GENERIC VANISHING AND MINIMAL COHOMOLOGY CLASSES ON ABELIAN FIVEFOLDS

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ABSTRACT. We classify GV-subschemes of five-dimensional ppavs, proving the main conjecture in [PP08b] in this case. This result is implied by a more general statement about subvarieties of minimal cohomology class whose sum is a theta divisor.

A. INTRODUCTION

The aim of this paper is to give a proof of the main conjecture in [PP08b] in the case of five-dimensional principally polarized complex abelian varieties (ppavs).

Theorem A. *Let* (A, Θ) *be an indecomposable ppav of dimension* 5. *If* $1 \le d \le 3$, *then* X *is a geometrically nondegenerate closed* GV-*subscheme of* A *of dimension* d *if and only if one of the following holds:*

(a) *There is a smooth genus* 5 *curve* C *and an isomorphism* $(A, \Theta) \cong (JC, \Theta_C)$ *that identifies* X *with* $W_d(C)$.

(b) $d = 2$ *and there is a smooth cubic threefold* Y *and an isomorphism* $(A, \Theta) \cong (JY, \Theta_Y)$ *that identifies* X *with* F, *the Fano surface of lines on* Y.

Recall that $X \subset A$ is called a GV-subscheme if $\mathcal{I}_X(\Theta)$ is a GV-sheaf on A in the sense of [PP11]; concretely, this means that

$$\mathrm{codim}\ \{\alpha \in \mathrm{Pic}^0(A) \mid H^i(A, \mathcal{I}_X(\Theta) \otimes \alpha) \ne 0\} \ge i \quad \text{for all} \ \ i \ge 0.$$

This property is a formal analogue of 2-regularity in projective space, as we will recall below. When $\dim X = 1$ or 3, the result in Theorem A already follows from [PP08b, Theorem C]. The (much more difficult) case addressed here is that of $\dim X = 2$; this turns out to be a consequence of the following more general statement, which is the main result of the paper.

Theorem B. *Let* (A, Θ) *be an indecomposable ppav of dimension* 5, *and let* $V, W \subset A$ *be two-dimensional subvarieties, such that* $[V] = [W] = \frac{\theta^3}{3!}$ *and* $V + W = \Theta$. *Then one of the following holds:*

(a) *There is a smooth genus* 5 *curve* C *and an isomorphism* $(A, \Theta) \cong (JC, \Theta_C)$ *that identifies* V *with* $W_2(C)$.

(b) *There is a smooth cubic threefold* Y *and an isomorphism* $(A, \Theta) \cong (JY, \Theta_Y)$ *that identifies* V *with* F, *the Fano surface of lines on* Y.

In the statements of both theorems we are referring to the Abel–Jacobi embedding of the Fano surface of lines in JY, which also happens to be isomorphic to $\mathrm{Alb}(F)$; see [CG72].

Context and previous work. Recall the combination of the main conjectures in [Deb95] and [PP08b].

Conjecture C. *Let (A, Θ) be an indecomposable ppav of dimension g, and let X be a closed subscheme of A of dimension $1 \leq d \leq g - 2$. The following are equivalent:*

(1) *X is reduced of pure dimension and has minimal cohomology class, i.e. $[X] = \frac{\theta^{g-d}}{(g-d)!}$.*

(2) *X is a geometrically nondegenerate GV-subscheme, i.e. X is geometrically nondegenerate and $\mathcal{I}_X(\Theta)$ is a GV-sheaf on A.*

(3) *One of the following holds:*

    (a) *There is a smooth genus g curve C and an isomorphism $(A, \Theta) \cong (JC, \Theta_C)$ that identifies X with $W_d(C)$.*

    (b) *$g = 5$, $d = 2$, and there is a smooth cubic threefold Y and an isomorphism $(A, \Theta) \cong (JY, \Theta_Y)$ that identifies X with F, the Fano surface of lines on Y.*

Establishing the equivalence between (1) and (3) would be a vast generalization of the Matsusaka–Ran criterion, which is the special case of the conjecture where $d = 1$; this was proposed by Debarre [Deb95], building also on previous questions of Beauville and Ran. The equivalence with (2) was proposed in [PP08b], in order to provide a more manageable cohomological bridge between the two conditions. Note that in $\mathbf{P}^n$ there are quite familiar analogues of these equivalences; we recall this next:

(i) A formal analogy between the subvarieties of (A, Θ) in (3) and subvarieties of minimal degree in $\mathbf{P}^n$ is described in [PP08a, §2(6)]: the $W_d(C)$ in Jacobians correspond to rational normal scrolls, while in the five-dimensional case the Fano surface of lines corresponds to the Veronese surface. Thus the equivalence of (1) and (3) corresponds under this formal analogy to the classification of subvarieties of minimal degree in $\mathbf{P}^n$.

(ii) On the other hand, as described in [PP08a, §2(7)], work in [PP03] shows that condition (2) corresponds formally to the 2-regularity of $\mathcal{I}_X$ in the sense of Castelnuovo–Mumford, for a subvariety $X \subset \mathbf{P}^n$. Therefore, the equivalence of (2) with the other conditions is the analogue of another well-known property in projective space: a subvariety in $\mathbf{P}^n$ has minimal degree if and only if its ideal sheaf is 2-regular.

