a relative setting: Proposition 5.5 below is the relative analogue of the exchange of Pontrjagin and tensor product under cohomology while Proposition 5.6 provides the formulas for the pullback via multiplication by an integer (both in the usual and the relative setting).

Terminology/Notation 5.4. (Pontrjagin product relative with respect to the second variable, relative P.I.T. and P.W.I.T.) We denote by p_i and p_{ij} respectively, the projections and the intermediate projections of $X \times X \times \hat{X}$. Consider the bifunctor

$$? \hat{*}_{rel} ? = p_{13*}((p_1 + p_2)^* ? \otimes p_2^* ? \otimes p_{23}^* \mathcal{P})$$

from $\text{Mod}(X) \times \text{Mod}(X)$ to $\text{Mod}(X \times \hat{X})$, and let $\hat{*}_{rel}$ be its derived functor. As usual, we have corresponding notions of Index Theorem and Weak Index Theorem: e.g. relative P.I.T. with $p_{rel}(\mathcal{E}, \mathcal{G}) = k$ means that $H^i((T_x^* \mathcal{E}) \otimes \mathcal{G} \otimes P_\xi) = 0$ for any $x \in X$, $\xi \in \hat{X}$ and $i \neq k$. As above, if relative P.W.I.T. holds, we write $\hat{*}_{rel}$ rather than $\hat{*}_{rel}$. We denote by Γ the global sections functor.

Proposition 5.5. (Exchange of Pontrjagin and tensor product under (relative) cohomology.) (a) Assume that G and H are locally free sheaves on X and $?$ is either an object or a morphism in $\mathbf{D}(X)$. Then

$$(i) \mathbf{R}\Gamma((? \hat{*}^{\mathbf{R}} G) \otimes H) \cong \mathbf{R}\Gamma((? \hat{*}_{rel}^{\mathbf{R}} H) \otimes p_X^* G).$$

$$(ii) \mathbf{R}\hat{S}((? \hat{*}^{\mathbf{R}} G) \otimes H) \cong \mathbf{R}p_{\hat{X}*}((? \hat{*}_{rel}^{\mathbf{R}} H) \otimes p_X^* G).$$

(b) Let in addition \mathcal{E} be a sheaf on X such that (\mathcal{E}, G) satisfies P.I.T. with $p(\mathcal{E}, G) = k$. Then

(i) If (\mathcal{E}, H) satisfies P.I.T. with $p(\mathcal{E}, H) = k$, then for any i ,

$$H^i((\mathcal{E} \hat{*} G) \otimes H) \cong H^i((\mathcal{E} \hat{*} H) \otimes G).$$

(ii) If (\mathcal{E}, H) satisfies relative P.I.T. with $p_{rel}(\mathcal{E}, H) = k$, then for any i ,

$$R^i \hat{S}((\mathcal{E} \hat{*} G) \otimes H) \cong R^i p_{\hat{X}*}((\mathcal{E} \hat{*}_{rel} H) \otimes p_X^* G).$$

Proof. Part (b)(i) is precisely Lemma 3.2 of [Pa]. The rest of the proof follows the same argument in the derived setting. Note that we are suppressing part of the symbols showing that we are working with the derived functors, in order to simplify the notation. For the reader's convenience we prove (a)(ii) and (b)(ii). The left hand side of (a)(ii) can be written as

$$\begin{aligned} \mathbf{R}\hat{S}((? \hat{*}^{\mathbf{R}} G) \otimes H) &= \mathbf{R}p_{\hat{X}*}(\mathcal{P} \otimes p_X^*(H \otimes \mathbf{R}p_{1*}((p_1 + p_2)^{*?} \otimes p_2^* G))) \\ &\cong \mathbf{R}p_{\hat{X}*}(\mathcal{P} \otimes p_X^*(\mathbf{R}p_{1*}(p_1^* H \otimes (p_1 + p_2)^{*?} \otimes p_2^* G))) \\ &\cong \mathbf{R}p_{\hat{X}*}(\mathbf{R}p_{13*}(p_{13}^* \mathcal{P} \otimes p_1^* H \otimes (p_1 + p_2)^{*?} \otimes p_2^* G)) \\ &\cong \mathbf{R}p_{3*}(p_{13}^* \mathcal{P} \otimes p_1^* H \otimes (p_1 + p_2)^{*?} \otimes p_2^* G) \end{aligned}$$

On the other hand, working out the right hand side we have

$$\begin{aligned} \mathbf{R}p_{\hat{X}*}((? \hat{*}_{rel}^{\mathbf{R}} H) \otimes p_X^* G) &= \mathbf{R}p_{\hat{X}*}(\mathbf{R}p_{13*}((p_1 + p_2)^{*?} \otimes p_2^* H \otimes p_{23}^* \mathcal{P}_{23}) \otimes p_X^* G) \\ &\cong \mathbf{R}p_{\hat{X}*}(\mathbf{R}p_{13*}(p_1^* G \otimes (p_1 + p_2)^{*?} \otimes p_2^* H \otimes p_{23}^* \mathcal{P})) \\ &\cong \mathbf{R}p_{3*}(p_1^* G \otimes (p_1 + p_2)^{*?} \otimes p_2^* H \otimes p_{23}^* \mathcal{P}) \end{aligned}$$

