POSITIVITY FOR HODGE MODULES AND GEOMETRIC APPLICATIONS

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ABSTRACT. This is a survey of vanishing and positivity theorems for Hodge modules, and their recent applications to birational and complex geometry.

CONTENTS

A. INTRODUCTION

Over the past few years a substantial body of work has been devoted to applications of Morihiko Saito's theory of mixed Hodge modules to the geometry of complex algebraic varieties, as well as to questions in singularities, arithmetic geometry, representation theory, and other areas. One of the main reasons for this is

the realization that generalizations of classical constructions in Hodge theory, like Hodge bundles and de Rham complexes, arising in this context, satisfy analogues of well-known vanishing and positivity theorems.

This survey has two main aims. On one hand, I will review the vanishing, injectivity and weak positivity results that are known in the context of Hodge modules; it turns out that there is by now a complete package of such theorems extending the standard results of this type in birational geometry. On the other hand, I will describe applications of these results in, loosely speaking, the realm of birational geometry: results on families of varieties, holomorphic forms, generic vanishing theorems, and singularities, that at the moment we only know how to approach using the theory of Hodge modules.

Here is a brief review of the contents of the paper; each section contains a more detailed description of the topic treated there. First, in §1 and §2 I include a quick tour of the basic facts from the theory of filtered $\mathscr{D}$-modules and Hodge modules, respectively, that are used in the paper. There are also references to general material on these subjects.

Ch.C reviews the package of vanishing and positivity theorems for Hodge modules mentioned above. Ch.D contains applications of Hodge modules to generic vanishing. This is an area where their impact has been substantial, both by allowing to extend the range of applications in the projective setting, and by providing basic new results in the Kähler setting.

Both standard and generic vanishing for Hodge modules, as well as weak positivity, have proven to be useful in the study of families of varieties. The applications are obtained by means of a general construction, inspired by objects introduced by Viehweg and Zuo in a somewhat more restrictive context, which is described in §7. In §8 I explain how this construction is used in studying zeros of holomorphic one-forms, while in §9 this is applied to families of varieties of general type over abelian varieties. On the other hand, §10 contains applications of this construction to producing Viehweg-Zuo sheaves for families with maximal variation whose geometric generic fiber has a good minimal model; this in particular leads to a strengthening of Viehweg's hyperbolicity conjecture.

Ch.F is devoted to an introduction to recent joint work with M. Mustaţă, whose focus are objects called Hodge ideals, intimately linked to Saito's Hodge filtration on the $\mathscr{D}$-module of functions with poles along a given divisor. They are a useful tool in studying the singularities and Hodge theory of hypersurfaces and their complements in smooth complex varieties. Both global methods like vanishing theorems, and local techniques from the study of Hodge modules, are important here.

The paper is certainly not intended to be a balanced survey of all recent applications of Hodge modules, which is something beyond my abilities. I have however included in Ch.G a (non-exhaustive) list, with brief explanations, of notable uses of Hodge modules to areas other than those treated here, and references to further literature.

invitation and for suggesting that I submit a paper. Many results described here were obtained together with Christian Schnell, whom I would like to thank for numerous discussions and explanations on the theory of Hodge modules. I would also like to thank Mircea Mustaţă and Lei Wu for useful conversations on the topic.

B. Preliminaries on Hodge modules

The next two sections contain the minimal amount of information necessary in order to navigate the rest of the paper, for the reader who is not closely acquainted to the theory of Hodge modules; those who are can skip directly to Ch.C. The original, and most comprehensive, references for Hodge modules are Saito's papers [Sai88] and [Sai90b]. Recently Saito also wrote a more informal introduction in [Sai16]. Another general overview of the theory can be found in [Sch14b]. A detailed source in its development stages, but already containing plenty of useful information, is the project [SS16] by Sabbah and Schnell.

1. Background on $\mathscr{D}$-modules. This section contains a very quick review of the necessary tools from the theory of $\mathscr{D}$-modules. A comprehensive reference for the material here is [HTT08].

Filtered $\mathscr{D}$-modules. Let X be a smooth complex variety of dimension n. A filtered left $\mathscr{D}$-module on X is a $\mathscr{D}_X$-module $\mathcal{M}$ with an increasing filtration $F = F_\bullet\mathcal{M}$ by coherent $\mathscr{O}_X$-modules, bounded from below and satisfying

$$F_k\mathscr{D}_X \cdot F_\ell\mathcal{M} \subseteq F_{k+\ell}\mathcal{M} \ \text{ for all } k,\ell \in \mathbb{Z}.$$

In addition, the filtration is *good* if the inclusions above are equalities for $k \gg 0$. This condition is equivalent to the fact that the total associated graded object

$$\mathrm{gr}_\bullet^F \mathcal{M} = \bigoplus_k \mathrm{gr}_k^F \mathcal{M} = \bigoplus_k F_k\mathcal{M}/F_{k-1}\mathcal{M}$$

is finitely generated over $\mathrm{gr}_\bullet^F \mathscr{D}_X \simeq \mathrm{Sym}\, T_X$, i.e. induces a coherent sheaf on the cotangent bundle T^*X. Similar definitions apply to right $\mathscr{D}_X$-modules.

While the simplest example of a filtered left $\mathscr{D}_X$-module is $\mathscr{O}_X$, with the trivial filtration $F_k\mathscr{O}_X = \mathscr{O}_X$ for $k \geq 0$ and $F_k\mathscr{O}_X = 0$ for $k < 0$, the canonical bundle ω_X is naturally endowed with a filtered right $\mathscr{D}_X$-module structure; locally, if $z_1, \ldots, z_n$ are local coordinates on X, for any $f \in \mathscr{O}_X$ and any $P \in \mathscr{D}_X$ the action is

$$(f \cdot dz_1 \wedge \cdots \wedge dz_n) \cdot P = {}^t P(f) \cdot dz_1 \wedge \cdots \wedge dz_n.$$

Here, if $P = \sum_\alpha g_\alpha \partial^\alpha$, then ${}^t P = \sum_\alpha (-\partial)^\alpha g_\alpha$ is its formal adjoint. This structure leads to a one-to-one correspondence between left and right $\mathscr{D}_X$-modules given by

$$\mathcal{M} \mapsto \mathcal{N} = \mathcal{M} \otimes_{\mathscr{O}_X} \omega_X \ \text{ and } \ \mathcal{N} \mapsto \mathcal{M} = \mathcal{H}om_{\mathscr{O}_X}(\omega_X, \mathcal{N}),$$

while on filtrations the left-right rule is

$$F_p\mathcal{M} = F_{p-n}\mathcal{N} \otimes_{\mathscr{O}_X} \omega_X^{-1}.$$

In particular $F_k\omega_X = \omega_X$ for $k \geq -n$ and 0 for $k < -n$.

de Rham complex. Given a left $\mathscr{D}_X$-module $\mathcal{M}$, its de Rham complex is:

$$\mathrm{DR}(\mathcal{M}) = \left[\mathcal{M} \to \Omega_X^1 \otimes \mathcal{M} \to \cdots \to \Omega_X^n \otimes \mathcal{M}\right].$$

This is a $\mathbb{C}$-linear complex with differentials induced by the corresponding integrable connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes \Omega_X^1$. By placing it in degrees $-n, \ldots, 0$, we consider it to be the de Rham complex associated to the corresponding right $\mathscr{D}$-module. A filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$ induces a filtration on $\mathrm{DR}(\mathcal{M})$ by the formula

$$F_k \mathrm{DR}(\mathcal{M}) = \Big[F_k\mathcal{M} \to \Omega_X^1 \otimes F_{k+1}\mathcal{M} \to \cdots \to \Omega_X^n \otimes F_{k+n}\mathcal{M} \Big].$$

For any integer k, the associated graded complex for this filtration is

$$\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M}) = \Big[\mathrm{gr}_k^F \mathcal{M} \to \Omega_X^1 \otimes \mathrm{gr}_{k+1}^F \mathcal{M} \to \cdots \to \Omega_X^n \otimes \mathrm{gr}_{k+n}^F \mathcal{M} \Big].$$

This is now a complex of coherent $\mathscr{O}_X$-modules in degrees $-n, \ldots, 0$, providing an object in $\mathbf{D}^b(X)$, the bounded derived category of coherent sheaves on X.

The lowest non-zero graded piece of a filtered $\mathscr{D}$-module is of particular interest. For one such left $\mathscr{D}_X$-module $(\mathcal{M}, F)$ define

$$p(\mathcal{M}) := \min \{p \mid F_p\mathcal{M} \neq 0\} \quad\text{and}\quad S(\mathcal{M}) := F_{p(\mathcal{M})}\mathcal{M}.$$

For the associated right $\mathscr{D}_X$-module we then have

$$p(\mathcal{N}) = p(\mathcal{M}) - n \quad\text{and}\quad S(\mathcal{N}) = S(\mathcal{M}) \otimes \omega_X.$$

Pushforward. Let $f \colon X \to Y$ be a morphism of smooth complex varieties. The associated transfer module

$$\mathscr{D}_{X \to Y} := \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y$$

has the structure of a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule, and has a filtration given by $f^* F_k \mathscr{D}_Y$. The pushforward via f is a natural operation for right $\mathscr{D}_X$-modules; it is defined at the level of derived categories by the formula

$$f_+ : \mathbf{D}(\mathscr{D}_X) \longrightarrow \mathbf{D}(\mathscr{D}_Y), \quad \mathcal{N}^\bullet \mapsto \mathbf{R}f_*\big(\mathcal{N}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y}\big).$$

See [HTT08, §1.5] for more details, where this functor is denoted by $\int_f$.

Given a proper morphism of smooth varieties $f \colon X \to Y$, Saito has also constructed in [Sai88, §2.3] a filtered direct image functor

$$f_+ : \mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_X)\big) \to \mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_Y)\big).$$

Here the two categories are the bounded derived categories of filtered $\mathscr{D}$-modules on X and Y respectively. Without filtration, it is precisely the functor above; some details about the filtration appear below, for the special $\mathscr{D}$-modules considered here.

2. Background on Hodge modules. We now move to a review of the main notions and results from the theory of mixed Hodge modules that are used in this paper.

Hodge modules and variations of Hodge structure. Let X be a smooth complex variety of dimension n, and let Z be an irreducible closed subset. Let $\mathbb{V} = (\mathcal{V}, F^\bullet, \mathbb{V}_{\mathbb{Q}})$ be a polarizable variation of $\mathbb{Q}$-Hodge structure of weight ℓ on an open set U in the smooth locus of Z, where $(\mathcal{V}, \nabla)$ is a vector bundle with flat connection with underlying $\mathbb{Q}$-local system $\mathbb{V}_{\mathbb{Q}}$, and Hodge filtration $F^\bullet\mathcal{V}$. Following [Sai88], one can think of it as being smooth pure Hodge module of weight $\dim Z + \ell$ on U, whose main constituents are:

(1) The left $\mathscr{D}_U$-module $\mathcal{M} = \mathcal{V}$ with filtration $F_p\mathcal{M} = F^{-p}\mathcal{V}$.
(2) The $\mathbb{Q}$-perverse sheaf $P = \mathbb{V}_{\mathbb{Q}}[n]$.

According to Saito's theory, this extends uniquely to a *pure polarizable Hodge module* M of weight $\dim Z + \ell$ on X, whose support is Z. This has an underlying perverse sheaf, which is the (shifted) intersection complex $\mathrm{IC}_Z(\mathbb{V}_{\mathbb{Q}})[n] = {}^p j_{!*} \mathbb{V}_{\mathbb{Q}}[n]$ associated to the given local system. It also has an underlying $\mathscr{D}_X$-module, namely the minimal extension of $\mathcal{M}$, corresponding to $\mathrm{IC}_Z(\mathbb{V}_{\mathbb{C}})$ via the Riemann-Hilbert correspondence. Its filtration is (nontrivially) determined by the Hodge filtration on U; see e.g. Example 2.9 for the case of simple normal crossings boundary.

More generally, in [Sai88] Saito introduced an abelian category $\mathrm{HM}(X, \ell)$ of pure polarizable Hodge modules on X of weight ℓ. The main two constituents of one such Hodge module M are still:

(1) A filtered (regular holonomic) left $\mathscr{D}_X$-module $(\mathcal{M}, F)$, where $F = F_\bullet \mathcal{M}$ is a good filtration by $\mathscr{O}_X$-coherent subsheaves, so that $\mathrm{gr}_\bullet^F \mathcal{M}$ is coherent over $\mathrm{gr}_\bullet^F \mathscr{D}_X$.
(2) A $\mathbb{Q}$-perverse sheaf P on X whose complexification corresponds to $\mathcal{M}$ via the Riemann-Hilbert correspondence, so that there is an isomorphism

$$\alpha : \mathrm{DR}_X(\mathcal{M}) \xrightarrow{\simeq} P \otimes_{\mathbb{Q}} \mathbb{C}.$$

These are subject to a list of conditions, which are defined by induction on the dimension of the support of M. If X is a point, a pure Hodge module is simply a polarizable Hodge structure of weight ℓ. In general it is required, roughly speaking, that the nearby and vanishing cycles associated to M with respect to any locally defined holomorphic function are again Hodge modules, now on a variety of smaller dimension; for a nice discussion see [Sch14b, §12]. The definition of a polarization on M is quite involved, but at the very least it involves an isomorphism $\mathbf{D}P \simeq P(\ell)$, where $\mathbf{D}P$ is the Verdier dual of the perverse sheaf P, together of course with further properties compatible with the inductive definition of Hodge modules.