Regarding what is known, the equivalence of (1) and (3) for $d = 1$ is the Matsusaka–Ran criterion, while in dimension 4 it holds for all d by [Ran81]. Debarre [Deb95] also shows that it holds when (A, Θ) is already assumed to be the Jacobian of a curve. Similarly, Höring [Hör10, Theorem 1.2] shows that it holds when (A, Θ) is assumed to be a generic intermediate Jacobian of a cubic threefold. The main results of [PP08b] are that $(2) \implies (3)$ holds for $d = 1, g - 2$, and that (2) implies (1) apart from the reducedness assertion, which is proven in [Sch16]. Condition (3) is known to imply the others; see the introduction to [PP08b] for an explanation and references. In particular, Conjecture C is fully known in dimension up to 4, and for $d = 1$ in general.

As mentioned above, in this paper we prove the implication (2) $\implies$ (3) in dimension 5. This has a special significance: besides being the next case that was not known, it is also arguably the most interesting, due to the exceptional appearance of the Fano surface of lines. Theorem A and Theorem B provide the first known criteria which detect all Jacobians of smooth genus five curves and intermediate Jacobians of cubic threefolds at the same time. This also strengthens the hope for obtaining the predicted list of varieties of minimal class; we expect the implication (1) $\implies$ (2) to be a formal, albeit difficult, fact.

Ingredients and outline of the proof of Theorem B. The proof of Theorem B relies on a number of different ingredients. Here is a rough sketch of the main steps:

(1) If V or W is singular, we show in the proof of Theorem 4.1 that the singular locus Θ^{sing} of Θ is at least two-dimensional. Since (A, Θ) is a generalized Prym variety, and assumed to be indecomposable, a theorem of Beauville [Bea77] implies that it must be the Jacobian of a hyperelliptic curve C. Since V has minimal cohomology class, Debarre's theorem then implies that V corresponds to $W_2(C)$. We can therefore assume in what follows that V and W are smooth.

(2) If V is smooth and $V + V = \Theta$ (i.e., we assume $W = V$), then we prove in Theorem 5.1 that (A, Θ) is isomorphic to the Jacobian of a (necessarily nonhyperelliptic) curve. The outline is as follows. By [Sch16], V has property $(\mathcal{P})$ with respect to itself (see Definition 3.1), and we use this to show that V has a curve summand. Therefore Θ has a curve summand and so the main result of [Sch14] implies that (A, Θ) is isomorphic to the Jacobian of a smooth curve.

(3) If V is smooth and $V - V = \Theta$ (i.e., we assume $W = -V$), then we prove in Theorem 6.1 that (A, Θ) is isomorphic to the intermediate Jacobian of a smooth cubic threefold Y. This is done by analyzing the multiplicity of Θ at the origin, using intersection theory and Schubert calculus associated with the difference map $g : V \times V \to V - V$; we obtain a point of multiplicity 3, at which stage results from [CM08, CM09] imply the desired characterization of (A, Θ). Further arguments show that one can choose this isomorphism so as to identify V with the Fano surface of lines on Y.

(4) In Theorem 12.1, the final step of the proof, we reduce Theorem B to the cases treated above. To this end, we consider again the difference map

$$g : V \times V \longrightarrow V - V.$$

Since V has minimal class, the map g is generically finite of degree 1, 2, 3 or 6. We prove that $\deg(g) = 2$ or 3 is impossible, that $\deg(g) = 1$ implies $V + V = \Theta$ and that $\deg(g) = 6$ implies $V - V = \Theta$. The surface W plays a crucial role in the argument. We prove that $V - V = W - W$ and consider the pre-images $g^{-1}(W - w)$, for $w \in W$. The argument makes essential use of the fact from [Sch16] (see Theorem 3.3 below) that V has property $(\mathcal{P})$ with respect to W. The most difficult part is to rule out $\deg(g) = 2$ or 3. Here the main strategy is to construct a curve summand of V or W, which implies via [Sch14] that (A, Θ) is isomorphic to the Jacobian of a smooth curve C; Debarre's result [Deb95] for Jacobians implies that V is a translate of $W_2(C)$, which leads to a contradiction, since the difference map then has degree 1.

Proof of Theorem A. Once Theorem B is proven, Theorem A is obtained via the following argument; we recall that it suffices to treat the case of GV-subschemes of dimension $d = 2$, since the cases $d = 1$ and 3 are already known by [PP08b].