The result follows via the automorphism $(x, y, \xi) \mapsto (y, x, \xi)$ of $X \times X \times \hat{X}$. As for (b)(ii), under the hypothesis at hand $(\mathcal{E} \hat{*} G) \otimes H$ reduces to $(\mathcal{E} \hat{*} G) \otimes H[k]$ and $(\mathcal{E} \hat{*}_{rel}^{\mathbf{R}} H) \otimes p_X^* G$ reduces to $(\mathcal{E} \hat{*}_{rel}^{\mathbf{R}} H) \otimes p_X^* G[k]$. Therefore the assertion follows from (a)(ii). \square

Pulling back via multiplication by an integer. When line bundles are involved, Pontrjagin products usually look simpler when pulled back via multiplication by an appropriate integer. The purpose of this subsection is to generalize the results of [Pa] §3(b) to Pontrjagin products relative with respect to the second variable. Given a positive integer n , the map $x \mapsto nx$ will be denoted $n_X : X \rightarrow X$.

Proposition 5.6. (a) Let n be a positive integer and let L be a line bundle on X .

$$(i) n_X^*(L \hat{*}^{\mathbf{R}} ?) \cong (L^n \hat{*}^{\mathbf{R}} (? \otimes L^{-n+1})) \otimes n_X^*(L) \otimes p_X^* L^{-n}$$

$$(ii) (n_X, 1_{\hat{X}})^*(L \hat{*}_{rel}^{\mathbf{R}} ?) \cong (L^n \hat{*}_{rel}^{\mathbf{R}} (? \otimes L^{-n+1})) \otimes (n_X, 1_X)^*(L) \otimes p_X^* L^{-n}$$

(b) Skew Pontrjagin products with the structure sheaf can be expressed as follows:

$$(i) \ ?^{\mathbf{R}}\hat{*}\mathcal{O}_X \cong \mathbf{R}\Gamma(?) \otimes \mathcal{O}_X$$

$$(ii) \ ?^{\mathbf{R}}\hat{*}_{rel}\mathcal{O}_X \cong p_{\hat{X}}^*(\mathbf{RS}(?)) \otimes \mathcal{P}^\vee$$

(c) Let A be an ample line bundle on X and assume that a and $a+b$ are positive integers. Then

$$(i) \ (a+b)_X^*(A^a\hat{*}(A^b \otimes P_\xi)) \cong H^0(A^{a+b} \otimes P_\xi) \otimes (a+b)_X^*A^a \otimes a_X^*(A^{-a-b}) \otimes P_\xi^{-a}$$

$$(a+b)_X^*(A^{-a}\hat{*}(A^{-b} \otimes P_\xi)) \cong H^g(A^{-a-b} \otimes P_\xi) \otimes (a+b)_X^*A^{-a} \otimes a_X^*(A^{a+b}) \otimes P_\xi^a$$

$$(ii) \ ((a+b)_X \times 1_{\hat{X}})^*(A^a\hat{*}_{rel}A^b) \cong p_{\hat{X}}^*((A^{a+b})^\wedge) \otimes p_X^*((a+b)_X^*A^a \otimes a_X^*A^{-a-b}) \otimes \mathcal{P}^{-a}$$

$$((a+b)_X \times 1_{\hat{X}})^*(A^{-a}\hat{*}_{rel}A^{-b}) \cong p_{\hat{X}}^*((A^{-a-b})^\wedge) \otimes p_X^*((a+b)_X^*A^{-a} \otimes a_X^*A^{a+b}) \otimes \mathcal{P}^a$$

Proof. Note that (a)(i) is Proposition 3.4 of [Pa] and the proof of (a)(ii) identical. Furthermore, (b)(i) is Remark 3.5(b) of [Pa] and (b)(ii) is proved in the same way. Therefore all these proofs are omitted. The first isomorphism of (c)(i) is Proposition 3.6 of [Pa], but note that in that paper there is a misprint: the last factor of the right hand side of [Pa] Prop. 3.6 should read $a_X^*(A^{-a-b} \otimes \alpha^\vee)$ instead of $a_X^*(A^{-a-b}) \otimes \alpha^\vee$. The present formulation follows since $a_X^*P_\xi^\vee = P_\xi^{-a}$. The second isomorphism of (c)(i) follows by duality. Finally (c)(ii) is proved exactly in the same way, using Lemma 5.7(a) below, and therefore its proof is omitted too. \square

Lemma 5.7. (a) $(n_X, 1_{\hat{X}})^*\mathcal{P} = (1_X, n_{\hat{X}})^*\mathcal{P} = \mathcal{P}^{\otimes n}$.

(b) $R^i p_{\hat{X}*}(\mathcal{P}^{\otimes n}) = 0$ for $i < g$ and $= \mathcal{O}_{\hat{X}_n}$ for $i = g$.

Proof. (a) By double duality it is enough to prove the first equality. We prove it by induction on n . For $n = 2$, we apply the Theorem of the Cube ([M1] Cor.2 p.58) with $f(x, \xi) = g(x, \xi) = (x, 0)$, $h(x, \xi) = (0, \xi)$ (all maps $X \times \hat{X} \rightarrow X \times \hat{X}$) and $L = \mathcal{P}$. We get, using that $\mathcal{P}_{|\{0\} \times \hat{X}} = \mathcal{O}_{\hat{X}}$ and $\mathcal{P}_{|X \times \{0\}} = \mathcal{O}_X$, that $(2_X, 1_{\hat{X}})^*\mathcal{P} = \mathcal{P}^{\otimes 2}$. The general formula follows by induction, applying the same method with $f(x, y) = ((n-1)_X, 0)$, $g(x, y) = (x, 0)$, $h(x, y) = (0, y)$.