Furthermore, M. Saito introduced in [Sai90b] the abelian category $\mathrm{MHM}(X)$ of (graded-polarizable) *mixed Hodge modules* on X. In addition to data as in (1) and (2) above, in this case a third main constituent is:

(3) A finite increasing weight filtration $W_\bullet M$ of M by objects of the same kind, compatible with α, such that the graded quotients $\mathrm{gr}_\ell^W M = W_\ell M / W_{\ell-1} M$ are pure Hodge modules in $\mathrm{HM}(X, \ell)$.

Again, a mixed Hodge module on a point is a graded-polarizable mixed Hodge structure, while in general these components are subject to several conditions defined by induction on the dimension of the support of M, involving the graded quotients of the nearby and vanishing cycles of M; for a further discussion of the definition see [Sch14b, §20]. Only one class of examples of mixed (as opposed to pure) Hodge modules, described below, will appear explicitly in this paper.

Let D be a divisor in X with complement U, and assume that we are given a variation of Hodge structure $\mathbb{V}$ on U. Besides the pure Hodge module extension to X described above, it is also natural to consider a mixed Hodge module extension, denoted $j_* j^{-1} M$ in [Sai90b]. Its underlying perverse sheaf is simply the direct

image $j_*\mathbb{V}_\mathbb{Q}$. More precisely,

$$j_*j^{-1}M = \big((\mathcal{V}(*D), F); j_*\mathbb{V}_\mathbb{Q}\big),$$

where $\mathcal{V}(*D)$ is Deligne's meromorphic connection extending $\mathcal{V}$ (see e.g. [HTT08, §5.2]). Further details are given in Example 2.8.

Strictness. From the point of view of applications, we mainly think here of Hodge modules as being filtered $\mathscr{D}$-modules with extra structure. Part of this extra structure is the *strictness* of the filtration; this property of $\mathscr{D}$-modules underlying Hodge modules naturally appearing in birational geometry is crucial for applications.

Concretely, a morphism of filtered $\mathscr{D}_X$-modules $f\colon (\mathcal{M}, F) \to (\mathcal{N}, F)$, compatible with the filtrations, is called *strict* if

$$f(F_k\mathcal{M}) = F_k\mathcal{N} \cap f(\mathcal{M}) \quad \text{for all } k.$$

A complex $(\mathcal{M}^\bullet, F_\bullet\mathcal{M}^\bullet)$ is strict if all of its differentials are strict. This condition is equivalent to the injectivity, for every $i, k \in \mathbb{Z}$, of the induced morphism

$$H^i(F_k\mathcal{M}^\bullet) \longrightarrow H^i\mathcal{M}^\bullet.$$

It is only in this case that the cohomologies of $\mathcal{M}^\bullet$ can also be seen as filtered $\mathscr{D}_X$-modules. Moreover, if we denote by $\mathrm{FM}(\mathscr{D}_X)$ the category of filtered $\mathscr{D}_X$-modules, one can construct an associated bounded derived category $\mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_X)\big)$, and it continues to make sense to talk about the strictness of an object in $\mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_X)\big)$.

Recall now that given a proper morphism of smooth varieties $f\colon X \to Y$, we have a filtered direct image functor

$$f_+ : \mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_X)\big) \to \mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_Y)\big).$$

For a filtered right $\mathscr{D}_X$-module $(\mathcal{M}, F)$, the strictness of $f_+(\mathcal{M}, F)$ as an object in $\mathbf{D}^b\big(\mathrm{FM}(\mathscr{D}_Y)\big)$ is equivalent to the injectivity of the mapping

$$R^i f_*\big(F_k(\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y})\big) \to R^i f_*(\mathcal{M} \overset{\mathbf{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X \to Y})$$

for all integers i and k, where the filtration on the left is the natural tensor product filtration induced by those on $\mathcal{M}$ and $\mathscr{D}_{X \to Y}$.

Some fundamental theorems of Saito. It is time to state the most important results obtained by M. Saito in the theory of pure Hodge modules. They explain why Hodge modules are a natural and useful extension of the notion of variation of Hodge structure in the presence of singularities.

The first theorem describes the simple objects in the category of pure polarizable Hodge modules on X. A pure Hodge module supported precisely along a subvariety Z is said to have *strict support* if it has no nontrivial subobjects or quotient objects whose support is Z.

Theorem 2.1 (**Simple objects**, [Sai90b, Theorem 3.21]). *Let X be a smooth complex variety, and Z an irreducible closed subvariety of X. Then:*

   (1) *Every polarizable VHS of weight ℓ defined on a nonempty open set of Z extends uniquely to a pure polarizable Hodge module of weight $\ell + \dim Z$ with strict support Z.*
   (2) *Conversely, every pure polarizable Hodge module of weight $\ell + \dim Z$ with strict support Z is obtained in this way.*

The category $\mathrm{HM}(X, \ell)$ is in fact semi-simple, due to the existence of polarizations: each object admits a decomposition by support, and simple objects with support equal to an irreducible subvariety Z are precisely those in the theorem.

One of the most important results of the theory is Saito's theorem on the behavior of direct images of polarizable Hodge modules via projective morphisms, only a part of which is stated below.

Theorem 2.2 (**Stability Theorem**, [Sai88, Théorème 5.3.1]). *Let $f \colon X \to Y$ be a projective morphism of smooth complex varieties, and $M \in \mathrm{HM}(X, \ell)$ a polarizable pure Hodge module on X with underlying filtered* right $\mathscr{D}$-module $(\mathcal{M}, F)$. *Then the filtered direct image $f_+(\mathcal{M}, F)$ is strict, and $\mathcal{H}^i f_+ \mathcal{M}$ underlies a polarizable Hodge module $M_i \in \mathrm{HM}(Y, \ell + i)$.*

The strictness of the direct image is a key property of $\mathscr{D}$-modules underlying Hodge modules that is not shared by arbitrary filtered $\mathscr{D}$-modules. It is one of the most crucial inputs for the applications described in this survey, by means of the following consequences:

(i) (*Laumon's formula*, see e.g. [PS13, §2.2, Theorem 9] for a proof.) If $\mathcal{A}_X := \mathrm{Sym}\, T_X \simeq \mathrm{gr}^F_\bullet \mathscr{D}_X$, and $\mathcal{A}_Y$ is defined similarly on Y, then

$$\mathrm{gr}^F_\bullet \mathcal{H}^i f_+(\mathcal{M}, F) \simeq R^i f_* \big(\mathrm{gr}^F_\bullet \mathcal{M} \otimes_{\mathcal{A}_X} f^* \mathcal{A}_Y \big),$$

(ii) (*Saito's formula*, see [Sai88, 2.3.7].) The associated graded of the filtered de Rham complex satisfies

$$R^i f_* \mathrm{gr}^F_k \mathrm{DR}(\mathcal{M}, F) \simeq \mathrm{gr}^F_k \mathrm{DR}\big(\mathcal{H}^i f_+(\mathcal{M}, F) \big).$$

(iii) (*Degeneration of the Hodge-to-de Rham spectral sequence.*) In the case of the constant morphism $f \colon X \to \mathrm{pt}$, the strictness of $f_+(\mathcal{M}, F)$ is equivalent to the E_1-degeneration of the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = \mathbf{H}^{p+q}\big(X, \mathrm{gr}^F_{-q} \mathrm{DR}(\mathcal{M}) \big) \implies H^{p+q}\big(X, \mathrm{DR}(\mathcal{M}) \big).$$

Finally, a fundamental consequence of (the full version of) the Stability Theorem is the extension of the well-known Decomposition Theorem in topology to the setting of pure polarizable Hodge modules [Sai88]; here I state the filtered $\mathscr{D}$-modules version, used later.

Theorem 2.3 (**Saito Decomposition Theorem**). *Let $f \colon X \to Y$ be a projective morphism of smooth complex varieties, and let $M \in \mathrm{HM}(X, \ell)$, with underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F)$. Then*

$$f_+(\mathcal{M}, F) \simeq \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i f_+(\mathcal{M}, F)[-i]$$

in the derived category of filtered $\mathscr{D}_Y$-modules.

Examples. The Hodge modules that have proved most useful in birational geometry to date are among the simplest concrete examples of such objects that one can write down; nevertheless, they contain deep information.

Example 2.4 (**The trivial Hodge module**). If X is smooth of dimension n and $\mathbb{V} = \mathbb{Q}_X$ is the constant variation of Hodge structure, we have that $P = \mathbb{Q}_X[n]$, while $\mathcal{M} = \mathscr{O}_X$ with the natural left $\mathscr{D}_X$-module structure and trivial filtration, so that $\mathrm{gr}_k^F \mathscr{O}_X$ is $\mathscr{O}_X$ for $k = 0$ and 0 for $k \neq 0$. The associated Hodge module is usually denoted $\mathbb{Q}_X^H[n]$. The de Rham complex of $\mathcal{M}$ is

$$\mathrm{DR}_X(\mathscr{O}_X) = \left[\mathscr{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^n\right][n],$$

i.e. the usual de Rham complex of X considered in degrees $-n, \ldots, 0$. With the filtration induced from that on $\mathscr{O}_X$, note that

$$\mathrm{gr}_{-k}^F \mathrm{DR}(\mathscr{O}_X) = \Omega_X^k[n-k] \quad \text{for all } k.$$

In this example we have $p(\mathcal{M}) = 0$ and $S(\mathcal{M}) = \mathscr{O}_X$.

Example 2.5 (**Direct images**). Let $f \colon X \to Y$ be a projective morphism of smooth complex varieties of dimensions n and m respectively, and let $\mathbb{V}$ be a polarizable variation of $\mathbb{Q}$-Hodge structure of weight k on an open dense subset $U \subset X$, inducing a pure Hodge module M of weight $n+k$ with strict support X. Here it is convenient to use right $\mathscr{D}$-modules: if $(\mathcal{N}, F)$ is filtered right $\mathscr{D}_X$-module underlying M, Theorem 2.3 gives a decomposition

$$f_+(\mathcal{N}, F) \simeq \bigoplus_i (\mathcal{N}_i, F)[-i]$$

in the derived category of filtered $\mathscr{D}_Y$-modules. According to Theorem 2.2, each $(\mathcal{N}_i, F)$ underlies a pure Hodge module $M_i = \mathcal{H}^i f_* M$ on Y, of weight $n + k + i$. Furthermore, $f_+(\mathcal{N}, F)$ is strict; using Saito's formula (ii) in the previous subsection this implies for instance that

$$(2.6) \qquad \mathbf{R}f_* S(\mathcal{N}) \simeq F_{p(\mathcal{N})}(f_+\mathcal{N}) \simeq \bigoplus_i F_{p(\mathcal{N})}\mathcal{N}_i[-i]$$

in the bounded derived category of coherent sheaves on Y.

For example, when $\mathbb{V}$ is the constant variation of Hodge structure $\mathbb{Q}_X$, by Example 2.4 we have $p(\mathcal{N}) = -n$ and $S(\mathcal{N}) = \omega_X$. This implies for all i that

$$p(N_i) = m - n \quad \text{and} \quad F_{m-n}\mathcal{N}_i = R^i f_* \omega_X.$$

Consequently, formula (2.6) specializes to

$$\mathbf{R}f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i],$$

which is the well-known Kollár decomposition theorem [Kol86b].[1] In fact $R^i f_* S(\mathcal{N})$ satisfy other important properties known from [Kol86a] in the case of canonical bundles; for instance, answering a conjecture of Kollár, Saito proved the following:

Theorem 2.7 (Saito, [Sai91a])**.** *Let $f \colon X \to Y$ be a surjective projective morphism, with X smooth, and let $(\mathcal{M}, F)$ be the filtered $\mathscr{D}$-module underlying a pure Hodge*

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[1]To recover the full statement of Kollár's theorem note that we should allow Y to be singular, and we can indeed do this. One can work with Hodge modules on singular varieties using local embeddings into smooth varieties and Kashiwara's theorem; for details see [Sai90b, §2.1] or [Sch14a, §7].

module with strict support X that is generically a polarized variation of Hodge structure $\mathbb{V}$. For each $i \geq 0$, one has

$$R^i f_* S(\mathcal{M}) = S(Y, \mathbb{V}^i),$$

the lowest Hodge piece of the variation of Hodge structure $\mathbb{V}^i$ on the intersection cohomology of $\mathbb{V}$ along the fibers of f. Consequently, $R^i f_* S(\mathcal{M})$ are torsion-free, and in particular

$$R^i f_* S(\mathcal{M}) = 0 \quad \text{for} \quad i > \dim X - \dim Y.$$

Example 2.8 (**Localization**). Let $\mathcal{M}$ be a left $\mathscr{D}_X$-module and D an effective divisor on a smooth variety X, given locally by an equation f. One can define a new $\mathscr{D}_X$-module $\mathcal{M}(*D)$ by localizing $\mathcal{M}$ in f; in other words, globally we have

$$\mathcal{M}(*D) = j_* j^{-1} \mathcal{M},$$

where $j : U \hookrightarrow X$ is the inclusion of the complement $U = X \smallsetminus D$.

