DERIVED EQUIVALENCE AND NON-VANISHING LOCI

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To Joe Harris, with great admiration.

1. THE CONJECTURE AND ITS VARIANTS

The purpose of this note is to propose and motivate a conjecture on the behavior of cohomological support loci for topologically trivial line bundles under derived equivalence, to verify it in the case of surfaces, and to explain further developments. The reason for such a conjecture is the desire to understand the relationship between the cohomology groups of (twists of) the canonical line bundles of derived equivalent varieties. This in turn is motivated by the following well-known problem, stemming from a prediction of Kontsevich in the case of Calabi-Yau manifolds (and which would also follow from the main conjecture in Orlov [Or2]).

Problem 1.1. *Let X and Y be smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Is it true that $h^{p,q}(X) = h^{p,q}(Y)$ for all p and q?*

Here, given a smooth projective complex variety X, we denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$. For surfaces the answer is yes, for instance because of the derived invariance of Hochschild homology [Or1], [Ca]. This is also true for threefolds, again using the invariance of Hochschild homology, together with the behavior of the Picard variety under derived equivalence [PS]. In general, even the invariance of $h^{0,q}$ with $1 < q < \dim X$ is not known at the moment, and this leads to the search for possible methods for circumventing the difficult direct study of the cohomology groups $H^i(X, \omega_X)$.

More precisely, in [PS] it is shown that if $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are isogenous. This opens the door towards studying the behavior or more refined objects associated to irregular varieties (i.e. those with $q(X) = h^0(X, \Omega_X^1) > 0$) under derived equivalence. Among the most important such objects are the *cohomological support loci* of the canonical bundle: given a smooth projective X, for $i = 0, \ldots, \dim X$ one defines

$$V^i(\omega_X) := \{\alpha \mid H^i(X, \omega_X \otimes \alpha) \neq 0\} \subseteq \mathrm{Pic}^0(X).$$

By semicontinuity, these are closed algebraic subsets of $\mathrm{Pic}^0(X)$. It has become clear in recent years that these loci are the foremost tool in studying the special birational geometry of irregular varieties, with applications ranging from results about singularities of theta divisors [EL] to the proof of Ueno's conjecture [ChH]. They are governed by the following fundamental results of generic vanishing theory ([GL1], [GL2], [Ar], [Si]):

- If $a : X \to \mathrm{Alb}(X)$ is the Albanese map of X, then

(1.1) $$\mathrm{codim}\, V^i(\omega_X) \geq i - \dim X + \dim a(X) \quad \text{for all i,}$$

and there exists an i for which this is an equality.

- The irreducible components of each $V^i(\omega_X)$ are torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

- Each positive dimensional component of some $V^i(\omega_X)$ corresponds to a fibration $f : X \to Y$ onto a normal variety with $0 < \dim Y \leq \dim X - i$ and with generically finite Albanese map.

The main point of this note is the following conjecture, saying that cohomological support loci should be preserved by derived equivalence. In the next sections I will explain that the conjecture holds for surfaces, and that is almost known to hold for threefolds.

Conjecture 1.2. *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then*
$$V^i(\omega_X) \simeq V^i(\omega_Y) \quad \text{for all } i \geq 0.$$

Note that I am proposing isomorphism, even though the ambient spaces $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ may only be isogenous. There are roughly speaking three main reasons for this: (1) the conjecture is known to hold for surfaces and for most threefolds, as explained in §2 and §3; (2) it holds for V^0 in arbitrary dimension, as explained at the beginning of §3; (3) more heuristically, according to [PS] the failure of isomorphism at the level of Pic^0 is induced by the presence of abelian varieties in the picture, and for these all cohomological support loci consist only of the origin.

Furthermore, denote by $V^i(\omega_X)_0$ the union of the irreducible components of $V^i(\omega_X)$ passing through the origin. Generic vanishing theory tells us that in many applications one only needs to control well $V^i(\omega_X)_0$. In fact, for all applications I currently have in mind, the following variant of Conjecture 1.2 suffices.