As in [PP08b], given a closed reduced subscheme X of a ppav (A, Θ), we can consider the associated *theta-dual* scheme $V(X)$; set theoretically we have

$$V(X) = \{x \in A : X - x \subseteq \Theta\}.$$

If X is a geometrically nondegenerate closed GV-subscheme of dimension two, we know from [PP08b] that $[X] = \frac{\theta^3}{3!}$, and that $V(X)$ is two-dimensional and of the same minimal class. (In [PP08b], X is assumed to be pure-dimensional and reduced, but the same proofs work without these assumptions – pure-dimensionality turns out to be automatic, see also [Sch16, Section 2.5].) If (A, Θ) is indecomposable, then by [Sch16] both X and $V(X)$ are irreducible and reduced. This in turn implies $X - V(X) = \Theta$ by the nondegeneracy of X and the definition of $V(X)$; see also Lemma 2.1 below. At this point, Theorem B applies with $V = X$ and $W = -V(X)$ to give Theorem A.

Conventions. We work over the field of complex numbers $\mathbf{C}$. A *variety* is an integral separated scheme of finite type over $\mathbf{C}$. A *curve* (resp. *surface*) is a variety of dimension one (resp. two). Unless mentioned otherwise, all varieties will be assumed to be proper. A general point of a variety is a point of a Zariski open and dense subset.

B. Preliminaries

1. Nondegeneracy. We recall that, according to [Ran81], a closed subscheme X of dimension d of an abelian variety A of dimension g is called *nondegenerate* in A if the cup product map

$$(1.1) \qquad\qquad \cup [X] : H^{0,d}(A) \longrightarrow H^{g-d,g}(A)$$

is injective (hence an isomorphism). This definition depends only on the d-dimensional components of X, since $\omega \cup [X'] = 0$ for all $\omega \in H^{d,0}(A)$ if $\dim(X') < d$. If X is pure-dimensional and reduced, it is shown in [Ran81] that this is equivalent to the image of the Gauss map

$$\Gamma : X^{\mathrm{reg}} \to \mathbf{G}(d-1, g-1)$$

not being contained in any hyperplane via the Plücker embedding, which is further equivalent to the injectivity of the restriction map $H^0(A, \Omega_A^d) \to H^0(X^{\mathrm{reg}}, \Omega_{X^{\mathrm{reg}}}^d)$, where X^{reg} is the smooth locus of X.

Our main case of interest is the following: a codimension k subscheme of a ppav (A, Θ) is nondegenerate if its cohomology class is a multiple of the minimal class $\frac{\theta^k}{k!}$, as follows directly from the definition above.

Theorem A is phrased in terms of the weaker notion of *geometric nondegeneracy*, which means that the kernel of the above cup product map (1.1) contains no decomposable elements. This notion also originated in Ran's work [Ran81]; it is treated in the present generality in [Sch16]. It turns out that X is geometrically nondegenerate if and only if the reduced scheme X^{red} is. Moreover, if X is reduced, then it is geometrically nondegenerate if and only if the kernel of the restriction map

$$H^0(A, \Omega_A^d) \longrightarrow H^0(X^{\mathrm{reg}}, \Omega_{X^{\mathrm{reg}}}^d)$$

does not contain any decomposable elements, see [Sch16, Section 2.4]. However, geometrically nondegenerate GV-subschemes are known to represent minimal cohomology classes [PP08b]; it follows *a posteriori* that they are nondegenerate.

2. The sum of two surfaces of minimal class. Let V and W be closed subschemes of an abelian variety A. The sum $V + W$ denotes the image of the addition morphism

$$+ : V \times W \longrightarrow A;$$

it has the expected dimension if $\dim(V + W) = \dim(V) + \dim(W)$ or $V + W = A$.

The sum of a geometrically nondegenerate subscheme with any other subscheme of an abelian variety has the expected dimension; see [Deb05, Ch.8, Corollary 11] for the case of subvarieties and [Sch16, Lemma 5] in general. The following well-known lemma yields a more precise answer in the case where we consider the sum of two surfaces of minimal class.

Lemma 2.1. *Let (A, Θ) be a ppav of dimension 5, and let $V, W \subseteq A$ be surfaces of minimal cohomology class, i.e. $[V] = [W] = \frac{\theta^3}{3!}$. Then $[V + W] = \lambda \cdot \theta$ for some $\lambda \in \{1, 2, 3, 6\}$. Moreover, the addition morphism*

$$f : V \times W \longrightarrow V + W$$

is generically finite of degree $\deg(f) = \frac{6}{\lambda}$.

Proof. This follows from

$$6 \cdot \theta = \frac{\theta^3}{3!} \star \frac{\theta^3}{3!} = [V] \star [W] = \deg(f) \cdot [V + W],$$

where $\star$ denotes the Pontryagin product on the cohomology of A (see e.g., [BL04, Corollary 16.5.8]). $\square$

3. The property $(\mathcal{P})$. Let V and W be subvarieties of an abelian variety A. The following definition can be found in Debarre's paper [Deb95]; it originated in Ran's work [Ran81, Ran82].

Definition 3.1. Let $f : V \times W \to A$ be the addition map. We say that V has property $(\mathcal{P})$ with respect to W if for general $v \in V$, the only subvariety of $f^{-1}(v + W)$ which dominates W via the second projection and $v + W$ via f is $v \times W$.