(b) By flat base change, $R^i p_{\hat{X}*}((1_X \times n_{\hat{X}})^*(\mathcal{P})) = n_{\hat{X}}^* R^i p_X^* \mathcal{P} = 0$ if $i < g$ and $n_{\hat{X}}^* \mathcal{O}_0 = \mathcal{O}_{\hat{X}_n}$ if $i = g$. \square

A first application: theta-group-free proof of a theorem of Ohbuchi. We end this section by presenting a theta-group-free proof of (part of) a classical theorem of Ohbuchi ([Oh2], see also [Ke3] Theorem 10.4, [LB] Theorem 7.2.3 and [Kh] for a proof working in $\text{char}(k) \neq 2$) on the normal generation of a line bundle of the form A^2 , where A is an ample line bundle on an abelian variety. This is intended to be a toy version and an introduction to the new results on equations and syzygies in the next section, based on the techniques described above.

Within this framework, it is natural to state Ohbuchi's Theorem in a slightly more general way. (This can be however easily deduced from the usual statement: compare [LB] Theorem 7.2.3 and Exercise 7.2.) Given an (ample) line bundle A on X , let us denote $s(A) := A \otimes (-1)_X^* A^\vee$. The map $s : \text{Pic}^{c_1(A)}(X) \rightarrow \text{Pic}^0(X)$ is surjective and

flat $(s(A) \in \text{Pic}^0(X)$ classifies the "non-symmetry" of the line bundle A). Let also $t(A)$ denote a square root of $s(A)$.

Theorem 5.8. ($\text{char}(k) \neq 2$) *Let A be an ample line bundle on X . Then*

$$\mathcal{M}(A^2, A^2) = \bigcup_{\xi \in \hat{X}_2} B(A \otimes P_{t(A)} \otimes P_\xi).$$

Hence A^2 is normally generated if and only if $0 \notin \bigcup_{\xi \in \hat{X}_2} B(A \otimes P_{t(A)} \otimes P_\xi)$.

Proof. We will prove only the "positive part" of the result, i.e. the inclusion $\mathcal{M}(A^2, A^2) \subset \bigcup_{\xi \in \hat{X}_2} B(A \otimes P_{t(A)} \otimes P_\xi)$, since this is the part to be generalized in the next section. The opposite inclusion can be proved similarly. We have that

$$(1) \quad R^i \hat{\mathcal{S}}((A^{-2} \hat{*} A^{-2}) \otimes A) \cong \begin{cases} 0 & \text{if } i < g \\ \widehat{A^{-1}} \otimes \mathcal{O}_{\hat{X}_2 + t(A)} & \text{if } i = g, \end{cases}$$

where $\hat{X}_2 + t(A)$ denotes the set $\{\eta \mid \eta - t(A) \in \hat{X}_2\}$. Postponing the proof of (1) for a moment, let us show how it implies the statement. In fact (1) yields that the hypothesis of Theorem 4.1 are fulfilled by $(A^2 \hat{*} A^2) \otimes A^\vee$. Moreover (1) gives also that $J((A^2 \hat{*} A^2) \otimes A^\vee) = \hat{X}_2 + t(A)$ (this follows immediately from base change and Serre duality). Therefore, by Theorem 4.1, $B(A^2 \hat{*} A^2) \subset \bigcup_{\xi \in \hat{X}_2} B(A \otimes P_{t(A)} \otimes P_\xi)$. On the other hand, by Proposition 5.2, $\mathcal{M}(A^2, A^2) = B(A^2 \hat{*} A^2)$, and the statement is proved.

Proof of (1):

$$\begin{aligned} (2) \quad R^i \hat{\mathcal{S}}((A^{-2} \hat{*} A^{-2}) \otimes A) &\cong R^i p_{\hat{X}*}((A^{-2} \hat{*}_{rel} A) \otimes p_X^* A^{-2}) \\ (3) &\cong R^i p_{\hat{X}*}(p_X^* \widehat{A^{-1}} \otimes p_X^*(A^{-4} \otimes 2_X^* A) \otimes \mathcal{P}^2) \\ (4) &\cong R^i p_{\hat{X}*}(p_X^* \widehat{A^{-1}} \otimes p_X^*(A^{-1} \otimes (-1)_X^* A) \otimes \mathcal{P}^2) \\ (5) &= R^i p_{\hat{X}*}(p_X^* \widehat{A^{-1}} \otimes p_X^* P_{s(A)}^\vee \otimes \mathcal{P}^2) \\ (6) &\cong \begin{cases} 0 & \text{if } i < g \\ \widehat{A^{-1}} \otimes \mathcal{O}_{\hat{X}_2 + t(A)} & \text{if } i = g. \end{cases} \end{aligned}$$