When $\mathcal{M}$ underlies a mixed Hodge module M, $\mathcal{M}(*D)$ underlies the corresponding mixed Hodge module $j_* j^{-1} M$, and so continues to carry a natural Hodge filtration F. This is in general very complicated to compute; for instance, the case $\mathcal{M} = \mathscr{O}_X$, where $\mathscr{O}_X(*D)$ is the sheaf of meromorphic functions on X that are holomorphic on U and the corresponding Hodge module is $j_* \mathbb{Q}_U^H[n]$, is very subtly related to the singularities of D, and to Deligne's Hodge filtration on $H^\bullet(U, \mathbb{C})$. It is known for example that

$$S\big(\mathscr{O}_X(*D)\big) = \mathscr{O}_X(D) \otimes \mathcal{I}\big(X, (1-\epsilon)D\big),$$

where ideal in question is the multiplier ideal of the $\mathbb{Q}$-divisor $(1-\epsilon)D$, with $0 < \epsilon \ll 1$. Much more on the Hodge filtration on $\mathscr{O}_X(*D)$ is discussed in Section F.

Example 2.9 (**Normal crossings case**). Often one analyzes the Hodge filtration on a Hodge $\mathscr{D}$-module of interest by pushing forward a better understood filtration on a log resolution, and applying the Stability Theorem. This example shows that for both the pure and mixed Hodge module extensions of a VHS over simple normal crossings boundary, there are explicit formulas for the underlying filtered $\mathscr{D}$-module.

Let (X, D) be a pair consisting of a smooth projective variety X of dimension n, and a simple normal crossings divisor D. Denote $U = X \smallsetminus D$ and $j : U \hookrightarrow X$. We consider a polarizable VHS

$$\mathbb{V} = (\mathcal{V}, F_\bullet, \mathbb{V}_\mathbb{Q})$$

over U, with quasi-unipotent local monodromies along the components D_i of D. In particular the eigenvalues of all residues are rational numbers. For $\alpha \in \mathbb{Z}$, we denote by $\mathcal{V}^{\geq \alpha}$ (resp. $\mathcal{V}^{> \alpha}$) the Deligne extension with eigenvalues of residues along the D_i in $[\alpha, \alpha + 1)$ (resp. $(\alpha, \alpha + 1]$). Recall that $\mathcal{V}^{\geq \alpha}$ is filtered by

$$(2.10) \qquad\qquad F_p \mathcal{V}^{\geq \alpha} = \mathcal{V}^{\geq \alpha} \cap j_* F_p \mathcal{V},$$

while the filtration on $\mathcal{V}^{> \alpha}$ is defined similarly. The terms in the filtration are locally free by Schmid's nilpotent orbit theorem [Sch73] (see also e.g. [Kol86b, 2.5(iii)] for the quasi-unipotent case).

If M is the pure Hodge module with strict support X uniquely extending $\mathbb{V}$, it is proved in [Sai90b, §3.b] that

$$M = \big(\mathscr{D}_X \mathcal{V}^{> -1}, F_\bullet, j_{!*} \mathbb{V}_\mathbb{Q}\big),$$

where
$$F_p \mathscr{D}_X \mathcal{V}^{>-1} = \sum_i F_i \mathscr{D}_X \cdot F_{p-i} \mathcal{V}^{>-1}.$$

For the natural mixed Hodge module extension of $\mathbb{V}$, namely
$$j_* j^{-1} M = \big(\mathcal{V}(*D), F_\bullet, j_* \mathbb{V}_\mathbb{Q} \big),$$

recall that $\mathcal{V}(*D)$ is Deligne's meromorphic connection extending $\mathcal{V}$. It has a lattice defined by $\mathcal{V}^{\geq \alpha}$ for any $\alpha \in \mathbb{Q}$, i.e. $\mathcal{V}(*D) = \mathcal{V}^{\geq \alpha} \otimes \mathscr{O}_X(*D)$, and its filtration is given by
$$F_p \mathcal{V}(*D) = \sum_i F_i \mathscr{D}_X \cdot F_{p-i} \mathcal{V}^{\geq -1}.$$

C. Vanishing and positivity package for Hodge modules

One of the key aspects regarding the applications of Hodge module theory to birational geometry is the existence of a full package of vanishing and positivity theorems generalizing well-known results about (direct images of) canonical bundles; many of those had of course themselves been obtained using more classical methods in Hodge theory.

3. Vanishing theorems. We start with a series of vanishing results for mixed Hodge modules, which includes generalizations of most of the standard vanishing theorems.

Kodaira-Saito vanishing. Saito noted in his original paper [Sai90b] that the Kodaira-Nakano vanishing theorem has a far-reaching extension to the setting of mixed Hodge modules.

Theorem 3.1 ([Sai90b, §2.g]). *Let X be a complex projective variety, and L an ample line bundle on X. Consider an integer $m > 0$ such that $L^{\otimes m}$ is very ample and gives an embedding $X \subseteq \mathbb{P}^N$. Let $(\mathcal{M}, F)$ be the filtered $\mathscr{D}$-module underlying a mixed Hodge module M on $\mathbb{P}^N$ with support contained in X. Then:*

(1) $\mathrm{gr}_k^F \mathrm{DR}_{\mathbb{P}^N}(\mathcal{M})$ is an object in $\mathbf{D}^b(X)$ for each k, independent of the embedding of X in $\mathbb{P}^N$.

(2) We have the hypercohomology vanishing
$$\mathbf{H}^i \big(X, \mathrm{gr}_k^F \mathrm{DR}_{\mathbb{P}^N}(\mathcal{M}) \otimes L \big) = 0 \text{ for all } i > 0.$$
and
$$\mathbf{H}^i \big(X, \mathrm{gr}_k^F \mathrm{DR}_{\mathbb{P}^N}(\mathcal{M}) \otimes L^{-1} \big) = 0 \quad \text{for all } i < 0.$$

Remark 3.2. The sightly awkward formulation in the statement is due to the fact that X is allowed to be singular, which is important in applications (see for instance Example 3.3(ii) below). See [Sch14a, §7] and [Pop14, §8] for more on this. When X is smooth, one can of course work directly with a mixed Hodge module on X.

The proof given in [Sai90b] goes along the lines of Ramanujam's topological approach to Kodaira vanishing; a detailed account is given in [Pop14]. Another proof, this time along the lines of an approach to vanishing theorems due to Esnault-Viehweg, was given in [Sch14a].

Example 3.3. (i) (**Kodaira-Nakano vanishing.**) Let X be a smooth projective complex variety of dimension n. For the trivial Hodge module $M = \mathbb{Q}_X^H[n]$, according to Example 2.4 the corresponding left $\mathscr{D}$-module is $\mathscr{O}_X$, and

$$\mathrm{gr}_{-k}^F \mathrm{DR}_X(\mathscr{O}_X) = \Omega_X^k[n - k] \ \text{ for all } k.$$

In this case Theorem 3.1 is therefore the Kodaira-Nakano vanishing theorem.

As this is used repeatedly, it is worth recording the fact that the Kodaira vanishing theorem, which corresponds to the lowest non-zero graded piece of the filtration on ω_X, can be seen as an example of the simplest special case of Theorem 3.1.

Corollary 3.4. *If $(\mathcal{M}, F)$ is the filtered $\mathscr{D}$-module underlying a mixed Hodge module M on a projective variety X,[2] and L is an ample line bundle on X, then*

$$H^i\big(X, \omega_X \otimes S(\mathcal{M}) \otimes L\big) = 0 \ \text{ for all } i > 0.$$

(ii) (**Kollár vanishing.**) Let $f \colon X \to Y$ be a morphism of complex projective varieties, with X smooth of dimension n, and let L be an ample line bundle on Y. Considering the push-forward $f_* \mathbb{Q}_X^H[n]$, according to Example 2.5 for the underlying *right* $\mathscr{D}$-module we have

$$f_+(\omega_X, F) \simeq \bigoplus_i (\mathcal{N}_i, F)[-i]$$

in the derived category of filtered $\mathscr{D}_Y$-modules, and for each i we have $S(\mathcal{N}_i) = R^i f_* \omega_X$. Therefore here a special case of Corollary 3.4 is Kollár's vanishing theorem [Kol86a, Theorem 2.1(iii)], saying that for any ample line bundle L on Y,

$$H^i(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{ for all } i > 0 \text{ and all } j.$$

(iii) (**Nadel vanishing.**) If D is an effective divisor on a smooth projective variety X, recall from Example 2.8 that $\mathscr{O}_X(*D)$ underlies a mixed Hodge module, and that its lowest term in the Hodge filtration is $\mathscr{O}_X(D) \otimes \mathcal{I}\big(X, (1 - \epsilon)D\big)$. Using Corollary 3.4, it follows that

$$H^i\big(X, \omega_X \otimes L \otimes \mathcal{I}\big(X, (1 - \epsilon)D\big)\big) = 0 \quad \text{for all } i > 0,$$

for any line bundle L such that $L(-D)$ is ample. This is a special case of the Nadel vanishing theorem, see [Laz04, Theorem 9.4.8]. Nadel vanishing for arbitrary $\mathbb{Q}$-divisors cannot be seen directly as an example of Saito vanishing, but can be deduced from it using covering constructions.

Kawamata-Viehweg-type vanishing. In [Pop14, §11] it was noted that a vanishing theorem for big and nef divisors, analogous to the well-known Kawamata-Viehweg theorem, holds for the lowest term $S(\mathcal{M})$ of the Hodge filtration, under a transversality hypothesis involving the non-smooth locus of M and the augmented base locus of L. This hypothesis has been removed by Suh and Wu independently.

Theorem 3.5 ([Suh15], [Wu15]). *Let X be a complex projective variety, L a big and nef line bundle on X, and $(\mathcal{M}, F)$ the filtered left $\mathscr{D}$-module underlying a polarizable pure Hodge module with strict support X. Then*

$$H^i\big(X, \omega_X \otimes S(\mathcal{M}) \otimes L\big) = 0 \quad \text{for all } i > 0.$$

---

[2]In the sense of Theorem 3.1 if X is singular.

The approach in [Suh15] is in the spirit of the Esnault-Viehweg method using the degeneration of the Hodge-to-de Rham spectral sequence. In the process Suh obtains a related statement for Deligne canonical extensions that applies to all graded quotients of the Hodge filtration; this itself has interesting applications, see Ch.G. The approach in [Wu15] uses an induction based on adjunction-type formulas involving mixed Hodge modules of nearby cycles, more in line with Kawamata's approach to vanishing.

Remark 3.6. One cannot expect the statement in Theorem 3.5 to hold for other graded pieces $\mathrm{gr}_k^F \mathrm{DR}(\mathcal{M})$ with $k \neq p(\mathcal{M})$, as in Theorem 3.1. Indeed, in the case $M = \mathbb{Q}_X^H[n]$ we have seen in Example 3.3 that these graded pieces are (shifts of) various Ω_X^p; it is well-known however that the analogue of Nakano vanishing does not hold for big and nef divisors.

Injectivity. On a related note, Wu [Wu15] has shown that the vanishing results involving the lowest term $S(\mathcal{M})$ of the Hodge filtration admit a further improvement extending Kollár's injectivity theorem [Kol86a, Theorem 2.2] and its generalizations by Esnault-Viehweg [EV92, 5.1]. The proof relies on Suh's approach to vanishing for Deligne canonical extensions in [Suh15].

Theorem 3.7 ([Wu15, Theorem 1.4]). *Let X be a complex projective variety, D an effective divisor on X, and $(\mathcal{M}, F)$ the filtered left $\mathscr{D}$-module underlying a polarizable pure Hodge module with strict support X. Let L be a line bundle on X satisfying either one of the following properties:*

(1) L is nef and big
(2) L is semiample, and $H^0\big(X, L^{\otimes m}(-D)\big) \neq 0$ for some $m > 0$.

Then the natural mapping

$$H^i\big(X, \omega_X \otimes S(\mathcal{M}) \otimes L\big) \longrightarrow H^i\big(X, \omega_X \otimes S(\mathcal{M}) \otimes L(D)\big)$$

is injective for all i.

$\mathbb{Q}$-divisors. Just as in the usual setting, in view of future applications it is important to have extensions of the results above to the case of $\mathbb{Q}$-divisors. In [Pop14, Theorem 11.1], a Kawamata-Viehweg-type vanishing theorem for arbitrary $\mathbb{Q}$-divisors is proved under the transversality hypothesis mentioned above. On the other hand, [Wu15, §7] contains $\mathbb{Q}$-versions of Theorem 3.5 and Theorem 3.7 with no such hypotheses; these necessarily require extra conditions on the $\mathbb{Q}$-coefficients in the abstract Hodge theoretic setting, but do recover the standard vanishing theorems for $\mathbb{Q}$-divisors. Recently Wu [Wu16] has announced a further generalization of the Kawamata-Viehweg vanishing theorem to the setting of Hodge modules, involving arbitrary $\mathbb{Q}$-divisors.