Variant 1.3. *Under the same hypothesis,*
$$V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0 \quad \text{for all } i \geq 0.$$

The key fact implied by this variant is that, excepting perhaps surjective maps to abelian varieties, roughly speaking two derived equivalent varieties must have the same types of fibrations onto lower dimensional irregular varieties (see Corollary 3.4). This would hopefully allow for further geometric tools in the classification of irregular derived partners. Even weaker versions of Conjecture 1.2 and Variant 1.3 are of interest, as they are all that is needed in other applications.

Variant 1.4. *Under the same hypothesis,*
$$\dim V^i(\omega_X) = \dim V^i(\omega_Y) \quad \text{for all } i \geq 0.$$

Variant 1.5. *Under the same hypothesis,*
$$\dim V^i(\omega_X)_0 = \dim V^i(\omega_Y)_0 \quad \text{for all } i \geq 0.$$

For instance, Variant 1.5 implies the derived invariance of the Albanese dimension (see Corollary 3.2). Other numerical applications, and progress due to Lombardi [Lo] in the case of V^0 and V^1, and on the full conjecture for threefolds, are described in §3. In §2 I present a proof of Conjecture 1.2 in the case of surfaces.

2. A PROOF OF CONJECTURE 1.2 FOR SURFACES

Due to the classification of Fourier-Mukai equivalences between surfaces, in this case the conjecture reduces to a calculation of all possible cohomological support loci via a case by case analysis, combined with some elliptic surface theory, generic vanishing theory, well-known results of Kollár on higher direct images of dualizing sheaves, and of Beauville on the positive dimensional components of $V^1(\omega_X)$. This of course does not have much chance to generalize to higher dimensions. In the next section I will point to more refined techniques developed by Lombardi [Lo], which recover the case of surfaces, but do address higher dimensions as well.

Theorem 2.1. *Conjecture 1.2 holds when X and Y are smooth projective surfaces.*

Proof. The first thing to note is that, due to the work of Bridgeland-Maciocia [BM] and Kawamata [Ka], Fourier-Mukai equivalences of surfaces are completely classified. According to [Ka] Theorem 1.6, the only non-minimal surfaces that can have derived partners are rational elliptic, and therefore regular. Hence we can restrict to minimal surfaces. Among these on the other hand, according to [BM] Theorem 1.1, only abelian, $K3$ and elliptic surfaces can have distinct derived partners.

Now $K3$ surfaces are again regular, hence for these the problem is trivial. On the other hand, on any abelian variety A (of arbitrary dimension) one has

$$V^i(\omega_A) = \{0\} \quad \text{for all } i,$$

and since the only derived partners of abelian varieties are again abelian varieties (see [HN] Proposition 3.1; cf. also [PS], end of §3), the problem is again trivial. Therefore our question is truly a question about elliptic surfaces which are not rational. Moreover, according to [BM], bielliptic surfaces do not have non-trivial derived partners. We are left with certain elliptic fibrations over $\mathbf{P}^1$, and with elliptic fibrations over smooth projective curves of genus at least 2. I will try in each case to present the most elementary proof I am aware of.

Let first $f : X \to \mathbf{P}^1$ be an elliptic surface over $\mathbf{P}^1$. Since our problem is non-trivial only for irregular surfaces, requiring $q(X) \neq 0$ we must then have that $q(X) = 1$, which implies that f is isotrivial, and in fact that X is a $\mathbf{P}^1$-bundle

$$\pi : X \longrightarrow E$$

over an elliptic curve E. We can now compute the cohomological support loci $V^i(\omega_X)$ explicitly. Note first that $V^2(\omega_X) = \{0\}$ for any smooth projective surface, by Serre duality. In the case at hand, note also that π is the Albanese map of X. Therefore, identifying line bundles in $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(E)$, for every $\alpha \in \mathrm{Pic}^0(X)$ we have

$$H^0(X, \omega_X \otimes \alpha) \simeq H^0(E, \pi_*\omega_X \otimes \alpha),$$

which implies that $V^0(\omega_X) = V^0(E, \pi_*\omega_X)$.[1] But given that $\pi_*\omega_X$ must be torsion-free, and the fibers of π are rational curves, we have $\pi_*\omega_X = 0$, and so $V^0(\omega_X) = \emptyset$. We are left with computing $V^1(\omega_X)$. For this, recall that by [Ko2] Theorem 3.1, in $\mathbf{D}(E)$ we have the decomposition

$$\mathbf{R}\pi_*\omega_X \simeq \pi_*\omega_X \oplus R^1\pi_*\omega_X[-1] \simeq R^1\pi_*\omega_X[-1],$$