In the sequence of congruences above, (2) follows by Proposition 5.5(b)(ii), (3) follows by Proposition 5.6(c)(ii) (second part) with $a = 2$ and $b = -1$, (4) from the fact that $2_X^* A \cong A^3 \otimes (-1)_X^* A$ and (6) from (a slight variant of) Lemma 5.7(b) and the projection formula. \square

6. Equations defining abelian varieties and their syzygies

Putting together the machinery of the previous paragraphs, in this section we address the question of bounding the degrees of the generators (and their syzygies) of the homogeneous ideal $I_{X,L}$ of an abelian variety X embedded by a complete linear series $|L|$, where L is a suitable power of an ample line bundle A . Our main result is:

Theorem 6.1. ($\text{char}(k) \neq 2, 3$.) *Let A be an ample line bundle on X , with no base divisor. Then:*

- (a) If $k \geq 3$ then I_{X,A^k} is generated by quadrics.
 (b) I_{X,A^2} is generated by quadrics and cubics.

Theorem 6.1 turns out to be a special case of a more general result, extending in the now well-known language of Green [Gr] bounds on the degrees of generators of the ideal $I_{X,L}$ to a hierarchy of conditions about higher syzygies. Specifically, given a variety X embedded in projective space by a complete linear series $|L|$, the line bundle L is said to *satisfy property N_p* if the first p steps of the minimal graded free resolution of the algebra $R_L = \bigoplus H^0(L^n)$ over the polynomial ring $S_L = \bigoplus \text{Sym}^n H^0(L)$ are linear, i.e. of the form

$$S_L(-p-1)^{\oplus i_p} \rightarrow S_L(-p)^{\oplus i_{p-1}} \rightarrow \cdots \rightarrow S_L(-2)^{\oplus i_1} \rightarrow S_L \rightarrow R_L \rightarrow 0.$$

Thus N_0 means that the embedded variety is projectively normal (*normal generation* in Mumford's terminology), N_1 means that the homogeneous ideal is generated by quadrics (*normal presentation*), N_2 means that the relations among these quadrics are generated by *linear* ones and so on.

More generally even (cf. [Pa]), one can define properties measuring how far the first p steps of the resolution are from being linear. To do this, fix $p \geq 0$, and consider the first p steps of the minimal free resolution of R_L as an S_L -module:

$$E_p \rightarrow E_{p-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow R_L \rightarrow 0,$$

where $E_0 = S_L \oplus \bigoplus_j S_L(-a_{0j})$ with $a_{0j} \geq 2$ (since the linear series is complete), $E_1 = \bigoplus_j S_L(-a_{1j})$ with $a_{1j} \geq 2$ (since the embedding is non-degenerate) and so on, up to $E_p = \bigoplus_j S_L(-a_{pj})$ with $a_{pj} \geq p+1$. Then L is said to *satisfy property N_p^r* if $a_{pj} \leq p+1+r$. In particular N_1^r means that $a_{1j} \leq 2+r$, i.e. the ideal $I_{X,L}$ is generated by forms of degree $\leq 2+r$, while property N_p^0 is the same as N_p .

With this terminology, the extension of Theorem 6.1 to arbitrary syzygies is the following:

Theorem 6.2. ($\text{char}(k)$ does not divide $(p+1)$ and $(p+2)$.) *Assume that A has no base divisor. Then:*

- (a) If $k \geq p+2$ then A^k satisfies property N_p .
 (b) More generally, if $(r+1)(k-1) \geq p+1$ then A^k satisfies property N_p^r .

A word about the proofs. Although Theorem 6.1 is subsumed in Theorem 6.2, we prefer to start by proving it separately. The reason is that a substantially higher degree of technicality in the proof of Theorem 6.2 might potentially make the main idea less transparent – with this separation, some of the similar arguments will not be repeated in the second proof.

Background material. We briefly recall some well-known facts about the relationship between condition N_p , or more generally N_p^r , and the surjectivity of suitable multiplication maps of vector bundles. For the facts surveyed here see e.g. [La1] and [Pa]. The main point is that condition N_p^r is equivalent to the exactness in the middle of the piece of the Koszul complex (cf. [Gr]):

$$(7) \quad \Lambda^{p+1} H^0(L) \otimes H^0(L^h) \rightarrow \Lambda^p H^0(L) \otimes H^0(L^{h+1}) \rightarrow \Lambda^{p-1} H^0(L) \otimes H^0(L^{h+2})$$

for all $h \geq r+1$. One can in turn express this as a vanishing condition for the cohomology of a suitable vector bundle. Specifically, for a globally generated line bundle L , let M_L be the kernel of the evaluation map:

$$(8) \quad 0 \rightarrow M_L \rightarrow H^0(L) \otimes \mathcal{O}_X \rightarrow L \rightarrow 0$$

It follows easily that the exactness of (7) is equivalent to the surjectivity of the map $\Lambda^{p+1}H^0(L) \otimes H^0(L^h) \rightarrow H^0(\Lambda^p M_L \otimes L^{h+1})$ arising from the exact sequence (obtained by taking exterior powers in (8)):

$$0 \rightarrow \Lambda^{p+1}M_L \otimes L^h \rightarrow \Lambda^{p+1}H^0(L) \otimes L^h \rightarrow \Lambda^p M_L \otimes L^{h+1} \rightarrow 0.$$

Therefore N_p^r holds as soon as

$$(9) \quad H^1(\Lambda^{p+1}M_L \otimes L^h) = 0, \quad \forall h \geq r+1.$$

(On abelian varieties the converse is also true since $H^1(L^h) = 0$ for $h \geq 1$.) This leads to:

Proposition 6.3. (a) ($\text{char}(k)$ does not divide $(p+1)$) If $H^1(M_L^{\otimes(p+1)} \otimes L^h) = 0$ for all $h \geq r+1$, then L satisfies condition N_p^r .