The condition in the above definition implies that for general $v \in V$, the only component of the reduced preimage $f^{-1}(v + W)^{\mathrm{red}}$ which dominates W via the second projection and $v + W$ via f is $v \times W$. In particular, f is generically finite. Moreover, if $\deg(f) = 1$, then the property $(\mathcal{P})$ is trivially satisfied. However, if $\deg(f) \geq 2$, then the condition is quite restrictive. A guiding example is given by $W_d(C)$ inside the Jacobian JC of a smooth genus g curve C, which has property $(\mathcal{P})$

with respect to $W_{g-e}(C)$ for any $e \geq d$, see [Deb95, Example 2.2]. The following result of Debarre yields a different interpretation of this notion.

Lemma 3.2 ([Deb95, Lemma 2.3]). *Let V and W be two subvarieties of an abelian variety A and let $g : V \times V \to A$ be the difference map. Then W has property $(\mathcal{P})$ with respect to V if and only if, for $w \in W$ general, the only subvariety of $g^{-1}(W - w)$ which dominates both factors of $V \times V$ is the diagonal.*

The following theorem, which is a special case of the main result in [Sch16], is one of the key ingredients of this paper.

Theorem 3.3 ([Sch16, Corollary 21]). *Let (A, Θ) be an indecomposable ppav, and let $V, W \subseteq A$ be subvarieties of dimensions d and $g - d - 1$ respectively, whose cohomology classes are minimal, i.e.*

$$[V] = \frac{\theta^{g-d}}{(g-d)!} \quad \text{and} \quad [W] = \frac{\theta^{d+1}}{(d+1)!} \,.$$

If $V + W = \Theta$, then V has property $(\mathcal{P})$ with respect to W and vice versa.

C. Detecting Jacobians of hyperelliptic curves

4. The purpose of this section is to deal with the case where V or W in Theorem B is singular. We show that this corresponds to the case of hyperelliptic Jacobians.

Theorem 4.1. *Let (A, Θ) be an indecomposable ppav of dimension 5. Suppose that there exist two surfaces $V, W \subseteq A$ of minimal cohomology class $\frac{\theta^3}{3!}$, such that $V + W = \Theta$. If either V or W is singular, then there is a smooth hyperelliptic curve C and an isomorphism $(A, \Theta) \cong (J(C), \Theta_C)$ which identifies V with $W_2(C)$.*

Proof. By symmetry, it suffices to deal with the case where V is singular. We claim that

(4.2) $(V^{\text{sing}} + W)^{\text{red}} \subseteq \Theta^{\text{sing}},$

where Θ^{sing} denotes the singular locus of Θ and $(V^{\text{sing}} + W)^{\text{red}}$ denotes the reduced scheme structure on the image $V^{\text{sing}} + W$. In order to prove the above inclusion, let $v_0 \in V^{\text{sing}}$ be a singular point of V. Then the Zariski tangent space

$$T_{v_0}V \subseteq T_0A$$

is at least three-dimensional.

Let us consider the Gauss map

$$\Gamma : W^{\text{reg}} \longrightarrow G(2,5), \quad w \mapsto [T_wW \subseteq T_0A],$$

where we see the tangent space T_wW as being translated inside the tangent space of A at the origin. Since W has minimal cohomology class, it is nondegenerate, and so the image of Γ via the Plücker embedding $G(2,5) \hookrightarrow \mathbf{P}^9$ is not contained in any hyperplane. This implies that for general $w \in W$,

$$T_{v_0}V + T_wW = T_0A.$$

Hence, $v_0 + w \in \Theta^{\text{sing}}$ for general $w \in W$ and so $v_0 + W \subseteq \Theta^{\text{sing}}$. This proves (4.2).

Since V contains a singular point by assumption, (4.2) implies $\dim(\Theta^{\text{sing}}) \geq 2$. Now as (A, Θ) is five-dimensional, it is a generalized Prym variety. Therefore, (A, Θ) indecomposable with $\dim(\Theta^{\text{sing}}) \geq 2$ implies by a result of Beauville [Bea77,

Theorem 4.10, Theorem 5.4] that (A, Θ) is isomorphic to the Jacobian of a smooth hyperelliptic curve. The identification of V with $W_2(C)$ follows then from Debarre's proof of the minimal class conjecture for Jacobians [Deb95, Theorem 5.1]. $\qquad\square$

Remark 4.3. More generally, the above arguments relate the locus of ppavs with decomposable theta divisor $\Theta = V + W$ to the Andreotti–Mayer loci N_ℓ; i.e., the loci of ppavs (A, Θ) of dimension g with $\dim(\Theta^{\mathrm{sing}}) \geq \ell$. Indeed, if (A, Θ) is a ppav of dimension g, admitting subvarieties $V, W \subseteq A$ with $V + W = \Theta$, then V and W are nondegenerate by [Sch16, Theorem 1]. If V is singular the arguments above therefore show

$$(A, \Theta) \in N_{g-1-\dim V}.$$

In particular, if V is a singular surface, then $(A, \Theta) \in N_{g-3}$, and if V is a singular threefold, then $(A, \Theta) \in N_{g-4}$. Andreotti and Mayer have shown that the closure of the Jacobian locus $\bar{J}_g \subseteq \mathcal{A}_g$ is an irreducible component of N_{g-4} (although it is never equal to N_{g-4} for $g \geq 4$ [Bea77]). Moreover, for indecomposable Jacobians and generalized Prym varieties, $(A, \Theta) \in N_{g-3}$ implies that (A, Θ) is the Jacobian of a hyperelliptic curve [Bea77].