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[1]In general, for any coherent sheaf $\mathcal{F}$ on a smooth projective variety Z, and any integer i, we denote $V^i(Z, \mathcal{F}) := \{\alpha \in \mathrm{Pic}^0(Z) \mid h^i(Z, \mathcal{F} \otimes \alpha) \neq 0\}$.

where the second isomorphism follows from what we said above. Therefore, for any $\alpha \in \text{Pic}^0(X)$, we have

$$H^1(X, \omega_X \otimes \alpha) \simeq H^0(E, R^1\pi_*\omega_X \otimes \alpha).$$

Finally, [Ko1] Proposition 7.6 implies that, as the top non-vanishing higher direct image,

$$R^1\pi_*\omega_X \simeq \omega_E \simeq \mathcal{O}_E,$$

which immediately gives that $V^1(\omega_X) = \{0\}$. In conclusion, we have obtained that for the type of surface under discussion we have

$$V^0(\omega_X) = \emptyset, \quad V^1(\omega_X) = \{0\}, \quad V^2(\omega_X) = \{0\}.$$

Finally, if Y is another smooth projective surface such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then due to [BM] Proposition 4.4 we have that Y is another elliptic surface over $\mathbf{P}^1$ with the same properties as X, which leads therefore to the same cohomological support loci.

Assume now that $f : X \to C$ is an elliptic surface over a smooth projective curve C of genus $g \geq 2$ (so that $\kappa(X) = 1$). By the same [BM] Proposition 4.4, if Y is another smooth projective surface such that $\mathbf{D}(Y) \simeq \mathbf{D}(X)$, then Y has an elliptic fibration structure $h : Y \to C$ over the same curve (and with isomorphic fibers over a Zariski open set in C; in fact it is a relative Picard scheme associated to f). There are two cases, namely when f is isotrivial, and when it is not. It is well known (see e.g. [Be1] Exercise IX.1 and [Fr] Ch.7) that f is isotrivial if and only if $q(X) = g + 1$ (in which case the only singular fibers are multiple fibers with smooth reduction), and it is not isotrivial if and only if $q(X) = g$. Since we know that $q(X) = q(Y)$, we conclude that h must be of the same type as f. We will again compute all $V^i(\omega_X)$ in the two cases.

Let's assume first that f is not isotrivial. As mentioned above, in this case $q(X) = g$, and in fact $f^* : \text{Pic}^0(C) \to \text{Pic}^0(X)$ is an isomorphism. To compute $V^1(\omega_X)$, we use again [Ko2] Theorem 3.1, saying that in $\mathbf{D}(C)$ there is a direct sum decomposition

$$\mathbf{R}f_*\omega_X \simeq f_*\omega_X \oplus R^1f_*\omega_X[-1],$$

and therefore for each $\alpha \in \text{Pic}^0(X)$ one has

$$H^1(X, \omega_X \otimes \alpha) \simeq H^1(C, f_*\omega_X \otimes \alpha) \oplus H^0(C, R^1f_*\omega_X \otimes \alpha).$$

Once again using [Ko1] Proposition 7.6, we also have $R^1f_*\omega_X \simeq \omega_C$. This, combined with the decomposition above, gives the inclusion

$$f^* : \text{Pic}^0(C) = V^0(C, \omega_C) \hookrightarrow V^1(\omega_X),$$

finally implying $V^1(\omega_X) = \text{Pic}^0(X)$. Finally, note that by the Castelnuovo inequality [Be1] Theorem X.4, we have $\chi(\omega_X) \geq 0$. Now the Euler characteristic is a deformation invariant, hence $\chi(\omega_X \otimes \alpha) \geq 0$ for all $\alpha \in \text{Pic}^0(X)$. For $\alpha \neq \mathcal{O}_X$, this gives

$$h^0(X, \omega_X \otimes \alpha) \geq h^1(X, \omega_X \otimes \alpha),$$

so that $V^1(\omega_X) \subset V^0(\omega_X)$. By the above, we obtain $V^0(\omega_X) = \text{Pic}^0(X)$ as well. We rephrase the final result as saying that

$$V^0(\omega_X) = V^1(\omega_X) \simeq \text{Pic}^0(C), \quad V^2(\omega_X) = \{0\}.$$

The preceding paragraph says that the exact same calculation must hold for a Fourier-Mukai partner Y.