(b) Assume that $H^1(M_L^{\otimes p} \otimes L^h) = 0$. Then $H^1(M_L^{\otimes(p+1)} \otimes L^h) = 0$ if and only if the multiplication map

$$H^0(L) \otimes H^0(M_L^{\otimes p} \otimes L^h) \rightarrow H^0(M_L^{\otimes p} \otimes L^{h+1})$$

is surjective.

Proof. Part (a) follows from (9) since, under the assumption on the characteristic, $\Lambda^{p+1}M_L$ is a direct summand of $M_L^{\otimes(p+1)}$. Part (b) follows from the exact sequence

$$(10) \quad 0 \rightarrow M_L^{\otimes(p+1)} \otimes L^h \rightarrow H^0(L) \otimes M_L^{\otimes p} \otimes L^h \rightarrow M_L^{\otimes p} \otimes L^{h+1} \rightarrow 0.$$

□

Proof of Theorem 6.1. (a) The result is known for $k \geq 4$, so it is enough to prove it for $k = 3$. By Proposition 6.3(a), it suffices to show that

$$H^1(M_{A^3}^{\otimes 2} \otimes A^{3h}) = 0, \quad \forall h \geq 1.$$

Moreover, we know that $H^1(M_{A^3} \otimes A^{3h}) = 0$ for $h \geq 1$ – by (10) for $p = 0$, this is equivalent to the *normal generation* of A^3 , i.e. Koizumi's Theorem. Therefore, by Proposition 6.3(b), it is enough to prove that the multiplication map

$$H^0(A^3) \otimes H^0(M_{A^3} \otimes A^{3h}) \rightarrow H^0(M_{A^3} \otimes A^{3(h+1)})$$

is surjective for $h \geq 1$. Again, this is well known for $h \geq 2$ – it is equivalent to the fact that the homogeneous ideal of X embedded by $|A^3|$ is generated by forms of degree 2 and 3 (cf. [Ke1],[LB] or, in this interpretation, [Pa]). Therefore the only case to be examined is $h = 1$.

We prove more generally that the locus $\mathcal{M}(A^3, M_{A^3} \otimes A^3)$ is empty (cf. Proposition 5.2). By Proposition 5.2 we have $\mathcal{M}(A^3, M_{A^3 \otimes A^3}) = B(A^{3*}(M_{A^3} \otimes A^3))$, since from the defining sequence 8 it is not hard to see that the pair $(A^3, M_{A^3} \otimes A^3)$ satisfies P.I.T. with index 0. To this end we make the following:

Claim. If A has no base divisor, then $(A^3 \hat{*} (M_{A^3} \otimes A^3)) \otimes A^\vee$ is M -regular.

By Theorem 2.7 this yields that $A^3 \hat{*} (M_{A^3} \otimes A^3)$ is globally generated, and hence the theorem.

Proof of Claim. Recall from [PP] §3 that it is enough to prove that the cohomological support loci

$$V^i := \{ \xi \in \hat{X} \mid h^i((A^3 \hat{*} (M_{A^3} \otimes A^3)) \otimes A^\vee \otimes P_\xi) > 0 \}$$

have codimension $> i$ for all $i > 0$. By Proposition 5.5(b)(i) we have that

$$V^i = \{ \xi \in \hat{X} \mid h^i((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes M_{A^3} \otimes A^3) > 0 \}.$$

Let us consider the exact sequence obtained from 8

$$\begin{aligned} 0 \rightarrow (A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes M_{A^3} \otimes A^3 &\rightarrow H^0(A^3) \otimes (A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^3 \rightarrow \\ &\rightarrow (A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^6 \rightarrow 0 \end{aligned}$$

Subclaim. $h^i(((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^n)) = 0$ for any $n \geq 3$, $\xi \in \hat{X}$ and $i \geq 1$.

Proof. This is again a standard application of Proposition 5.6(c)(i): taking $a = 3$ and $b = -1$ we have that $2_X^*(A^3 \hat{*} (A^\vee \otimes P_\xi)) \cong H^0(A^2 \otimes P_\xi) \otimes 2_X^* A^3 \otimes 3_X^* A^{-2} \otimes P_\xi^{-3}$. This implies that

$$2_X^*(A^3 \hat{*} (A^\vee \otimes P_\xi) \otimes A^n) \cong H^0(A^2 \otimes P_\xi) \otimes 2_X^* A^3 \otimes 3_X^* A^{-2} \otimes P_\xi^{-3} \otimes 2_X^* A^n,$$

which is isomorphic to a sum of copies of line bundles algebraically equivalent to $A^{(4n-6)}$, thus certainly ample for $n \geq 3$. As we are in characteristic $\neq 2$, $H^i((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^n)$ is a direct summand of $H^i(2_X^*((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^n))$, which proves the subclaim. \square

Passing to cohomology in the exact sequence above, by the Subclaim we have that

- (i) V^i is empty for $i \geq 2$.
- (ii) V^1 coincides with the locus where the multiplication map

$$H^0(A^3) \otimes H^0((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^3) \rightarrow H^0((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^6)$$

is not surjective.