4. Weak positivity. A general property of the lowest term of the Hodge filtration of a generically defined polarizable variation of (mixed) Hodge structure, or of an extension of such over simple normal crossings boundary, is its semipositivity. This was studied in a celebrated series of works by Griffiths, Fujita, Kawamata, Kollár and others; for the most recent results in this direction, and detailed references, see Fujino-Fujisawa-Saito [FFS14].

To deal with more general situations, Viehweg introduced a more relaxed notion of positivity, still sufficient for many applications: a torsion-free coherent sheaf $\mathscr{F}$ on a quasiprojective variety X is *weakly positive* on a non-empty open set $U \subseteq X$ if for every ample line bundle A on X and every $a \in \mathbb{N}$, the sheaf $\hat{S}^{ab}\mathscr{F} \otimes A^{\otimes b}$ is generated by global sections at each point of U for b sufficiently large.[3] For line bundles for instance, while semipositivity is analogous to nefness in algebraic language, weak positivity is equivalent to pseudo-effectivity. Viehweg [Vie83a] showed for instance that for any morphism of smooth projective varieties $f \colon Y \to X$, and any $m \geq 1$, the sheaf $f_*\omega_{Y/X}^{\otimes m}$ is weakly positive.

These results have useful versions at the level of Hodge modules. For the rest of the section, we fix a polarizable pure Hodge module M with strict support X, and consider the filtered left $\mathscr{D}_X$-module $(\mathcal{M}, F)$ underlying M. It is not hard to see that $S(\mathcal{M})$ is a torsion-free sheaf. First, along the lines of results regarding the lowest term in the Hodge filtration, one has the following extension to this setting:

Theorem 4.1 ([Sch13, Theorem 1.4], [Pop14, Theorem 10.4])**.** *The sheaf $S(\mathcal{M})$ is weakly positive torsion-free sheaf.*

Example 4.2. When $f \colon Y \to X$ is a morphism of smooth projective varieties, taking $M = M_0$ in Example 2.6, one recovers Viehweg's result above for $m = 1$, i.e. the fact that $f_*\omega_{Y/X}$ is weakly positive. Moreover, taking $M = M_i$, one gets the same result for $R^i f_*\omega_{Y/X}$ for all i.

A sketch of the proof of Theorem 4.1 may be instructive here, as it shows the efficiency of considering Hodge modules, as well as an approach originally due to Kollár using vanishing; it provides a quicker argument even in the setting of Viehweg's theorem. In brief, considering for any $m > 0$ the diagonal embedding

$$i \colon X \hookrightarrow X \times \cdots \times X,$$

where the product is taken m times, and pulling back the box-product $M \boxtimes \cdots \boxtimes M$ via i, it is not too hard to show that there exists an abstract Hodge module M' on X, together with a morphism

$$S(\mathcal{M}') \longrightarrow S(\mathcal{M})^{\otimes m},$$

which is an isomorphism on the open set where M is a VHS. Now taking any very ample line bundle L on X, Corollary 3.4 combined with Castelnuovo-Mumford regularity shows that $S(\mathcal{M}') \otimes \omega_X \otimes L^{\otimes(n+1)}$ is globally generated, where $n = \dim X$. Since m is arbitrary, a standard argument implies the weak positivity of $S(\mathcal{M})$.

It turns out that there is a more general result, which holds at each step of the Hodge filtration; this is very useful for applications. It has its origin in previous results of Zuo [Zuo00] and Brunebarbe [Bru15], which are analogous Griffiths-type metric statements in the setting of Deligne canonical extensions over a simple normal crossings boundary. Concretely, for each p we have a natural Kodaira-Spencer type $\mathscr{O}_X$-module homomorphism

$$\theta_p : \mathrm{gr}_p^F \mathcal{M} \longrightarrow \mathrm{gr}_{p+1}^F \mathcal{M} \otimes \Omega_X^1$$

induced by the $\mathscr{D}_X$-module structure, and we denote

$$K_p(\mathcal{M}) := \ker \theta_p.$$

---

[3]Here $\hat{S}^p \mathscr{F}$ denotes the reflexive hull of the symmetric power $S^p \mathscr{F}$.

Theorem 4.3 ([PW16, Theorem A]). *The torsion-free sheaf $K_p(\mathcal{M})^\vee$ is weakly positive for any p.*

One can show that Theorem 4.3 implies Theorem 4.1 via duality arguments. On the other hand, the proof of Theorem 4.3 is done by reduction to the metric statements in the papers cited above; it would be interesting to know whether it has a proof based on vanishing theorems as well.

D. GENERIC VANISHING

Hodge modules have made an impact in generic vanishing theory. Besides theoretical results on the graded pieces of the Hodge filtration that are useful in applications, as we will see later on, their study has led to previously inaccessible results on bundles of holomorphic forms (in the projective setting) and higher direct images of canonical bundles, as well as structural results (in the Kähler setting).

Let T be a compact complex torus. Given a coherent sheaf $\mathscr{F}$ on T, and an integer $i \geq 1$, one defines

$$V^i(\mathscr{F}) = \left\{ L \in \mathrm{Pic}^0(T) \mid H^i(T, \mathscr{F} \otimes L) \neq 0 \right\}.$$

Recall the following definitions from [PP11a] and [PP03] respectively; the sheaf $\mathscr{F}$ is called a *GV-sheaf* if the inequality

$$\mathrm{codim}_{\mathrm{Pic}^0(T)} V^i(\mathscr{F}) \geq i$$

is satisfied for every integer $i \geq 0$. It is called *M-regular* if the inequality

$$\mathrm{codim}_{\mathrm{Pic}^0(T)} V^i(\mathscr{F}) \geq i + 1$$

is satisfied for every integer $i \geq 1$. The notion of GV-sheaf is meant to formalize the behavior of canonical bundles of compact Kähler manifolds with generically finite Albanese maps, which were shown to have this property in [GL87]. Using vanishing theorems, Hacon [Hac04, §4] showed the stronger fact that if $f \colon X \to A$ is a morphism from a smooth projective variety to an abelian variety, then the higher direct image sheaves $R^j f_* \omega_X$ are GV-sheaves on A; see also [PP11a] for further developments. On the other hand, the stronger notion of M-regularity turns out to be very useful in the study of syzygies and of linear series on irregular varieties; see e.g [PP11b] and the references therein.

5. Generic vanishing on smooth projective varieties. Given a morphism $f \colon X \to A$ to an abelian variety, unlike in the case of $R^j f_* \omega_X$, with the techniques developed in the papers above it has not been possible to obtain optimal results regarding the GV-property of other bundles of holomorphic forms or, in an appropriate sense, of rank 1 local systems. This was overcome in [PS13] by a series of generic vanishing results involving mixed Hodge modules; here is a first version:

Theorem 5.1 ([PS13, Theorem 1.1]). *Let A be a complex abelian variety, and M a mixed Hodge module on A with underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F)$. Then, for each $k \in \mathbb{Z}$, the coherent sheaf $\mathrm{gr}_k^F \mathcal{M}$ is a GV-sheaf on A.*

Note that by Examples 2.4 and 2.5, this result includes the theorems of Green-Lazarsfeld and Hacon described above. For a subsequent generalization, see Theorem 6.2 below. The main reason for Theorem 5.1 is the fact that on abelian varieties Kodaira-Saito vanishing holds in a much stronger form than Theorem 3.1:

Proposition 5.2 ([PS13, Lemma 2.5]). *Under the assumptions of Theorem 5.1, if L is an ample line bundle, then*

$$H^i\big(A, \mathrm{gr}_k^F \mathcal{M} \otimes L\big) = 0 \ \text{ for all } i > 0.$$

This has a quite simple explanation, due to the triviality of the cotangent bundle: if $g = \dim A$, then we have

$$\mathrm{gr}_k^F \mathrm{DR}_A(\mathcal{M}) = \Big[\mathrm{gr}_k^F \mathcal{M} \to \Omega_A^1 \otimes \mathrm{gr}_{k+1}^F \mathcal{M} \to \cdots \to \Omega_A^g \otimes \mathrm{gr}_{k+g}^F \mathcal{M} \Big],$$

supported in degrees $-g, \ldots, 0$. According to Theorem 3.1, this complex satisfies

$$\mathbf{H}^i\big(A, \mathrm{gr}_k^F \mathrm{DR}_A(\mathcal{M}) \otimes L\big) = 0 \ \text{ for all } i > 0.$$

On the other hand, using the fact that $\Omega_A^1 \simeq \mathcal{O}_A^{\oplus g}$, and $\mathrm{gr}_k^F \mathcal{M} = 0$ for $k \ll 0$, one has for each ℓ a distinguished triangle

$$E_\ell \to \mathrm{gr}_\ell^F \mathrm{DR}_A(\mathcal{M}) \to \mathrm{gr}_{\ell+g}^F \mathcal{M} \to E_\ell[1],$$

where it can be assumed inductively that E_ℓ is an object satisfying $H^i(A, E_\ell \otimes L) = 0$. One can then deduce the statement of Proposition 5.2 by induction on k. Theorem 5.1 can be deduced from Proposition 5.2 by using a cohomological criterion of Hacon [Hac04] for detecting the GV-property.

Theorem 5.1 implies an optimal Nakano-type generic vanishing theorem, by taking $(\mathcal{M}, F)$ to be the various constituents of the push-forward $f_+(\omega_X, F)$ via the Decomposition Theorem, as in Example 2.5. Here $f\colon X \to A$ is the Albanese map of X. Recall that *defect of semismallness* of f is defined by the formula

$$\delta(f) = \max_{\ell \in \mathbb{N}}\big(2\ell - \dim X + \dim A_\ell\big),$$

where $A_\ell = \big\{ y \in A \mid \dim f^{-1}(y) \ge \ell \big\}$ for $\ell \in \mathbb{N}$. The condition $\delta(f) = 0$ is equivalent to f being semismall.

Theorem 5.3 ([PS13, Theorem 1.2]). *Let X be a smooth complex projective variety of dimension n. Then*[4]

$$\mathrm{codim}\, V^q(\Omega_X^p) \ge |p + q - n| - \delta(f) \quad \text{for all} \ \ p, q \in \mathbb{N},$$

and there exist p, q for which equality is attained. In particular, f is semismall if and only if X satisfies the generic Nakano vanishing theorem, i.e.

$$\mathrm{codim}\, V^q(\Omega_X^p) \ge |p + q - n| \quad \text{for all} \ \ p, q \in \mathbb{N}.$$

Working with Hodge modules allows one however to obtain a much more complex picture than Theorem 5.1, putting in some sense topologically trivial line bundles and holomorphic 1-forms on equal footing. If $(\mathcal{M}, F)$ is the filtered $\mathscr{D}$-module underlying a mixed Hodge module on A, we denote by $\mathrm{gr}^F \mathcal{M}$ the coherent sheaf

---

[4]Here, by analogy with the definition for sheaves on tori, one sets

$$V^q(\Omega_X^p) = \big\{ L \in \mathrm{Pic}^0(X) \mid H^q(X, \Omega_X^p \otimes L) \ne 0 \big\}.$$

on $T^*A = A \times V$ corresponding to the total associated graded $\mathrm{gr}_\bullet^F \mathcal{M}$, where $V = H^0(A, \Omega_A^1)$. If P is the normalized Poincaré bundle on $A \times \hat{A}$, using its pull-back to $A \times \hat{A} \times V$ as a kernel, we define the (relative) Fourier-Mukai transform of $\mathrm{gr}^F \mathcal{M}$ to be the complex of coherent sheaves on $\hat{A} \times V$ given by

$$E = \mathbf{R}\Phi_P(\mathrm{gr}^F \mathcal{M}) = \mathbf{R}(p_{23})_* \Big(p_{13}^* \big(\mathrm{gr}^F \mathcal{M}\big) \otimes p_{12}^* P \Big).$$

Theorem 5.4 ([PS13, Theorem 1.6]). *With the notation above, let $(\mathcal{M}, F)$ be the filtered left $\mathscr{D}$-module associated to any of the direct summands of $f_+(\omega_X, F)$, where $f\colon X \to A$ is a morphism from a smooth projective variety. Then the complex E has the following properties:*

(1) *E is a perverse coherent sheaf, meaning that its cohomology sheaves $\mathcal{H}^\ell E$ are zero for $\ell < 0$, and*

$$\mathrm{codim}\ \mathrm{Supp}(\mathcal{H}^\ell E) \geq 2\ell \quad \text{for} \ \ \ell \geq 0.$$

(2) *The union of the supports of all the higher cohomology sheaves of E is a finite union of translates of triple tori in $\hat{A} \times V$.*

(3) *The dual complex $\mathbf{R}\mathcal{H}om(E, \mathscr{O})$ has the same properties.*

Here a triple torus, in Simpson's terminology [Sim93], is a subset of the form

$$\mathrm{im}\Big(\varphi^*\colon \hat{B} \times H^0(B, \Omega_B^1) \to \hat{A} \times H^0(A, \Omega_A^1) \Big),$$

where $\varphi\colon A \to B$ is a morphism to another abelian variety. The properties of Hodge $\mathscr{D}_A$-modules of geometric origin provided by the theorem are very useful in applications; for instance they are key to the proof of Theorem 8.1 below.