Let's now assume that f is isotrivial. First note that for such an X we have $q(X) = g+1$, and in fact the Albanese variety of X is an extension of abelian varieties

$$1 \to F \to \mathrm{Alb}(X) \to J(C) \to 1$$

with F an elliptic curve isogenous to the general fiber of f, though this will not play an explicit role in the calculation. Moreover, we have $\chi(\omega_X) = 0$ (see [Fr] Ch.7, Lemma 14 and Corollary 17).

We now use a result of Beauville [Be2] [Be3], characterizing the positive dimensional irreducible components of $V^1(\omega_X)$. Concretely, by [Be3] Corollaire 2.3, any such positive dimensional component would have to come either from a fiber space $h : X \to B$ over a curve of genus at least 2, or from a fiber space $p : X \to F$ over an elliptic curve, with at least one multiple fiber. Regarding the first type, the union of all such components is shown in *loc. cit.* to be equal to

$$\mathrm{Pic}^0(X, h) := \mathrm{Ker}\big(\mathrm{Pic}^0(X) \xrightarrow{i^*} \mathrm{Pic}^0(F)\big),$$

where i^* is the restriction map to any smooth fiber F of h. But since f is an elliptic fibration, it is clear that there is exactly one such fiber space, namely f itself (otherwise the elliptic fibers of any other fibration would have to dominate C, which is impossible). Therefore the union of the components coming from fibrations over curves of genus at least 2 is $\mathrm{Pic}^0(X, f)$. On the other hand, for elliptic surfaces of the type we are currently considering, fibrations $p : X \to F$ over elliptic curves as described above do not exist. (Any such would have to come from a group action on a product between an elliptic curve F' and another of genus at least 2, with the action on the elliptic component having no fixed points, therefore leading to an étale cover $F' \to F$; in the language of [Be3], we are saying that $\Gamma^0(p) = \{0\}$.)

Using once more the deformation invariance of the Euler characteristic, we have $\chi(\omega_X \otimes \alpha) = 0$ for all $\alpha \in \mathrm{Pic}^0(X)$, which gives

$$(2.1) \qquad h^1(X, \omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha), \quad \text{for all } \alpha \neq \mathcal{O}_X.$$

This implies that $\mathrm{Pic}^0(X, f)$ is also the union of all positive dimensional components of $V^0(\omega_X)$.

We are left with considering nontrivial isolated points in $V^1(\omega_X)$ (or equivalently in $V^0(\omega_X)$ by (2.1)). These can be shown not to exist by means of a different argument: by a variant of the higher dimensional Castelnuovo-de Franchis inequality, see [LP] Remark 4.13, an isolated point $\alpha \neq 0$ in $V^1(\omega_X)$ forces the inequality

$$\chi(\omega_X) \geq q(X) - 1 = g \geq 2,$$

which contradicts the fact that $\chi(\omega_X) = 0$.[2]

Putting everything together, we obtain

$$V^0(\omega_X) = V^1(\omega_X) = \mathrm{Pic}^0(X, f), \quad V^2(\omega_X) = \{0\}.$$

Recall that a Fourier-Mukai partner of X must be an elliptic fibration of the same type over C. Now one of the main results of [Ph], Theorem 5.2.7, says that for derived equivalent

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[2]As L. Lombardi points out, a variant of the derivative complex argument in [LP] leading to this inequality can also be used, as an alternative to Beauville's argument, in order to show that positive dimensional components not passing through the origin do not exist in the case of surfaces of maximal Albanese dimension with $\chi(\omega_X) = 0$.

elliptic fibrations $f : X \to C$ and $h : Y \to C$ which are isotrivial with only multiple fibers, one has
$$\mathrm{Pic}^0(X, f) \simeq \mathrm{Pic}^0(Y, h),\,^3$$
which allows us to conclude that $V^i(\omega_X)$ and $V^i(\omega_Y)$ are isomorphic.   □