In view of (i), the Claim would be implied by the inequality $\text{codim}(V^1) > 1$. Again by Proposition 5.2, we have that

$$(11) \quad V^1 = \{ \xi \in \hat{X} \mid 0_X \in B(A^3 \hat{*} ((A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^3)) \}.$$

We will approach this by the same trick of twisting with A^\vee in order to try and apply Theorem 4.1. By relative duality we have

$$((A^3 \hat{*} (A^3 \hat{*} (A^\vee \otimes P_\xi)) \otimes A^3) \otimes A^\vee)^\vee \cong (A^{-3} \hat{*} (A^{-3} \hat{*} (A \otimes P_\xi^\vee)) \otimes A^{-3}) \otimes A.$$

By Proposition 5.5(b)(ii)

$$(12) \quad R^i \hat{S}((A^{-3} \hat{*} (A^{-3} \hat{*} (A \otimes P_\xi^\vee)) \otimes A^{-3}) \otimes A) \cong R^i p_{\hat{X}*}((A^{-3} \hat{*}_{rel} A) \otimes p_X^*((A^{-3} \hat{*} (A \otimes P_\xi^\vee)) \otimes A^{-3})).$$

The key point that W.I.T. with index g is satisfied goes through the following:

$$(13) \quad \begin{aligned} R^i p_{\hat{X}*}((2_X, 1_{\hat{X}})^*[(A^{-3} \hat{*}_{rel} A) \otimes p_X^*((A^{-3} \hat{*}(A \otimes P_\xi^\vee)) \otimes A^{-3})]) &\cong \\ &\cong \begin{cases} 0 & \text{if } i < g \\ H^g(A^{-2} \otimes P_\xi^\vee) \otimes \widehat{A^{-2}} \otimes \mathcal{O}_{\hat{X}_3 - s(A) - \xi} & \text{if } i = g \end{cases} \end{aligned}$$

Proof of (13). By Proposition 5.6(c)(ii) with $a = 3$ and $b = -1$:

$$(2_X, 1_{\hat{X}})^*(A^{-3} \hat{*}_{rel} A) \cong p_X^* \widehat{A^{-2}} \otimes p_X^*(2_X^* A^{-3} \otimes 3_X^* A^2) \otimes \mathcal{P}^3.$$

In conclusion, using also Proposition 5.6(c)(i),

$$\begin{aligned} &(2_X, 1_{\hat{X}})^*[(A^{-3} \hat{*}_{rel} A) \otimes p_X^*((A^{-3} \hat{*}(A \otimes P_\xi^\vee)) \otimes A^{-3})] \cong \\ &p_X^* \widehat{A^{-2}} \otimes p_X^*(2_X^* A^{-3} \otimes 3_X^* A^2 \otimes H^g(A^{-2} \otimes P_\xi^\vee) \otimes 2_X^* A^{-3} \otimes 3_X^* A^2 \otimes P_\xi^{-3} \otimes 2_X^* A^{-3}) \otimes \mathcal{P}^3 \cong \\ &\cong p_X^* H^g(A^{-2} \otimes P_\xi^\vee) \otimes p_X^* \widehat{A^{-2}} \otimes p_X^*(P_{s(A)}^\vee \otimes P_\xi^\vee)^3 \otimes \mathcal{P}^3 \end{aligned}$$

(Cf. the notation introduced before Theorem 5.8.) Therefore (13) follows from the projection formula and Lemma 5.7.

We are now able to conclude the proof of the Claim. As we are in $\text{char}(k) \neq 2$,

$$R^i p_{\hat{X}*}((A^{-3} \hat{*}_{rel} A) \otimes p_X^*((A^{-3} \hat{*}(A \otimes P_\xi^\vee)) \otimes A^{-3}))$$

is a direct summand of

$$R^i p_{\hat{X}*}((2_X, 1_{\hat{X}})^*[(A^{-3} \hat{*}_{rel} A) \otimes p_X^*((A^{-3} \hat{*}(A \otimes P_\xi^\vee)) \otimes A^{-3})]).$$

According to (12) and (13), it follows that $(A^3 \hat{*}(A^3 \hat{*}(A^\vee \otimes P_\xi) \otimes A^3)) \otimes A^\vee$ satisfies both hypotheses of Theorem 4.1. Moreover, by (13) it also follows, using relative duality and base change, that there is an inclusion

$$J((A^3 \hat{*}(A^3 \hat{*}(A^\vee \otimes P_\xi) \otimes A^3)) \otimes A^\vee) \subset \hat{X}_3 - s(A) - \xi.$$

Therefore Theorem 4.1 implies that

$$(14) \quad B(A^3 \hat{*}((A^3 \hat{*}(A^\vee \otimes P_\xi) \otimes A^3))) \subset \bigcup_{\eta \in \hat{X}_3} B(A \otimes P_{-s(A) - \xi} \otimes P_\eta).$$

From (11) and (14) it follows that $\text{codim}(S^1) = \text{codim}(B((A^3 \hat{*}(A^\vee \otimes P_\xi) \otimes A^3))) \geq \text{codim}(B(A))$ and this proves the Claim.