Theorem 5.4 turns out to hold for arbitrary holonomic $\mathscr{D}$-modules on A; this, as well as numerous other similar results for $\mathscr{D}$-modules on abelian varieties, were obtained by Schnell in [Sch15]. Moreover, similar results are obtained in [PS13] and [Sch15] involving a different type of Fourier-Mukai transform, namely the Laumon-Rothstein transform

$$\mathbf{R}\Phi_{P^\natural}\colon \mathbf{D}_{\mathrm{coh}}^b(\mathscr{D}_A) \to \mathbf{D}_{\mathrm{coh}}^b(\mathscr{O}_{A^\natural}),$$

where on the left we have coherent $\mathscr{D}_A$-modules, and on the right coherent sheaves on the moduli space $A^\natural$ of line bundle with integrable connection on A. This leads to generalizations of the generic vanishing results in [GL91].

6. Generic vanishing on compact Kähler manifolds. The original results on generic vanishing for ω_X were established by Green and Lazarsfeld [GL87], [GL91] by means of Hodge theory, and therefore apply to compact Kähler manifolds. On the other hand, two well-known theorems on projective varieties, Simpson's [Sim93] result that every irreducible component of every

$$\Sigma^k(X) = \big\{ \rho \in \mathrm{Char}(X) \ \big| \ H^k(X, \mathbb{C}_\rho) \neq 0 \big\}$$

contains characters of finite order, and Hacon's result [Hac04] that higher direct images $R^i f_* \omega_X$ are GV-sheaves, have eluded on Kähler manifolds proofs based on similar methods.

This difficulty can be overcome by the use of Hodge modules. The following extension of Hacon's theorem was also motivated by a result of Chen-Jiang [CJ13] for $f_* \omega_X$ on projective varieties of maximal Albanese dimension.

Theorem 6.1 ([PPS15, Theorem A])**.** *Let $f \colon X \to T$ be a holomorphic mapping from a compact Kähler manifold to a compact complex torus. Then for $j \geq 0$, one has a decomposition*

$$R^j f_* \omega_X \simeq \bigoplus_{k=1}^n \big(q_k^* \mathscr{F}_k \otimes L_k \big),$$

where $\mathscr{F}_k$ are M-regular (hence ample) coherent sheaves with projective support on the compact complex tori T_k, $q_k \colon T \to T_k$ are surjective morphisms with connected fibers, and $L_k \in \mathrm{Pic}^0(T)$ have finite order. In particular, $R^j f_* \omega_X$ is a GV-sheaf on T.

Thus the direct images $R^j f_* \omega_X$, being direct sums of pull-backs of ample sheaves, are essentially projective objects and have positivity properties that go beyond the usual weak positivity in the sense of Viehweg (cf. §4). Since they are the lowest nonzero term in filtrations on Hodge $\mathscr{D}$-modules with integral structure, according to Example 2.5, the result above is a consequence of the following statement generalizing Theorem 5.1. It is important to note that on Kähler manifolds the absence of a polarization defined over $\mathbb{Q}$ leads to the need for a more general context of polarizable Hodge modules with real structure. It was already suggested in [Sai90a] how to develop such a theory; for more details see [PPS15, Ch.B].

Theorem 6.2 ([PPS15, Theorem D])**.** *Let $M = \big((\mathcal{M}, F); M_{\mathbb{R}} \big)$ be a polarizable real Hodge module on a complex torus T. Then for each $k \in \mathbb{Z}$, the coherent $\mathscr{O}_T$-module $\mathrm{gr}_k^F \mathcal{M}$ decomposes as*

$$\mathrm{gr}_k^F \mathcal{M} \simeq \bigoplus_{j=1}^n \big(q_j^* \mathscr{F}_j \otimes_{\mathscr{O}_T} L_j \big),$$

where $q_j \colon T \to T_j$ is a surjective map with connected fibers to a complex torus, $\mathscr{F}_j$ is an M-regular coherent sheaf on T_j with projective support, and $L_j \in \mathrm{Pic}^0(T)$. If M admits an integral structure, then each L_j has finite order.

On the other hand, Simpson's result on points of finite order was extended to the Kähler setting by Wang [Wan13], using in the process results on Hodge modules on abelian varieties from [Sch15].

Theorem 6.3 ([Wan13, Theorem 1.5])**.** *If a polarizable Hodge module M of geometric origin on a compact complex torus T admits an integral structure, then the sets*

$$\Sigma_m^k(T, M) = \big\{ \, \rho \in \mathrm{Char}(T) \; \big| \; \dim H^k(T, M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}_\rho) \geq m \, \big\}$$

are finite unions of translates of linear subvarieties by points of finite order.

The condition of being of geometric origin essentially means that M is a direct summand in the pushforward of the trivial complex Hodge module via a holomorphic map $f \colon X \to T$, via the Decomposition Theorem. Wang's result was extended to arbitrary polarizable (real) Hodge modules on T in [PPS15, Theorem E].

Going back to the higher direct images $R^i f_* \omega_X$, their GV-property can be applied to classification results for compact Kähler manifolds, in line with well-known statements in the projective setting. For projective varieties, part (1) below is a theorem of Chen-Hacon [CH01], answering a conjecture of Kollár, while (2) is a theorem of Jiang [Jia11], providing an effective version of result of Kawamata.

Theorem 6.4 ([PPS15, Theorem B, Theorem 19.1]). *Let X be a compact Kähler manifold with $P_1(X) = P_2(X) = 1$. Then:*

> *(1) If $b_1(X) = 2\dim X$, then X is bimeromorphic to a compact complex torus.*
> *(2) More generally, the Albanese map of X is surjective, with connected fibers.*

Note that the more precise statement in Theorem 6.1 leads to simplified proofs of these results even in the projective setting. Further applications to the classification of irregular manifolds with small invariants can be found in [Par16].

E. FAMILIES OF VARIETIES

7. A Hodge module construction for special families. One of the essential motivations for the development of Hodge modules was to extend the theory of geometric variations of Hodge structure to morphisms with arbitrary singularities. Hodge modules are thus a very useful tool for attacking problems regarding families of varieties. They provide concrete additions to this already rich area of study from two main points of view:

> (i) One can sometimes work directly with morphisms of smooth projective varieties whose singular locus is arbitrary.
> (ii) The general fiber does not need to have strong positivity properties.

For the statement below, we need to introduce a locus which measures the singularities of a surjective morphism $f\colon X \to Y$ of smooth projective varieties. We use the following notation for the induced morphisms between the cotangent bundles:

$$
\begin{array}{ccccc}
X & \xleftarrow{\ p_2\ } & T^*Y \times_Y X & \xrightarrow{\ df\ } & T^*X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_1} & & \\
Y & \xleftarrow{\ p\ } & T^*Y & &
\end{array}
$$

Inside the cotangent bundle of Y, we consider the set of singular cotangent vectors

$$S_f = p_1\big(df^{-1}(0)\big) \subseteq T^*Y.$$

One can easily check that S_f is the union of the zero section and a closed conical subset of T^*Y whose image via p is equal to D_f, the singular locus of f.

In the problems we consider below, starting from diverse hypotheses one always reaches the following situation: there is a surjective morphism $f\colon X \to Y$ between two smooth projective varieties, and a line bundle L on Y, such that if we consider on X the line bundle

$$B = \omega_{X/Y} \otimes f^*L^{-1},$$

the following condition holds:

$$(7.1) \qquad\qquad H^0(X, B^{\otimes m}) \neq 0 \quad \text{for some } m \geq 1.$$

Starting from this data, one can construct a graded module over $\mathcal{A}_Y := \operatorname{Sym}\mathscr{T}_Y$ with the following properties.

Theorem 7.2 ([PS15, Theorem 9.2]). *Assuming (7.1), one can find a graded $\mathcal{A}_Y$-module $\mathscr{G}_\bullet$ that is coherent over $\mathcal{A}_Y$ and has the following properties:*

(a) *As a coherent sheaf on the cotangent bundle,* $\operatorname{Supp}\mathscr{G} \subseteq S_f$.

(b) *One has* $\mathscr{G}_0 \simeq L \otimes f_*\mathscr{O}_X$.

(c) *Each* $\mathscr{G}_k$ *is torsion-free on the open subset* $Y \setminus D_f$.

(d) *There exists a regular holonomic* $\mathscr{D}_X$*-module* $\mathcal{M}$ *with good filtration* $F_\bullet\mathcal{M}$, *and an inclusion of graded* $\mathcal{A}_Y$*-modules* $\mathscr{G}_\bullet \subseteq \operatorname{gr}^F_\bullet\mathcal{M}$.

(e) *The filtered* $\mathscr{D}$*-module* $(\mathcal{M}, F)$ *underlies a polarizable Hodge module* M *on* Y *with strict support* Y, *and* $F_k\mathcal{M} = 0$ *for* $k < 0$.

The Hodge module M and the graded submodule $\mathscr{G}_\bullet$ are constructed by applying results from the theory of Hodge modules to a new morphism determined by a section $0 \neq s \in H^0\big(X, B^{\otimes m}\big)$. Let's at least specify M: such a section defines a branched covering $\pi\colon X_m \to X$ of degree m, unramified outside the divisor $Z(s)$. If m is chosen to be minimal, X_m is irreducible, and we consider a resolution of singularities $\mu\colon Z \to X_m$ that is an isomorphism over the complement of $Z(s)$. We define $\varphi = \pi \circ \mu$ and $h = f \circ \varphi$ as in the diagram

(7.3)

$$
\begin{array}{ccccc}
 & & \varphi & & \\
Z & \xrightarrow{\ \mu\ } & X_m & \xrightarrow{\ \pi\ } & X \\
 & \searrow & & & \downarrow f \\
 & h & & & Y
\end{array}
$$

If $d = \dim X = \dim Z$, let

$$\mathcal{H}^0 h_*\mathbb{Q}_Z^H[d] \in \operatorname{HM}(Y, d)$$

be the polarizable Hodge module obtained by taking the direct image of the constant Hodge module on Z, provided by Theorem 2.2; restricted to the smooth locus of h, this is just the polarizable variation of Hodge structure on the middle cohomology of the fibers. One takes $M \in \operatorname{HM}(Y, d)$ to be the summand with strict support Y in the decomposition of $\mathcal{H}^0 h_*\mathbb{Q}_Z^H[d]$ by strict support. Constructing $\mathscr{G}_\bullet$ requires extra work, specifically using the section s and the strictness property in Theorem 2.2; see [PS15, §11] for details.

This type of construction was introduced by Viehweg and Zuo [VZ01], [VZ02], [VZ03] in the context of Higgs bundles, under the assumption that B is semiample, but it becomes both simpler and more flexible through the use of Hodge modules. It is also useful for applications to use the local properties of Hodge modules in order to construct a suitable Higgs sheaf from the graded module $\mathscr{G}_\bullet$; this is done in [PS15, Theorem 9.4].

8. Zeros of holomorphic one-forms. The following basic result, conjectured (and proved in some important special cases) by Hacon-Kovács [HK05] and Luo-Zhang [LZ05], was obtained in [PS14] as an application of part of the construction in Theorem 7.2, combined with results in §5.

Theorem 8.1 ([PS14, Theorem 2.1])**.** *Let* X *be a smooth complex projective variety, and assume that there is a linear subspace* $W \subseteq H^0(X, \Omega^1_X)$ *such that* $Z(\omega)$ *is empty for every nonzero* 1*-form* $\omega \in W$. *Then*

$$\dim W \leq \dim X - \kappa(X).$$

In particular, if X is of general type, then every holomorphic 1-form on X has at least one zero.

It is quite standard to reduce the statement to the last assertion, i.e. to the case when X is of general type. It is also possible to express the result in the theorem in terms of the Albanese map $f\colon X \to A$ of X. As such, one of the key points in the proof is to have access to a Hodge module M with underlying filtered $\mathscr{D}$-module $(\mathcal{M}, F)$ and a graded submodule $\mathscr{G}_\bullet \subseteq \mathrm{gr}^F_\bullet \mathcal{M}$ on A as in Theorem 7.2; but in this set-up it is almost immediate to see that, at least after passing to an étale cover of X, condition (7.1) is satisfied for any ample line bundle L on A.