3. FURTHER EVIDENCE AND APPLICATIONS

Progress. Progress towards the conjectures in §1 has been made by Lombardi [Lo]. The crucial point is to come up with an explicit mapping realizing the potential isomorphisms in Conjecture 1.2. This is done by means of the *Rouquier isomorphism*; namely, given a Fourier-Mukai equivalence $\mathbf{R}\Phi_\mathcal{E} : \mathbf{D}(X) \to \mathbf{D}(Y)$ induced by an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, Rouquier [Ro] Théorème 4.18 shows that there is an induced isomorphism of algebraic groups
$$F : \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \longrightarrow \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$
given by a concrete formula involving $\mathcal{E}$ (usually mixing the two factors), [PS] Lemma 3.1. A key result in [Lo] is that if $\alpha \in V^0(\omega_X)$ and
$$F(\mathrm{id}_X, \alpha) = (\varphi, \beta),$$
then in fact $\varphi = \mathrm{id}_Y$, $\beta \in V^0(\omega_Y)$, and moreover

(3.1) $H^0(X, \omega_X \otimes \alpha) \simeq H^0(Y, \omega_Y \otimes \beta).$

One of the main tools used there is the derived invariance of a generalization of Hochschild homology taking into account the Rouquier isomorphism. This implies the invariance of V^0, while further work using a variant of the Hochschild-Kostant-Rosenberg isomorphism gives the following, again the isomorphisms being induced by the Rouquier mapping.

Theorem 3.1 (Lombardi [Lo]). *Let X and Y be smooth projective varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Then:*

(i) $V^0(\omega_X) \simeq V^0(\omega_Y)$.

(ii) $V^1(\omega_X) \cap V^0(\omega_X) \simeq V^1(\omega_Y) \cap V^0(\omega_Y)$.

(iii) $V^1(\omega_X)_0 \simeq V^1(\omega_Y)_0$.

This result recovers Theorem 2.1 in a more formal way. In the case when $\dim X = \dim Y = 3$, with extra work one shows that this has the following consequences, verifying or getting close to verifying the various conjectures:

• Variant 1.3 holds.

• For any i, $V^i(\omega_X)$ is positive dimensional if and only if $V^i(\omega_Y)$ is positive dimensional, and of the same dimension. Therefore Variant 1.4 holds, except for the possible case where for some $i > 0$, $V^i(\omega_X)$ is finite, while $V^i(\omega_Y) = \emptyset$. This last case can possibly happen only when $q(X) = 1$.

• Conjecture 1.2 is true when:

   (1) X is of maximal Albanese dimension (i.e. the Albanese map of X is generically finite onto its image).

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[3]This also follows from [Lo], via Theorem 3.1 below.

(2) $V^0(\omega_X) = \mathrm{Pic}^0(X)$ – for instance, by [PP] Theorem E, this condition holds whenever the Albanese image $a(X)$ is not fibered in subtori of $\mathrm{Alb}(X)$, and $V^0(\omega_X) \neq \emptyset$.

(3) $\mathrm{Aut}^0(X)$ is affine – this holds for varieties which are not isotrivially fibered over a positive dimensional abelian variety (see [Br] p.2 and §3), for instance again when the Albanese image is not fibered in subtori of $\mathrm{Alb}(X)$ according to a theorem of Nishi (cf. [Ma] Theorem 2).

These conditions together impose very strong restrictions on the threefolds for which the conjecture is not yet known. Note finally that in [Lo] there are further extensions involving cohomological support loci for $\omega_X^{\otimes m}$ with $m \geq 2$, and for Ω_X^p with $p < \dim X$.

Some first applications. Let X be a smooth projective complex variety of dimension d, and let $a : X \to A = \mathrm{Alb}(X)$ be the Albanese map of X. A first consequence of the weakest version of the conjectures would be the derived invariance of the Albanese dimension $\dim a(X)$.