(b) The proof is completely similar. This time one has to prove that the multiplication map

$$H^0(A^2) \otimes H^0(M_{A^2} \otimes A^4) \rightarrow H^0(M_{A^2} \otimes A^6)$$

is surjective. As above, the result follows from the following statement, proved in the same way:

Claim. *If A has no base divisor then $(A^2 \hat{*}(M_{A^2} \otimes A^4)) \otimes A^\vee$ is M -regular.*

Proof of Theorem 6.2. The argument is a combination between the proof of Lazarsfeld's conjecture in [Pa] and the idea of the proof of Theorem 6.1. We prove only (a), since the proof of (b) is completely analogous. First of all the theorem is known for $k \geq p + 3$ and so we need to prove it only for $k = p + 2$. For $L = A^{p+2}$, the exactness of the complex (7) is known to hold for $h \geq 2$. (This means that the syzygies at the p -th step are generated at most in degree 2, i.e. condition N_p^1 in terminology of [Pa], and so it follows from [Pa] Theorem 4.3.) Putting everything together, by Proposition 6.3 it follows that it suffices to prove that the multiplication map

$$H^0(A^{p+2}) \otimes H^0(M_{A^{p+2}}^{\otimes p} \otimes A^p) \rightarrow H^0(M_{A^{p+2}}^{\otimes p} \otimes A^{2p})$$

is surjective. Given $\xi \in \hat{X}$, we denote

$$F_\xi^{(n,m)} := A^{n*}(A^m \otimes P_\xi)$$

We will prove the following:

Claim 1. For every integer k such that $1 \leq k \leq p$ and every $\xi_1, \dots, \xi_k \in \hat{X}$, the locally free sheaf

$$A^{p+2*}(M_{A^{p+2}}^{\otimes k} \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})$$

is globally generated.

For $k = p$ this, together with Proposition 5.2, proves the theorem (again, the fact that P.I.T. with index 0 is verified follows from [Pa] Proposition 4.2).

Proof of Claim 1. This goes by induction on k . Let us assume for a moment that we know the initial step $k = 1$, and show that the statement for $k - 1$ implies the statement for k , for all $k \geq 2$.

We fix $\xi_1, \dots, \xi_k \in \hat{X}$. By Theorem 2.7, it is enough to prove that the vector bundle

$$(A^{p+2*}(M_{A^{p+2}}^{\otimes k} \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \otimes A^\vee$$

is M -regular. In fact, for $k \geq 2$, it will even satisfy I.T. with index 0, i.e.

$$(15) \quad H^i((A^{p+2*}(M_{A^{p+2}}^{\otimes k} \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \otimes A^\vee \otimes P_\xi) = 0$$

for any $i > 0$ and any $\xi \in \hat{X}$. By Proposition 5.5(b)(i) we have that

$$\begin{aligned} & H^i((A^{p+2*}(M_{A^{p+2}}^{\otimes k} \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \otimes A^\vee \otimes P_\xi) \cong \\ & \cong H^i((A^{p+2*}(A^\vee \otimes P_\xi)) \otimes M_{A^{p+2}}^{\otimes k} \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2}). \end{aligned}$$

(As usual, the hypotheses of Proposition 5.5(b)(ii) are fulfilled because of [Pa] Proposition 4.2.) Then, as in the proof of the previous theorem (and we won't go through all the details), the sequence

$$0 \rightarrow M_{A^{p+2}}^{\otimes k} \rightarrow H^0(A^{p+2}) \otimes M_{A^{p+2}}^{\otimes k-1} \rightarrow M_{A^{p+2}}^{\otimes k-1} \otimes A^{p+2} \rightarrow 0$$

twisted by $(A^{p+2}\hat{*}(A^\vee \otimes P_\xi)) \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2}$ gives that the cohomology groups of (15) are zero, except for H^1 which vanishes if only if the multiplication map

$$\begin{aligned} H^0(A^{p+2}) \otimes H^0(M_{A^{p+2}}^{\otimes k-1} \otimes (A^{p+2}\hat{*}(A^\vee \otimes P_\xi)) \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2}) \rightarrow \\ \rightarrow H^0(M_{A^{p+2}}^{\otimes k-1} \otimes (A^{p+2}\hat{*}(A^\vee \otimes P_\xi)) \otimes \bigotimes_{i=1}^{p-k} F_{\xi_i}^{(p+2,-1)} \otimes A^{2(p+2)}) \end{aligned}$$

is surjective. But this follows from the inductive hypothesis and Proposition 5.2.

We are left with proving Claim 1 for $k = 1$. To this end we will apply the same reasoning as in the previous paragraph, only this time we use Theorem 2.7 with the weaker input that the sheaf in question is just M -regular.