Moreover, one can show that the support of $\mathscr{G}_\bullet$, seen as a coherent sheaf on $T^*A \simeq A \times H^0(X, \Omega^1_X)$, satisfies the property

$$\mathrm{Supp}\, \mathscr{G}_\bullet \subseteq (f \times \mathrm{id})(Z_f),$$

where

$$Z_f = \{(x, \omega) \mid \omega(x) = 0\} \subseteq X \times H^0(X, \Omega^1_X).$$

Thus to prove the theorem it suffices to show that $\mathrm{Supp}\, \mathscr{G}_\bullet$ projects surjectively onto the space of holomorphic 1-forms. One obtains this by using the full package of generic vanishing theorems described in §5, applied to the Hodge module M.

While not directly a result about families of varieties, Theorem 8.1 has consequences to the study of families over abelian varieties via the following corollary, obtained by taking $W = f^* H^0(A, \Omega^1_A)$.

Corollary 8.2 ([PS14, Corollary 3.1]). *If $f\colon X \to A$ is a smooth morphism from a smooth complex projective variety onto an abelian variety, then*

$$\dim A \leq \dim X - \kappa(X).$$

This implies for example that there are no nontrivial smooth morphisms from a variety of general type to an abelian variety; this fact was proved in [VZ01] when the base is an elliptic curve, and can be deduced from [HK05] in general.

9. Families of varieties of general type over abelian varieties. The main application of Corollary 8.2 is to deduce the birational isotriviality of smooth families of varieties of general type over abelian varieties. Note that when the fibers are canonically polarized, Kovács [Kov97] and Zhang [Zha97] have shown that the family must be isotrivial.

Theorem 9.1 ([PS14, Corollary 3.2]). *If $f\colon X \to A$ is a smooth morphism onto an abelian variety, with fibers of general type, then f is birationally isotrivial.*

The conclusion means that X becomes birational to a product after a generically finite base-change, i.e. that $\mathrm{Var}(f) = 0$ in Viehweg's terminology. It follows by combining the bound $\kappa(X) \leq \dim F$ from Corollary 8.2 with the solution to the $C^+_{n,m}$ conjecture for fibers of general type in [Kol87], which in this case gives

$$\kappa(F) + \mathrm{Var}(f) \leq \kappa(X).$$

10. Families of maximal variation and Viehweg-Zuo sheaves. Arguments based on the Hodge module construction in Theorem 7.2 also allow, at the other end of the (variation) spectrum, to study families $f \colon X \to Y$ of varieties of general type (or more generally having good minimal models) with maximal variation $\mathrm{Var}(f) = \dim Y$. Recall that this means that the general fiber can be birational to only countably many other fibers.

It was conjectured by Viehweg that if Y° is smooth and quasi-projective, and $f^\circ : X^\circ \to Y^\circ$ is a smooth family of canonically polarized varieties with maximal variation, then Y° must be of log-general type. More precisely, if the data is extended to compactifications $f \colon X \to Y$, and $D = Y \smallsetminus Y^\circ$ is a divisor, then $\omega_Y(D)$ is big. This conjecture in its general form was settled by Campana and Păun [CP15b], [CP15a], after a series of papers by Kovács [Kov00], Kebekus-Kovács [KK08a], [KK08b], [KK10], and Patakfalvi [Pat12] proved important special cases. One of the fundamental inputs in all of these works is the existence of what are called Viehweg-Zuo sheaves [VZ02, Theorem 1.4] for families of canonically polarized varieties.

The Hodge module construction in §7 can be brought into play in order to prove the existence of Viehweg-Zuo sheaves for more general families.

Theorem 10.1 ([PS15, Theorem B]). *Let $f \colon X \to Y$ be an algebraic fiber space between smooth projective varieties, such that the singular locus of f is a divisor $D_f \subseteq Y$ with simple normal crossings. Assume that for every generically finite $\tau : \tilde{Y} \to Y$ with $\tilde{Y}$ smooth, and for every resolution $\tilde{X}$ of $X \times_Y \tilde{Y}$, there is an integer $m \geq 1$ such that $\det \tilde{f}_* \omega_{\tilde{X}/\tilde{Y}}^{\otimes m}$ is big. Then there exists a big coherent sheaf $\mathcal{H}$ on Y and an integer $s \geq 1$, together with an inclusion*

$$\mathcal{H} \hookrightarrow \left(\Omega_Y^1(\log D_f)\right)^{\otimes s}.$$

Viehweg's $Q_{n,m}$ conjecture [Vie83b, Remark 3.7] (see also [Mor87, (7.3)]) predicts that the hypothesis of the theorem is satisfied by any algebraic fiber space $f \colon X \to Y$ such that $\mathrm{Var}(f) = \dim Y$. This conjecture was proved by Kollár [Kol87] (see also [Vie89]) when the smooth fibers of f are of general type, and also by Kawamata [Kaw85] when the geometric generic fiber has a good minimal model.[5]

The point here is that the existence of an integer m as in the statement of Theorem 10.1 leads to the fact that (7.1) holds for a new family over $\tilde{Y}$ constructed from f by means of a general strategy introduced by Viehweg, based on his fiber product trick and semistable reduction. Very roughly speaking, the Hodge module and graded submodule produced by Theorem 7.2 can then be used in conjunction with Theorem 4.3 on weak positivity to reach the desired conclusion.

Together with the main result of Campana-Păun [CP15a] on the pseudo-effectivity of quotients of powers of log-cotangent bundles, Theorem 10.1 implies that Viehweg's conjecture also holds under weaker assumptions on the fibers of the family.

Theorem 10.2 ([PS15, Theorem A]). *Let $f \colon X \to Y$ be an algebraic fiber space between smooth projective varieties, and let $D \subseteq Y$ be any divisor containing the singular locus of f. Assume that $\mathrm{Var}(f) = \dim Y$. Then:*

---

[5]One of the main conjectures in birational geometry is of course that all varieties with $\kappa(X) \geq 0$ should do so.

(1) *If the general fiber of f is of general type, then the pair (Y, D) is of log-general type, i.e. $\omega_Y(D)$ is big.*

(2) *More generally, the same conclusion holds if the geometric generic fiber of f admits a good minimal model.*

F. Hodge ideals

11. Motivation. Let X be a smooth complex variety of dimension n, and D a reduced effective divisor on X. The left $\mathscr{D}_X$-module

$$\mathscr{O}_X(*D) = \bigcup_{k \geq 0} \mathscr{O}_X(kD)$$

of functions with arbitrary poles along D underlies the mixed Hodge module $j_*\mathbb{Q}_U^H[n]$, where $U = X \smallsetminus D$ and $j : U \hookrightarrow X$ is the inclusion map. It therefore comes equipped with a Hodge filtration $F_\bullet \mathscr{O}_X(*D)$. Saito [Sai93] showed that this filtration is contained in the pole order filtration, namely

$$F_k \mathscr{O}_X(*D) \subseteq \mathscr{O}_X\big((k+1)D\big) \quad \text{for all} \ \ k \geq 0,$$

and the problem of how far these are from being different is of interest both in the study of the singularities of D and in understanding the Hodge structure on the cohomology of the complement $H^\bullet(U, \mathbb{C})$. The inclusion above leads to defining for each $k \geq 0$ a coherent sheaf of ideals $I_k(D) \subseteq \mathscr{O}_X$ by the formula

$$F_k \mathscr{O}_X(*D) = \mathscr{O}_X\big((k+1)D\big) \otimes I_k(D).$$

This section is devoted to a brief introduction to work with Mustaţă [MP16], in which we study and apply the theory of these ideals using birational geometry methods. This involves redefining them by means of log-resolutions.

12. Alternative definition and local properties. We consider a log-resolution $f : Y \to X$ of the pair (X, D), and denote $E = \tilde{D} + F$, where F is the reduced exceptional divisor. Because we are dealing with pushforwards, it is more convenient to work in the setting of right $\mathscr{D}$-modules; in other words we will look at a filtration on $\omega_X(*D)$.

The starting point is the existence of a filtered complex of right $f^{-1}\mathscr{D}_X$-modules

$$0 \longrightarrow f^*\mathscr{D}_X \longrightarrow \Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \longrightarrow \cdots$$

$$\cdots \longrightarrow \Omega_Y^{n-1}(\log E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \longrightarrow \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*\mathscr{D}_X \longrightarrow 0$$

which is exact except at the rightmost term, where the cohomology is $\omega_Y(*E) \otimes_{\mathscr{D}_Y} \mathscr{D}_{Y \to X}$. Denoting it by $A^\bullet$, it has a filtration with subcomplexes $F_{k-n}A^\bullet$, for $k \geq 0$, given by

$$0 \to f^*F_{k-n}\mathscr{D}_X \to \Omega_Y^1(\log E) \otimes_{\mathscr{O}_Y} f^*F_{k-n+1}\mathscr{D}_X \to \cdots \to \omega_Y(E) \otimes_{\mathscr{O}_Y} f^*F_k\mathscr{D}_X \to 0.$$

Definition 12.1. The filtration on $\omega_X(*D)$, and in particular the k-*th Hodge ideal* $I_k(D)$ associated to D, are defined by the formula

$$F_k\omega_X(*D) := \omega_X\big((k+1)D\big) \otimes I_k(D) = \text{Im}\left[R^0 f_* F_{k-n}A^\bullet \longrightarrow R^0 f_* A^\bullet\right],$$

after proving that this image is contained in $\omega_X\big((k+1)D\big)$. It is shown in [MP16] that this definition is independent of the choice of log resolution, and that it indeed coincides with Saito's Hodge filtration, as in the previous paragraph.

Example 12.2. (i) If D is smooth, then $I_k(D) = \mathscr{O}_X$ for all $k \geq 0$.

(ii) More generally, when D is a simple normal crossings divisor, one has
$$\omega_X\big((k+1)D\big) \otimes I_k(D) = \omega_X(D) \cdot F_k \mathscr{D}_X \quad \text{for all } k \geq 0.$$

(iii) For $k = 0$ and D arbitrary, we have
$$I_0(D) = \mathcal{I}\big(X, (1-\epsilon)D\big),$$
the multiplier ideal associated to the $\mathbb{Q}$-divisor $(1-\epsilon)D$ on X, for any $0 < \epsilon \ll 1$. In particular, $I_0(D) = \mathscr{O}_X$ if and only if (X, D) is log-canonical; see [Laz04, 9.3.9].

This last example explains one of the main motivations for studying the higher ideals $I_k(D)$; taken together, they provide a generalization of the notion of multiplier ideal, and a finer invariant of singularities. It turns out that there is a sequence of inclusions
$$\cdots I_k(D) \subseteq \cdots \subseteq I_1(D) \subseteq I_0(D).$$
(Note that the fact that $F_k \omega_X(*D)$ is a filtration in the sense of $\mathscr{D}_X$-modules yields reverse inclusions $I_k(D) \cdot \mathscr{O}_X(-D) \subseteq I_{k+1}(D)$.)

Definition 12.3. The pair (X, D) is *k-log-canonical* if $I_k(D) = \mathscr{O}_X$. Thus 0-log-canonical is equivalent to the usual notion of a log-canonical pair. The general notion refines the property of having rational singularities:

Theorem 12.4 ([MP16, Theorem B]). *For every $k \geq 1$ we have an inclusion*
$$I_k(D) \subseteq \mathrm{adj}(D),$$
where $\mathrm{adj}(D)$ is the adjoint ideal of the divisor D; see [Laz04, §9.3.E]. Hence if (X, D) is k-log canonical for some $k \geq 1$, then D is normal with rational singularities.

Recall from the example above that if D is smooth, all Hodge ideals are trivial. It turns out that the two conditions are equivalent; in fact, more precisely, any level of log-canonicity beyond $(n-1)/2$ implies smoothness.

Theorem 12.5 ([MP16, Theorem A]). *The following are equivalent:*

(i) D is smooth.

(ii) the Hodge filtration and pole order filtration on $\mathscr{O}_X(*D)$ coincide.

(iii) $I_k(D) = \mathscr{O}_X$ for all $k \geq 0$.

(iv) $I_k(D) = \mathscr{O}_X$ for some $k \geq \frac{n-1}{2}$.

The statements above are consequences of general results regarding the order of vanishing of $I_k(D)$ along exceptional divisors on birational models over X, and along closed subsets in X. This in particular applies to provide criteria for the nontriviality of the Hodge ideals in terms of the multiplicity of D at singular points; one such criterion is the following:

Theorem 12.6 (cf. [MP16, Theorem E]). *Let D be a reduced effective divisor on a smooth variety X, and let $x \in X$. If $m = \mathrm{mult}_x(D)$, then for every k we have*
$$I_k(D) \subseteq \mathfrak{m}_x^q, \quad \text{where} \quad q = \min\{m-1, (k+1)m - n\},$$

where $\mathfrak{m}_x^q = \mathscr{O}_X$ if $q \leq 0$.