D. DETECTING JACOBIANS OF NONHYPERELLIPTIC CURVES

5. In this section we prove the following special case of Theorem B, which isolates the case of nonhyperelliptic Jacobians.

Theorem 5.1. *Let V be a smooth surface in a five-dimensional indecomposable ppav (A, Θ), of minimal cohomology class $[V] = \frac{\theta^3}{3!}$. If $V + V = \Theta$, then there is a smooth nonhyperelliptic curve C and an isomorphism $(A, \Theta) \cong (JC, \Theta_C)$ which identifies V with $W_2(C)$.*

Proof. What follows uses ideas from [Ran81, Theorem 5]. Let $f : V \times V \to \Theta$ be the addition morphism. Since V has minimal cohomology class, f is generically finite of degree 6 by Lemma 2.1.

Let us pick a general point $v \in V$. The reduced preimage of $v + V$ decomposes as

$$f^{-1}(v + V)^{\mathrm{red}} = (v \times V) \cup (V \times v) \cup R \cup Q,$$

where $f(Q) \subsetneq v + V$ while each component of R dominates $v + V$ via f. The restriction of f to R has degree 4, because for general $v_2 \in V$, $f^{-1}(v + v_2)$ is given by six distinct points. In particular, we can pick a two dimensional component $R' \subseteq R$ which dominates $v + V$ via f.

By Theorem 3.3, V has property $(\mathcal{P})$ with respect to itself. Hence, $\mathrm{pr}_2(R') \subsetneq V$. After swapping the factors of $V \times V$, the same argument shows $\mathrm{pr}_1(R') \subsetneq V$. Since R' is an irreducible surface in $V \times V$, these observations identify R' with the product $\mathrm{pr}_1(R') \times \mathrm{pr}_2(R')$ in $V \times V$:

$$R' = \mathrm{pr}_1(R') \times \mathrm{pr}_2(R').$$

Applying f shows that $V = \mathrm{pr}_1(R') + \mathrm{pr}_2(R')$ has a curve summand, hence so does Θ. It therefore follows from [Sch14] that there is a smooth curve C and an isomorphism $(A, \Theta) \cong (JC, \Theta_C)$.

Since V has minimal cohomology class, it again follows from [Deb95, Theorem 5.1] that we can choose this isomorphism in such a way that it identifies V with

$W_2(C)$. Since V is smooth, C must be nonhyperelliptic, which concludes the proof.
□

E. Detecting intermediate Jacobians of cubic threefolds

6. The aim of this section is to prove the following result, which is the next step towards the proof of Theorem B.

Theorem 6.1. *Let $V \subset (A, \Theta)$ be a smooth surface in an indecomposable ppav of dimension 5, of minimal cohomology class $[V] = \frac{\theta^3}{3!}$. If $V - V = \Theta$, then there is a smooth cubic threefold Y and an isomorphism $(A, \Theta) \cong (JY, \Theta_Y)$ which identifies V with the Fano surface of lines on Y.*

The rest of this section is devoted to the proof of Theorem 6.1; the notation will always be that in the statement of the theorem.

7. The difference map. We start by considering the natural difference map

$$g : V \times V \longrightarrow V - V = \Theta \subset A.$$

Since V has minimal class, $\deg(g) = 6$ by Lemma 2.1.

Lemma 7.1. *The fiber $g^{-1}(0)$ is scheme-theoretically isomorphic to the diagonal $\Delta \subseteq V \times V$, which is isomorphic to V.*

Proof. This is clear set-theoretically. On the other hand, if we fix a point $x \in V$, the differential of the difference map at (x, x) is given by vector difference, i.e.

$$dg_{(x,x)} : T_{(x,x)}V \times V \longrightarrow T_0 A, \quad (v, w) \mapsto v - w$$

where we consider $T_x V$ inside $T_0 A$ by translation. It follows that Ker $dg_{(x,x)}$ is two-dimensional, and so the scheme-theoretic fiber is smooth.   □

We can consider then the normal bundle N to $g^{-1}(0)$ in $V \times V$, and we have

$$N = N_{\Delta/V \times V} \simeq TV.$$

For Segre classes, this means that the coefficient of $[V]$ in $s(\Delta, V \times V)$ is

$$s_2(TV) = c_1(V)^2 - c_2(V).$$

Recall also (see [Ful98, §4.3]) that $s(0, \Theta) = \mathrm{mult}_0 \Theta$. From the general push-forward formula [Ful98, Proposition 4.2(a)], combined with Lemma 7.1, we immediately obtain:

Lemma 7.2. *The multiplicity of $\Theta = V - V$ at the origin satisfies*

$$\deg(g) \cdot \mathrm{mult}_0 \Theta = c_1(V)^2 - c_2(V).$$

8. The Euler characteristic of V. General results imply the following lower bound:

Proposition 8.1. *The holomorphic Euler characteristic of V satisfies*

$$\chi(\mathcal{O}_V) \geq 4.$$

Proof. Since V is smooth and nondegenerate in A, it is canonically polarized; see e.g. [Deb05, p.98–101]. Being in particular of general type, it then follows from [Bea82] that $p_g(V) \geq 2q(V) - 4$. In other words $q(V) \leq \frac{1}{2}p_g(V) + 2$. We then have that

$$\chi(\mathscr{O}_V) = 1 - q(V) + p_g(V) \geq 1 - \left(\frac{1}{2}p_g(V) + 2\right) + p_g(V) = \frac{p_g(V)}{2} - 1.$$

Since V is nondegenerate, we have $p_g(V) \geq \binom{5}{2} = 10$ (in fact, equality holds by [Sch16, Corollary 15]), from which it follows that $\chi(\mathscr{O}_V) \geq 4$. $\square$

9. The Gauss map. Recall that the inclusion of V in A provides a Gauss map

$$\Gamma : V \longrightarrow \mathbf{G}(1,4), \quad s \mapsto [\mathbf{P}T_sV \subseteq \mathbf{P}T_0A],$$

where we see all the tangent spaces to points of V as being translated inside the tangent space of A at the origin. Since V is smooth and nondegenerate in A, the Gauss map is finite; see e.g. [Deb05, Proposition 8.12, Corollary 15, p.101].

Remark 9.1. In the cohomology ring of $\mathbf{G}(1,4)$ we have the identification

$$(9.2) \qquad \Gamma_*[V] = \left(c_1(V)^2 - c_2(V)\right) \cdot \sigma_2^\vee + c_2(V) \cdot \sigma_{1,1}^\vee,$$

where $(-)^\vee$ denotes the Poincaré dual; concretely, $\sigma_2^\vee = \sigma_{31}$, $\sigma_{1,1}^\vee = \sigma_{22}$. Indeed, by definition, if T is the tautological bundle on $\mathbf{G}(1,4)$, then $TV \simeq N \simeq \Gamma^*T$. It follows that the Segre class of TV satisfies

$$s(TV) = \Gamma^*(1 + \sigma_1 + \sigma_2 + \cdots) \in H^*(V, \mathbf{Z}).$$

In particular we have that $s_2(TV) = \Gamma^*\sigma_2$, and so by the projection formula we get

$$\Gamma_*[V] \cdot \sigma_2 = s_2(TV) = c_1(V)^2 - c_2(V).$$

This gives the first coefficient. The second can be obtained similarly, noting that $c_2(V) = \Gamma^*\sigma_{1,1}$.

Considering now the difference map $g : V \times V \to V - V \subseteq A$, blowing up 0 in $\Theta = V - V$ and Δ in $V \times V$, we obtain a surjective morphism on exceptional divisors

$$dg : \mathbf{P}N \longrightarrow \mathbf{P}C_0\Theta \subseteq \mathbf{P}T_0A,$$

which is generically finite, and induced by the differential of g. Here $C_0\Theta$ is the tangent cone to Θ at the origin. Considering dg on fibers $\mathbf{P}(N_v)$ of the projectivized normal bundle, there is an induced map $V \to \mathbf{G}(1,4)$, which agrees with the Gauss map. More precisely, under the identification of the bundles $N = N_{\Delta/V \times V} \simeq T_V$, the assignments

$$v \mapsto \mathbf{P}dg_v(N_v) \subseteq \mathbf{P}T_0A$$

$$v \mapsto \mathbf{P}T_vV \subseteq \mathbf{P}T_vA = \mathbf{P}T_0A$$

agree. It follows that if we denote by F be the Fano variety of lines in $\mathbf{P}C_0\Theta \subseteq \mathbf{P}^4$, then the Gauss map factors as

$$(9.3) \qquad \Gamma : V \longrightarrow F \hookrightarrow \mathbf{G}(1,4).$$

Moreover, the lines parameterized by V cover $\mathbf{P}C_0\Theta$.

10. The case of quadric tangent cones. In this section, in order to understand what would happen if

$$Q = \mathbf{P}C_0\Theta \subseteq \mathbf{P}^4$$

were a quadric hypersurface, we recall some elementary facts about Fano varieties of lines on quadric threefolds.

Proposition 10.1. *Let $Q \subseteq \mathbf{P}^4$ be a quadric threefold.*

  (1) If Q is singular, then every irreducible component of the Fano variety F of lines on Q is contained in some Schubert variety $\Sigma_1 \subset \mathbf{G}(1,4)$. In other words, there exists a plane $\mathbf{P}^2 \subset \mathbf{P}^4$ meeting each line parametrized by the respective component.