Claim 2. The sheaf $A^{p+2}\hat{}(M_{A^{p+2}} \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})$ is M -regular.*

Proof. As before, the claim follows if we show that the locus of ξ such that the multiplication map

$$\begin{aligned} H^0(A^{p+2}) \otimes H^0((A^{p+2}\hat{*}(A^\vee \otimes P_\xi)) \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2}) \rightarrow \\ \rightarrow H^0((A^{p+2}\hat{*}(A^\vee \otimes P_\xi)) \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{2(p+2)}) \end{aligned}$$

is surjective has codimension at least 2. By Proposition 5.2, this locus is precisely

$$\left\{ \xi \mid 0_X \in B(A^{p+2}\hat{*}(F_{\xi}^{(p+2,-1)} \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \right\}.$$

By relative duality the dual of

$$(A^{p+2}\hat{*}(F_{\xi}^{(p+2,-1)} \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \otimes A^\vee$$

is

$$(A^{-p-2}\hat{*}(F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})) \otimes A$$

By Proposition 5.5 we have that

$$\begin{aligned} R^i \hat{\mathcal{S}}((A^{-p-2}\hat{*}(F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})) \otimes A) \cong \\ (16) \quad R^i p_{\hat{X}*}((A^{-p-2}\hat{*}_{rel} A) \otimes p_X^*(F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})) \end{aligned}$$

The key point, analogous to (13) of the previous proof, is that

$$\begin{aligned} R^i p_{\hat{X}*}(((p+1)_{X, 1_{\hat{X}}})^*((A^{-p-2}\hat{*}_{rel} A) \otimes F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})) \cong \\ (17) \quad \cong \begin{cases} 0 & \text{if } i < g \\ V \otimes \widehat{A^{-p-1}} \otimes \mathcal{O}_{\hat{X}_{p+2-\xi-\sum_{i=1}^{p-1} \xi_i - \frac{(p+1)(p+2)}{2}} s(A)}} & \text{if } i = g \end{cases} \end{aligned}$$

where V is a suitable vector space.

Proof of (17). By Proposition 5.6(c)(i) and (ii) we have that

$$\begin{aligned} ((p+1)_{X, 1_{\hat{X}}})^*((A^{-p-2}\hat{*}_{rel} A) \otimes F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2}) \cong \\ p_X^* \widehat{A^{-p-1}} \otimes V \otimes p_X^*((p+1)_{X, 1_{\hat{X}}})^* A^{-(p+2)(p+1)} \otimes (p+2)_{X, 1_{\hat{X}}}^* A^{(p+1)^2} \otimes P_{-\xi}^{p+2} \otimes \bigotimes_{i=1}^{p-1} P_{-\xi_i}^{p+2} \otimes \mathcal{P}^{-p-2} \end{aligned}$$

where V is the vector space $\bigotimes_{i=1}^{p-1} H^g(A^{-p-1} \otimes P_{\xi_i}^\vee) \otimes H^g(A^{-p-1} \otimes P_\xi^\vee)$. Therefore (17) follows from Lemma 5.7, noting that, by a standard calculation,

$$(p+1)_{\hat{X}}^*(A^{-(p+2)(p+1)} \otimes A^{-p-2}) \otimes (p+2)_{\hat{X}}^* A^{(p+1)^2} \cong P_{-s(A)}^{(p+1)^2(p+2)/2}.$$

The argument goes now as in the previous proof: we have first that

$$R^i p_{\hat{X}*}((A^{-p-2} \hat{*}_{rel} A) \otimes F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})$$

is a direct summand of

$$R^i p_{\hat{X}*}((2_X, 1_{\hat{X}})^*((A^{-p-2} \hat{*}_{rel} A) \otimes F_{-\xi}^{(-p-2,1)} \otimes \bigotimes_{i=1}^{p-1} F_{-\xi_i}^{(-p-2,1)} \otimes A^{-p-2})).$$

Summing up, the sheaf

$$(A^{p+2} \hat{*}_{\xi} (F_{\xi}^{(p+2,-1)} \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \otimes A^\vee$$

satisfies the hypotheses of Theorem 4.1, and its Fourier jump locus is included in $\hat{X}_{p+2} - \xi - \sum_{i=1}^{p-1} \xi_i - \frac{(p+1)(p+2)}{2} s(A)$. Thus, by Theorem 4.1, we finally have that

$$B(A^{p+2} \hat{*}_{\xi} (F_{\xi}^{(p+2,-1)} \otimes \bigotimes_{i=1}^{p-1} F_{\xi_i}^{(p+2,-1)} \otimes A^{p+2})) \subset \bigcup_{\eta \in \hat{X}_{p+2}} B(A \otimes P_{-\frac{(p+1)(p+2)}{2} s(A) - \sum \xi_i - \xi} \otimes P_\eta),$$

and the Claim follows since the base locus of A is of codimension at least 2.

A conjecture based on the M -regularity index. As already mentioned in Section 3, Theorem 6.2 raises a natural question about a potentially general relationship between the equations and syzygies of X in the embedding given by a power of a line bundle A , and the higher order properties of A , reflected in the M -regularity index $m(A)$ defined in §3.

Conjecture 6.4. *Let $p \geq m$ be non-negative integers. If A is ample and $m(A) \geq m$, then $A^{\otimes k}$ satisfies N_p for any $k \geq p + 3 - m$.*

This conjecture is a refinement of Lazarsfeld's conjecture, proved in [Pa], which is the case $m(A) = 0$, i.e. no conditions on A . Theorem 6.2 gives an affirmative answer to the conjecture for $m(A) = 1$, which by Example 3.6 happens precisely when A has no base divisor. We remark though that the methods of this paper fail to apply for powers A^k with $k \leq p + 1$, so a new idea seems to be needed for the case of higher regularity indices.

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