As an example, this says that for $q \leq m - 1$ one has

$$m \geq \frac{n+q}{2} \implies I_1(D) \subseteq \mathfrak{m}_x^q.$$

13. Examples. Here are some concrete calculations or statements that can be made at the moment. It is useful to start by introducing another related class of ideals. By analogy with the simple normal crossings case, one can define for each $k \geq 0$ auxiliary ideal sheaves $J_k(D)$ by the formula

$$\mathscr{O}_X\big((k+1)D\big) \otimes J_k(D) = F_k \mathscr{D}_X \cdot \big(\mathscr{O}_X(D) \otimes I_0(D)\big) = F_k \mathscr{D}_X \cdot F_0 \mathscr{O}_X(*D).$$

Since $F_k \mathscr{O}_X(*D)$ is a filtration in the sense of $\mathscr{D}_X$-modules, it is clear that for each $k \geq 0$ there is an inclusion

$$J_k(D) \subseteq I_k(D).$$

In dimension at least three this inclusion is usually strict, and the Hodge ideals are much more difficult to compute than $J_k(D)$. For instance, for $k = 1$, using the log resolution notation above, one can verify the existence of a short exact sequence

$$0 \longrightarrow \omega_X(2D) \otimes J_1(D) \longrightarrow \omega_X(2D) \otimes I_1(D) \longrightarrow R^1 f_* \Omega_Y^{n-1}(\log E) \longrightarrow 0.$$

Example 13.1 (Surfaces). When $\dim X = 2$ however, and more generally outside of a closed subset of codimension ≥ 3 in any dimension, it is shown in [MP16] that $I_k(D) = J_k(D)$ for all $k \geq 0$. Here are some concrete calculations:

• If $D = (xy = 0) \subset \mathbb{C}^2$ is a node, then $I_k(D) = (x, y)^k$ for all $k \geq 0$.

• If $D = (x^2 + y^3 = 0) \subset \mathbb{C}^2$ is a cusp, then $I_0(D) = (x, y)$, $I_1(D) = (x^2, xy, y^3)$, and $I_2(D) = (x^3, x^2 y^2, xy^3, y^5, y^4 - 3x^2 y)$.

• If $D = (xy(x + y) = 0) \subset \mathbb{C}^2$ is a triple point, then $I_0(D) = (x, y)$, while $I_1(D) = (x, y)^3$.

Note for instance how I_1 distinguishes between singularities for which I_0 is the same. This is one of the ways in which Hodge ideals become important in applications. For details on all of the examples that come next, see [MP16, §20].

Example 13.2 (Ordinary singularities). Let $x \in D$ be a point of multiplicity $m \geq 2$, with the property that the projectivized tangent cone of D at x is smooth; these are sometimes called ordinary singularities. For instance, D could be the cone over a smooth hypersurface of degree m in $\mathbf{A}^{n-1}$. One can show:

• $I_k(D)_x = \mathscr{O}_{X,x} \iff k \leq \left[\frac{n}{m}\right] - 1$.

• If $\frac{n}{2} \leq m \leq n - 1$, then $I_1(D)_x = \mathfrak{m}_x^{2m-n}$.

• If $m \geq n$, then

$$\mathscr{O}_X(-D) \cdot \mathfrak{m}_x^{m-n-1} + \mathfrak{m}_x^{2m-n} \subseteq I_1(D) \subseteq \mathscr{O}_X(-D) \cdot \mathfrak{m}_x^{m-n-2} + \mathfrak{m}_x^{2m-n-1},$$

with $\dim_{\mathbb{C}} I_1(D)/(\mathscr{O}_X(-D) \cdot \mathfrak{m}_x^{m-n-1} + \mathfrak{m}_x^{2m-n}) = m\binom{m-2}{n-2}$.

Example 13.3 (**Determinantal varieties and theta divisors**). Regarding Theorem 12.4, we see already from the example above that there can be divisors D with rational singularities such that $I_k(D) \neq \mathscr{O}_X$ for all $k \geq 1$. Here are two more celebrated examples where things can go both ways:

• If (A, Θ) is an irreducible principally polarized abelian variety, a result of Ein-Lazarsfeld [EL97] says that Θ is normal, with rational singularities, i.e. $\mathrm{adj}(\Theta) = \mathscr{O}_A$. On the other hand, it can happen that $I_1(\Theta) \neq \mathscr{O}_A$. For example, when (A, Θ) is the intermediate Jacobian of a smooth cubic threefold, it follows from the example above that $I_1(\Theta) = \mathfrak{m}_0$; indeed, the origin $0 \in A$ is the unique point singular point of Θ, and is an ordinary singularity of multiplicity 3.

• If $D \subset X = \mathbf{A}^{n^2}$ is the generic determinantal variety of codimension 1, one can show that $I_1(D) = \mathscr{O}_X$. This shows that D has rational singularities, which is of course a well-known result. Moreover, it turns out that D is 1-log-canonical but not more, i.e. $I_2(D) \neq \mathscr{O}_X$.

Example 13.4 (**Roots of Bernstein-Sato polynomial and diagonal hypersurfaces**). Suppose that f is a local equation of D. Recall that the Bernstein-Sato polynomial of D at x is the monic polynomial $b_{f,x} \in \mathbb{C}[s]$ of smallest degree such that around x there is a relation

$$b_{f,x}(s) f^s = P(s) \bullet f^{s+1}$$

for some nonzero $P \in \mathscr{D}_X[s]$. It is known that $(s + 1)$ divides $b_{f,x}$ and all roots of $b_{f,x}(s)$ are negative rational numbers. One defines $\alpha_{f,x}$ to be $-\lambda$, where λ is the largest root of $b_{f,x}(s)/(s + 1)$. Saito showed in [Sai93, Theorem 0.11] that

$$I_k(D)_x = \mathscr{O}_{X,x} \quad \text{for all} \quad k \leq \alpha_{f,x} - 1.$$

When D is the divisor in $\mathbf{A}^n$ defined by $f = \sum_{i=1}^n x_i^{a_i}$, with $a_i \geq 2$, it is known that $\alpha_{f,0} = \sum_{i=1}^n \frac{1}{a_i}$, and so it follows that

$$I_k(D) = \mathscr{O}_X \quad \text{for all} \quad k \leq -1 + \sum_{i=1}^n \frac{1}{a_i}.$$

14. Vanishing and applications. As it is well understood from the study of multiplier ideals, criteria like Theorem 12.6 are most useful for applications when applied in combination with vanishing theorems. There is indeed such a general vanishing theorem for Hodge ideals. For $k = 0$, it is precisely the well-known Nadel Vanishing; for $k \geq 1$ it requires more careful hypotheses.

Theorem 14.1. *Let X be a smooth projective variety of dimension n, D a reduced effective divisor, and L a line bundle on X. Then, for each $k \geq 1$, assuming that the pair (X, D) is $(k - 1)$-log-canonical we have:*

(i) If L is a line bundle such that $L(pD)$ is ample for all $0 \leq p \leq k$,[6] then

$$H^i\big(X, \omega_X((k + 1)D) \otimes L \otimes I_k(D)\big) = 0$$

---

[6]If $k \geq \frac{n+1}{2}$, then D is in fact smooth by Theorem 12.5, and so $I_k(D) = \mathscr{O}_X$. In this case, if L is a line bundle such that $L(kD)$ is ample, then

$$H^i\big(X, \omega_X((k + 1)D) \otimes L\big) = 0 \quad \text{for all} \quad i > 0.$$

for all $i \geq 2$. Moreover,

$$H^1\big(X, \omega_X((k+1)D) \otimes L \otimes I_k(D)\big) = 0$$

holds if $H^j\big(X, \Omega_X^{n-j} \otimes L((k-j+1)D)\big) = 0$ for all $1 \leq j \leq k$.

(ii) If D is ample, then (1) and (2) also hold with $L = \mathscr{O}_X$.

The main ingredient in the proof is the Kodaira-Saito vanishing theorem for Hodge modules, Theorem 3.1. In important classes of examples, the extra hypotheses in Theorem 14.1 are either automatically satisfied, or can be discarded, which greatly increases the range of applications. For instance, in all of the examples below the borderline Nakano-type hypothesis in (i) holds automatically:

• *Toric varieties*: due to the Bott-Danilov-Steenbrink vanishing theorem.

• *Projective space $\mathbb{P}^n$*: in addition to the above, one can also discard the assumption that D be $(k-1)$-log canonical, essentially because of the existence of a Koszul resolution for all $\Omega_{\mathbb{P}^n}^j$.

• *Abelian varieties*: due to the fact that all Ω_X^j are trivial; for the same reason, one can also discard the hypothesis that D be $(k-1)$-log canonical.

As mentioned above, when combined with nontriviality criteria like Theorem 12.6, these stronger versions of the vanishing theorem have interesting consequences regarding the behavior of isolated singular points on hypersurfaces in $\mathbb{P}^n$, or on theta divisors on ppav's. Here is a sampling of applications from [MP16]; for further results and context see Ch.H in *loc. cit.*

Theorem 14.2. *Let D be a reduced hypersurface of degree d in $\mathbb{P}^n$, with $n \geq 3$, and denote by S_m the set of isolated singular points on D of multiplicity $m \geq 2$. Then S_m imposes independent conditions on hypersurfaces of degree at least $([\frac{n}{m}]+1)d - n - 1$.*

This turns out to be an improvement of the bounds found in the literature for most $n \geq 5$ and $m \geq 3$.

A well-known result of Kollár [Kol95, Theorem 17.3] states that if (A, Θ) is a principally polarized abelian variety (ppav) of dimension g, then the pair (A, Θ) is log-canonical, and so $\text{mult}_x(\Theta) \leq g$ for any $x \in \Theta$. It is well known however that when Θ is irreducible one can do better; see e.g. [EL97] and the references therein. It is a folklore conjecture that the bound should actually be $\frac{g+1}{2}$.

Theorem 14.3. *Let (A, Θ) be ppav of dimension g such that Θ has isolated singularities. Then:*

(i) For every $x \in \Theta$ we have $\text{mult}_x(\Theta) \leq \frac{g+1}{2}$, and also $\text{mult}_x(\Theta) \leq \epsilon(\Theta) + 2$, where $\epsilon(\Theta)$ is the Seshadri constant of the principal polarization.

(ii) Moreover, there can be at most one point $x \in \Theta$ such that $\text{mult}_x(\Theta) = \frac{g+1}{2}$.

This is obtained using the ideal $I_1(\Theta)$, except for the bound involving the Seshadri constant, which uses all $I_k(\Theta)$. Results of a similar flavor have also been obtained recently by Codogni-Grushevsky-Sernesi.

G. Applications of Hodge modules in other areas

While my lack of expertise does not allow me to go into much detail, I will at least try to indicate some other areas in which constructions based on Hodge modules have made a substantial impact in recent years. The list is by no means supposed to be exhaustive; I apologize for any omissions. Many more references regarding the topics below, as well as others, can be found in [Sai16, §3.7].

Hodge theory. By its very nature, M. Saito's study of Hodge modules [Sai88], [Sai90b] has initially provided fundamental applications to core Hodge theory, like for instance the existence of mixed Hodge structures on the cohomology of complex varieties or intersection cohomology, or the enhancement of the Decomposition Theorem involving Hodge filtrations. More recently, Hodge modules have found new applications internal to Hodge theory, especially to the study of (admissible) normal functions. In [BFNP09], the authors used mixed Hodge modules to show that the Hodge conjecture is equivalent to the existence of singularities in certain admissible normal functions. The article [Sch12] contains the construction of complex analytic Néron models that extend families of intermediate Jacobians associated to complex variations of Hodge structure of odd weight over Zariski open subsets of complex manifolds; Hodge modules provide a functorial construction, with no extra requirements at the boundary.

Singularities, characteristic classes. It also goes without saying that Hodge modules have made an impact in the study of singularities via Hodge theoretic methods, not far from the spirit of Ch.F above. Besides [Sai93] invoked in the text, I will highlight the papers [Sai91b], which proves a conjecture of Steenbrink on the spectrum of certain isolated singularities, [BS05], which makes a connection between multiplier ideals and the V-filtration, and [DMS11], which extends the latter to arbitrary subvarieties; see also the references therein. Hodge modules have also proved useful in the study of characteristic classes of singular varieties; see the survey [MS13].

Donaldson-Thomas theory. Mixed Hodge modules have recently been used successfully in Donaldson-Thomas theory. For instance, the cohomological DT invariants (the hypercohomology of a certain perverse sheaves of vanishing cycles on moduli spaces of simple sheaves on Calabi-Yau threefolds) can be endowed with mixed Hodge structures via Hodge module theory. They are also used in the categorification of Donaldson-Thomas invariants. I will highlight [KL12] and [BBD+15], and refer to [Sze15] for a survey and for an extensive list of references.

Representation theory. Hodge modules have become an important tool in geometric representation theory. They have been used in the study of unitary representations of reductive Lie groups [SV11]. A categorical sl$_2$-action on a special category of sheaves on Grassmannians is related to the calculation in [CDK16] of the associated graded with respect to the Hodge and weight filtration for a representation-theoretically relevant example of localization along a hypersurface. See also [AK14] a relationship with Koszul duality.

Automorphic forms. In [Suh15], J. Suh has been able to apply a vanishing theorem for the associated graded quotients of de Rham complexes of Deligne canonical extensions, proved using tools from Hodge module theory (and the main ingredient

in his proof of Theorem 3.5) in order to obtain vanishing theorems for the cohomology of Shimura varieties. The use of Hodge modules has allowed for treating the coherent cohomology groups of automorphic bundles on arbitrary Shimura varieties, with general weights, extending previous results of Faltings, Li-Schwermer, and Lan-Suh.