  (2) If Q is smooth, then the Fano variety F of lines on Q satisfies $F \simeq \mathbf{P}^3$. Moreover, if we denote by $\gamma : F \hookrightarrow \mathbf{G}(1,4)$ the inclusion map, and by H a hyperplane in $\mathbf{P}^3$, via this isomorphism we have

$$\gamma_*[H] = 2\sigma_{31} + 2\sigma_{22} \in H^4\big(\mathbf{G}(1,4), \mathbf{Z}\big).$$

Proof. Everything besides the last calculation is folklore; we sketch the proofs for completeness, and direct the reader to [Har95, Ch. 22] for some of the standard results on Fano varieties of linear spaces on quadrics.

(1) If Q is a rank four quadric, then it is a cone over a smooth quadric surface. The Fano variety of lines then consists of two irreducible components F' and F'', one for each family of lines on the smooth quadric. These components meet along the family of lines through the vertex. It is clear from this description that any line on Q corresponding to a point in F' meets the plane spanned by the vertex of Q and a line on the smooth quadric in the "opposite" ruling, i.e. the ruling defining F''.

If Q is a rank three quadric, then it is a cone over a singular irreducible quadric surface. Every line on Q meets the line through the vertex and the singular point of the quadric surface, and in particular any plane containing it.

If Q is the union of hyperplanes, then the conclusion is obvious.

(2) Let Q be a smooth quadric threefold with Fano variety of lines F. We start by recalling a standard construction of the isomorphism $F \cong \mathbf{P}^3$ (see e.g. [Har95, Exercise 22.6, p.290]). Let V be a four dimensional $\mathbf{C}$-vector space, and let $\Omega : V \times V \to \mathbf{C}$ be a non-degenerate skew symmetric bilinear form. Then the set

$$Q' = \{\Lambda \in G(2,V) : \Omega(\Lambda, \Lambda) = 0\}$$

is a smooth hyperplane section of the Plücker embedding of $G(2,V)$ in $\mathbf{P}^5$. As Q' is a smooth quadric hypersurface, we can take $Q' = Q$. The identification of $\mathbf{P}V$ with F is made as follows: for $[v] \in \mathbf{P}V$, the set

$$\ell_{[v]} = \{\Lambda \in G(2,V) : \Lambda \supseteq \langle v \rangle, \ \Omega(\Lambda, \Lambda) = 0\} \subseteq Q$$

is a line in Q. Conversely, given any line ℓ on Q, there is a unique $[v] \in \mathbf{P}V$ so that the ray $\langle v \rangle$ is contained in every 2-plane corresponding to a point of ℓ.

Now using Ω, there is an identification $\mathbf{P}V = \mathbf{P}V^\vee$. Therefore, for $[v] \in \mathbf{P}V$, we have the associated hyperplane

$$H_{[v]} := \mathbf{P}\{v' \in V : \Omega(v', v) = 0\} \subseteq \mathbf{P}V.$$

Note that there is a bijection

$$H_{[v]} \longrightarrow \{\ell' \subseteq Q : \ell' \cap \ell_{[v]} \neq \emptyset\} \subset F, \quad [v'] \mapsto \ell_{[v']}.$$

Moving on to the computation of the class $\gamma_*[H]$ for a hyperplane $H \subseteq \mathbf{P}V$, for dimension reasons we have

$$\gamma_*[H] = a\sigma_{31} + b\sigma_{22},$$

where $a = \gamma_*[H] \cdot \sigma_2$ and $b = \gamma_*[H] \cdot \sigma_{11}$. We first claim that

$$\gamma_*[H] \cdot \sigma_{11} = 2.$$

Indeed, assume $H = H_{[v]}$ is the set of lines on the quadric meeting the line $\ell_{[v]}$. Recall that σ_{11} corresponds to lines in $\mathbf{P}^4$ contained in a given hyperplane. A general such hyperplane cuts Q in a smooth quadric surface, and cuts $\ell_{[v]}$ at a single point, which lies on Q. Therefore, we are asking for the set of lines on a smooth quadric, through a given point, which proves the claim. Next we claim that

$$\gamma_*[H] \cdot \sigma_2 = 2.$$

Assume again that $H = H_{[v]}$ is the set of lines on the quadric meeting $\ell_{[v]}$. Recall that σ_2 corresponds to lines in $\mathbf{P}^4$ meeting a given line ℓ. Take ℓ general, and consider the 3-plane $\langle \ell, \ell_{[v]} \rangle$. This cuts Q in a smooth quadric surface, that meets ℓ in two points. Therefore, we are asking for the set of lines on a smooth quadric, that meet a given line and pass through one of two general points. There are exactly two, one for each point. This proves the claim. □

11. Proof of Theorem 6.1. We can now put together all the ingredients discussed in the previous subsections.

11.1. (A, Θ) *is isomorphic to the intermediate Jacobian of a smooth cubic threefold.* Our aim will be to show that

(11.1) $$\mathrm{mult}_0\Theta = \deg \mathbf{P}C_0(\Theta) \geq 3.$$

We can then apply [CM09, Theorem 3], [CM08, Corollary 2.2.4] in order to deduce that (A, Θ) is either the intermediate Jacobian of a cubic threefold, or the Jacobian of a hyperelliptic curve C. In the latter case it follows from [Deb95, Theorem 5.1] that V must be a translate of $W_2(C)$ or $-W_2(C)$. However, in this case V is singular, which contradicts the hypothesis.