Corollary 3.2 (**assuming Variant 1.5**). *If X and Y are smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, then*

$$\dim a(X) = \dim a(Y).$$

This follows from the fact that, according to [LP] Remark 2.4, the Albanese dimension can be computed from the dimension of the cohomological support loci around the origin, according to the formula

$$\dim a(X) = \min_{i=0,\ldots,d} \{d - i + \mathrm{codim}\, V^i(\omega_X)_0\}.$$

Note that Lombardi [Lo] is in fact able to prove Corollary 3.2 when $\kappa(X) \geq 0$ by relying on different tools from birational geometry. The only progress when $\kappa(X) = -\infty$, namely a solution for surfaces and threefolds that can also be found in *loc. cit.*, involves the approach described here.

Another numerical application involves the holomorphic Euler characteristic. While the individual Hodge numbers are not yet known to be preserved by derived equivalence, the Euler characteristic can be attacked in some cases by using generic vanishing theory and the derived invariance of $V^0(\omega_X)$ established in Theorem 3.1.

Corollary 3.3 (*of Theorem 3.1*, [Lo]). *If X and Y are smooth projective complex varieties with $\mathbf{D}(X) \simeq \mathbf{D}(Y)$, and X is of maximal Albanese dimension, then $\chi(\omega_X) = \chi(\omega_Y)$.*

This follows from the fact that, according to (1.1), for generic $\alpha \in \mathrm{Pic}^0(X)$ one has

$$\chi(\omega_X) = \chi(\omega_X \otimes \alpha) = h^0(X, \omega_X \otimes \alpha),$$

combined with (3.1). The argument is extended in [Lo] to other cases as well. Going back to Hodge numbers, this implies for instance that if X and Y are derived equivalent 4-folds of maximal Albanese dimension, then

$$h^{0,2}(X) = h^{0,2}(Y),^4$$

since in the case of 4-folds all the other $h^{0,q}$ Hodge numbers are known to be preserved.

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[4]This also automatically implies $h^{1,3}(X) = h^{1,3}(Y)$ by the invariance of Hochschild homology.

Perhaps the main point in this picture is the fact that the positive dimensional components of the cohomological support loci $V^i(\omega_X)$ reflect the nontrivial fibrations of X over irregular varieties. Therefore, roughly speaking, the key geometric significance of Conjecture 1.2 is that derived equivalent varieties should have the same type of fibrations over lower dimensional irregular varieties, thus allowing for more geometric tools in the study of Fourier-Mukai partners. One version of this principle can be stated as follows:

Corollary 3.4 (**assuming Variant 1.3**). *Let X and Y be smooth projective varieties such that $\mathbf{D}(X) \simeq \mathbf{D}(Y)$. Fix an integer $m > 0$, and assume that X admits a morphism $f : X \to Z$ with connected fibers, onto a normal irregular variety of dimension m whose Albanese map is not surjective. Then Y admits a morphism $h : Y \to W$ with connected fibers, onto a positive dimensional normal irregular variety of dimension $\leq m$. Moreover, if $m = 1$, then W can also be taken to be a curve of genus at least 2.*

This is due to the fact that, by the degeneration of the Leray spectral sequence for $\mathbf{R}f_*\omega_X$ due to Kollár [Ko1], one has

$$f^* V^0(\omega_Z) \subset V^{n-m}(\omega_X),$$

where $n = \dim X = \dim Y$. Now in [EL] Proposition 2.2 it is shown that if 0 is an isolated point in $V^0(\omega_Z)$, then the Albanese map of Z must be surjective. Thus the hypothesis implies that we obtain a positive dimensional component in $V^{n-m}(\omega_X)_0$, hence by Variant 1.3 also in $V^{n-m}(\omega_Y)_0$. Going in reverse, recall now from §1 that, according to one of the main results of [GL2], a positive dimensional component of $V^{n-m}(\omega_Y)$ produces a fiber space $h : Y \to W$, with W a positive dimensional normal irregular variety (with generically finite Albanese map) and $\dim W \leq m$. The slightly stronger statement in the case of fibrations over curves follows from the precise description of the positive dimensional components of $V^{n-1}(\omega_X)$ given in [Be3].

I suspect that one should be able to remove the non-surjective Albanese map hypothesis (in other words allow maps onto abelian varieties), but this must go beyond the methods described here.

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