References

[AK14]   Pramod N. Achar and Sarah Kitchen, *Koszul duality and mixed Hodge modules*, Int. Math. Res. Not. IMRN (2014), no. 21, 5874–5911. MR 3273066

[BBD$^+$15] C. Brav, V. Bussi, D. Dupont, D. Joyce, and B. Szendrői, *Symmetries and stabilization for sheaves of vanishing cycles*, J. Singul. **11** (2015), 85–151, With an appendix by Jörg Schürmann. MR 3353002

[BFNP09] Patrick Brosnan, Hao Fang, Zhaohu Nie, and Gregory Pearlstein, *Singularities of admissible normal functions*, Invent. Math. **177** (2009), no. 3, 599–629, With an appendix by Najmuddin Fakhruddin. MR 2534102 (2010h:14013)

[Bru15]   Yohan Brunebarbe, *Symmetric differentials and variations of Hodge structures*, to appear, 2015.

[BS05]   Nero Budur and Morihiko Saito, *Multiplier ideals, V-filtration, and spectrum*, J. Algebraic Geom. **14** (2005), no. 2, 269–282. MR 2123230

[CDK16]   Sabin Cautis, Christopher Dodd, and Joel Kamnitzer, *Associated graded of Hodge modules and categorical* sl_2 *actions*, preprint arXiv:1603.07402, 2016.

[CH01]   Jungkai A. Chen and Christopher D. Hacon, *Characterization of abelian varieties*, Invent. Math. **143** (2001), no. 2, 435–447. MR 1835393 (2002c:14063)

[CJ13]   Jungkai A. Chen and Zhi Jiang, *Positivity in varieties of maximal albanese dimension*, to appear in J. reine angew. Math., 2013.

[CP15a]   Frédéric Campana and Mihai Păun, *Foliations with positive slopes and birational stability of orbifold cotangent bundles*, preprint arXiv:1508.02456, 2015.

[CP15b]   Frédéric Campana and Mihai Păun, *Orbifold generic semi-positivity: an application to families of canonically polarized manifolds*, Ann. Inst. Fourier (Grenoble) **65** (2015), no. 2, 835–861. MR 3449168

[DMS11]   Alexandru Dimca, Philippe Maisonobe, and Morihiko Saito, *Spectrum and multiplier ideals of arbitrary subvarieties*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 4, 1633–1653 (2012). MR 2951747

[EL97]   Lawrence Ein and Robert Lazarsfeld, *Singularities of Theta divisors and the birational geometry of irregular varieties*, J. Amer. Math. Soc. **10** (1997), no. 1, 243–258. MR 1396893 (97d:14063)

[EV92]   Hélène Esnault and Eckart Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR 1193913 (94a:14017)

[FFS14]   Osamu Fujino, Taro Fujisawa, and Morihiko Saito, *Some remarks on the semipositivity theorems*, Publ. Res. Inst. Math. Sci. **50** (2014), no. 1, 85–112. MR 3167580

[GL87]   Mark Green and Robert Lazarsfeld, *Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville*, Invent. Math. **90** (1987), no. 2, 389–407. MR 910207 (89b:32025)

[GL91]   Mark Green and Robert Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991), no. 1, 87–103. MR 1076513 (92i:32021)

[Hac04]   Christopher D. Hacon, *A derived category approach to generic vanishing*, J. Reine Angew. Math. **575** (2004), 173–187. MR 2097552 (2005m:14026)

[HK05]   Christopher D. Hacon and Sándor J. Kovács, *Holomorphic one-forms on varieties of general type*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 4, 599–607. MR 2172952 (2006g:14031)

[HTT08]   Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361 (2008k:32022)

[Jia11]   Zhi Jiang, *An effective version of a theorem of Kawamata on the Albanese map*, Commun. Contemp. Math. **13** (2011), no. 3, 509–532. MR 2813500 (2012g:14014)

[Kaw85]   Yujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. **363** (1985), 1–46. MR 814013 (87a:14013)

[KK08a]   Stefan Kebekus and Sándor J. Kovács, *Families of canonically polarized varieties over surfaces*, Invent. Math. **172** (2008), no. 3, 657–682. MR 2393082 (2010b:14072)

[KK08b]   Stefan Kebekus and Sándor J. Kovács, *Families of varieties of general type over compact bases*, Adv. Math. **218** (2008), no. 3, 649–652. MR 2414316 (2009d:14042)

[KK10]   Stefan Kebekus and Sándor J. Kovács, *The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties*, Duke Math. J. **155** (2010), no. 1, 1–33. MR 2730371 (2011i:14060)

[KL12]   Young-Hoon Kiem and Jun Li, *Categorification of donaldson-thomas invariants via perverse sheaves*, preprint arXiv:1212.6444, 2012.

[Kol86a]   János Kollár, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42. MR 825838 (87c:14038)

[Kol86b]   János Kollár, *Higher direct images of dualizing sheaves. II*, Ann. of Math. (2) **124** (1986), no. 1, 171–202. MR 847955 (87k:14014)

[Kol87]   János Kollár, *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 361–398. MR 946244 (89i:14029)

[Kol95]   János Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995. MR 1341589

[Kov97]   Sándor J. Kovács, *Families over a base with a birationally nef tangent bundle*, Math. Ann. **308** (1997), no. 2, 347–359. MR 1464907

[Kov00]   Sándor J. Kovács, *Algebraic hyperbolicity of fine moduli spaces*, J. Algebraic Geom. **9** (2000), no. 1, 165–174. MR 1713524

[Laz04]   Robert Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR 2095472

[LZ05]   Tie Luo and Qi Zhang, *Holomorphic forms on threefolds*, Recent progress in arithmetic and algebraic geometry, Contemp. Math., vol. 386, Amer. Math. Soc., Providence, RI, 2005, pp. 87–94. MR 2182772 (2007f:14012)

[Mor87]   Shigefumi Mori, *Classification of higher-dimensional varieties*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 269–331. MR 927961 (89a:14040)

[MP16]   Mircea Mustaţă and Mihnea Popa, *Hodge ideals*, preprint arXiv:1605.08088, 2016.

[MS13]   Laurentiu Maxim and Joerg Schuermann, *Characteristic classes of mixed Hodge modules and applications*, preprint arXiv:1307.5152, 2013.

[Par16]   Giuseppe Pareschi, *Standard canonical support loci*, preprint, 2016.

[Pat12]   Zsolt Patakfalvi, *Viehweg's hyperbolicity conjecture is true over compact bases*, Adv. Math. **229** (2012), no. 3, 1640–1642. MR 2871152 (2012m:14072)

[Pop14]   Mihnea Popa, *Kodaira-Saito vanishing and applications*, preprint arXiv:1407.3294, to appear in L'Enseign. Math., 2014.

[PP03]   Giuseppe Pareschi and Mihnea Popa, *Regularity on abelian varieties. I*, J. Amer. Math. Soc. **16** (2003), no. 2, 285–302 (electronic). MR 1949161 (2004c:14086)

[PP11a]   Giuseppe Pareschi and Mihnea Popa, *GV-sheaves, Fourier-Mukai transform, and generic vanishing*, Amer. J. Math. **133** (2011), no. 1, 235–271. MR 2752940 (2012e:14043)

[PP11b]   Giuseppe Pareschi and Mihnea Popa, *Regularity on abelian varieties III: relationship with generic vanishing and applications*, Grassmannians, moduli spaces and vector bundles, Clay Math. Proc., vol. 14, Amer. Math. Soc., Providence, RI, 2011, pp. 141–167. MR 2807853 (2012h:14113)

[PPS15]   Giuseppe Pareschi, Mihnea Popa, and Christian Schnell, *Hodge modules on complex tori and generic vanishing for compact Kähler manifolds*, preprint arXiv:1505.00635, to appear in Geometry and Topology, 2015.

[PS13]   Mihnea Popa and Christian Schnell, *Generic vanishing theory via mixed Hodge modules*, Forum Math. Sigma **1** (2013), e1, 60pp. MR 3090229

[PS14]   Mihnea Popa and Christian Schnell, *Kodaira dimension and zeros of holomorphic one-forms*, Ann. of Math. (2) **179** (2014), 1–12.

[PS15]   Mihnea Popa and Christian Schnell, *Viehweg's hyperbolicity conjecture for families with maximal variation*, preprint arXiv:1511.00294, to appear in Invent. Math., 2015.

[PW16]   Mihnea Popa and Lei Wu, *Weak positivity for Hodge modules*, Math. Res. Lett. **23** (2016), no. 4, 1139–1155. MR 3554504

[Sai88]   Morihiko Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989). MR 1000123 (90k:32038)

[Sai90a] Morihiko Saito, *Decomposition theorem for proper Kähler morphisms*, Tohoku Math. J. (2) **42** (1990), no. 2, 127–147. MR 1053945 (91j:32042)

[Sai90b] Morihiko Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333. MR 1047415 (91m:14014)

[Sai91a] Morihiko Saito, *On Kollár's conjecture*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 509–517. MR 1128566 (92i:14007)

[Sai91b] Morihiko Saito, *On Steenbrink's conjecture*, Math. Ann. **289** (1991), no. 4, 703–716. MR 1103044 (92e:32021)

[Sai93]   Morihiko Saito, *On b-function, spectrum and rational singularity*, Math. Ann. **295** (1993), no. 1, 51–74. MR 1198841 (95b:32051)

[Sai16]   Morihiko Saito, *A young person's guide to mixed Hodge modules*, preprint arXiv:1605.00435, 2016.

[Sch73]   Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR 0382272 (52 #3157)

[Sch12]   Christian Schnell, *Complex analytic Néron models for arbitrary families of intermediate Jacobians*, Invent. Math. **188** (2012), no. 1, 1–81. MR 2897692

[Sch13]   Christian Schnell, *Weak positivity via mixed Hodge modules*, 2013, To appear in *Hodge Theory and Complex Algebraic Geometry (Columbus, 2013)*.

[Sch14a] Christian Schnell, *On Saito's vanishing theorem*, arXiv:1407.3295, to appear in Math. Res. Lett., 2014.

[Sch14b] Christian Schnell, *An overview of Morihiko Saito's theory of mixed Hodge modules*, preprint arXiv:1405.3096, 2014.

[Sch15]   Christian Schnell, *Holonomic D-modules on abelian varieties*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 1–55. MR 3349829

[Sim93]   Carlos T. Simpson, *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 3, 361–401. MR 1222278 (94f:14008)

[SS16]   Claude Sabbah and Christian Schnell, *The MHM project*, http://www.cmls.polytechnique.fr/perso/sabbah/MHMProject/mhm.html, 2016.

[Suh15]   Junecue Suh, *Vanishing theorems for mixed Hodge modules and applications*, preprint, to appear in J. Eur. Math. Soc., 2015.

[SV11]   Wilfried Schmid and Kari Vilonen, *Hodge theory and unitary representations of reductive Lie groups*, Frontiers of mathematical sciences, Int. Press, Somerville, MA, 2011, pp. 397–420. MR 3050836

[Sze15]   Balázs Szendröi, *Cohomological Donaldson-Thomas theory*, preprint arXiv:1503.07349, 2015.

[Vie83a] Eckart Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353. MR 715656 (85b:14041)

[Vie83b] Eckart Viehweg, *Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map*, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 567–589. MR 728619 (85i:14020)

[Vie89]   Eckart Viehweg, *Weak positivity and the stability of certain Hilbert points*, Invent. Math. **96** (1989), no. 3, 639–667. MR 996558 (90i:14037)

[VZ01]   Eckart Viehweg and Kang Zuo, *On the isotriviality of families of projective manifolds over curves*, J. Algebraic Geom. **10** (2001), no. 4, 781–799. MR 1838979 (2002g:14012)

[VZ02]   Eckart Viehweg and Kang Zuo, *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. MR 1922109 (2003h:14019)

[VZ03]   Eckart Viehweg and Kang Zuo, *On the Brody hyperbolicity of moduli spaces for canonically polarized manifolds*, Duke Math. J. **118** (2003), no. 1, 103–150. MR 1978884 (2004h:14042)

[Wan13]   Botong Wang, *Torsion points on the cohomology jump loci of compact Kähler manifolds*, preprint arXiv:1312.6619, to appear in Math. Res. Lett., 2013.

[Wu15]   Lei Wu, *Vanishing and injectivity theorems for Hodge modules*, preprint arXiv:1505.00881, to appear in Trans. Amer. Math. Soc., 2015.

[Wu16]   Lei Wu, *Multiplier subsheaves*, in preparation, 2016.

[Zha97]   Qi Zhang, *Global holomorphic one-forms on projective manifolds with ample canonical bundles*, J. Algebraic Geom. **6** (1997), no. 4, 777–787. MR 1487236

[Zuo00]   Kang Zuo, *On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications*, Asian J. Math. **4** (2000), no. 1, 279–301, Kodaira's issue. MR 1803724 (2002a:32011)

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