It remains then to prove (11.1). To this end, note first that since V is nondegenerate as it has minimal class, $\mathbf{P}C_0\Theta$ cannot be contained in a hyperplane in $\mathbf{P}^4$. Indeed, since the Gauss map Γ factors through the inclusion of the Fano variety of lines in $\mathbf{P}C_0\Theta$ (9.3), the image of the Gauss map would be contained in a Schubert cycle Σ_1, which would precisely mean that V was geometrically degenerate; see §1. This would be a contradiction.

Our task is therefore to show that $Q = \mathbf{P}C_0\Theta$ cannot be a quadric; assuming that it is, we will aim for a contradiction. If Q is a singular quadric, then by Proposition 10.1 every irreducible component of the Fano variety F is contained in some $\Sigma_1 \subset \mathbf{G}(1,4)$, again contradicting the nondegeneracy of V.

Suppose finally that Q is a smooth quadric. Since $\deg(g) = 6$ by Lemma 2.1, then according to Lemma 7.2 we have

(11.2) $$c_1(V)^2 - c_2(V) = \deg(g) \cdot \mathrm{mult}_0\Theta = 12.$$

At the same time, under the standard isomorphism $F \cong \mathbf{P}^3$ of the Fano variety of lines on a smooth quadric threefold with projective 3-space, and the factorization

of the Gauss map
$$V \longrightarrow F \hookrightarrow \mathbf{G}(1,4),$$
we must have that the push-forward of the class of $[V]$ to F is equal to $n[H]$ for some n, where H is a hyperplane in $\mathbf{P}^3$. Let $\gamma : F \hookrightarrow \mathbf{G}(1,4)$ be the inclusion. Since by Proposition 10.1 we have $\gamma_*[H] = 2\sigma_{31} + 2\sigma_{22}$, it follows that
$$\Gamma_*[V] = \gamma_* n[H] = 2n\sigma_{31} + 2n\sigma_{22}.$$
Therefore, in particular, $\sigma_{11} \cdot \Gamma_*[V] = \sigma_2 \cdot \Gamma_*[V]$. Combining (9.2) and (11.2), we have that $\sigma_2 \cdot \Gamma_*[V] = c_1(V)^2 - c_2(V) = 12$. From (9.2) we therefore also have $c_2(V) = \sigma_{11} \cdot \Gamma_*[V] = 12$. We conclude that $c_1(V)^2 = 24$. Noether's formula then implies $\chi(\mathscr{O}_V) = 3$, contradicting Proposition 8.1.

11.2. *V is isomorphic to the Fano surface.* We have established that there is an isomorphism $(A, \Theta) \cong (JY, \Theta_Y)$ for some cubic threefold Y. Let F be the Fano surface of lines on Y; we now wish to show that $V \cong F$.

We start by recalling a few facts about F. First, from Clemens–Griffiths [CG72] we have that $JY \cong \mathrm{Alb}(F)$, the Albanese embedding of F has class $[F] = \theta^3/3!$, and $F - F = \Theta$. In other words, $F \subseteq A$ also satisfies the hypotheses of the theorem.

From the (degree 6) difference map $g_V : V \times V \to \Theta_Y$, and the fact established by Mumford [Mum74] that $\mathbf{P}C_0 \Theta_Y \cong Y$, we obtain that the Gauss map Γ_V factors (9.3) as
$$\Gamma_V : V \longrightarrow F \overset{i}{\hookrightarrow} \mathbf{G}(1,4).$$
Since, as V is smooth and nondegenerate, we have seen earlier in §9 that Γ_V is finite, it follows that the image of V is F. Thus, in $H^4(\mathbf{G}(1,4), \mathbf{Z})$ we have $\Gamma_{V*}[V] = n_V i_*[F]$ for some number n_V, so that
$$\sigma_2 \cdot \Gamma_{V*}[V] = n_V \sigma_2 \cdot i_*[F].$$
In the special case that $V = F$, we know that the degree of the Gauss map is 1 (i.e, $n_F = 1$). In particular,
$$\sigma_2 \cdot i_*[F] = \sigma_2 \cdot \Gamma_{F*}[F] = c_1(F)^2 - c_2(F) = \deg(g_F) \cdot \mathrm{mult}_0 \Theta_Y = 6 \cdot 3.$$
Using Remark 9.1, the same computation shows that $\sigma_2 \cdot \Gamma_{V*}[V] = 18$, as well. Therefore $n_V = 1$, and $\Gamma_*[V] = i_*[F]$. In particular, the degree of the finite Gauss map $\Gamma : V \to F$ is 1, and so $V \cong F$ since F is smooth.

11.3. *V can be identified with a translate of the Fano surface.* Here we show that there is an isomorphism $(A, \Theta) \cong (JY, \Theta_Y)$ which induces the isomorphism between V and F constructed in §11.2 above. This will finish the proof of Theorem 6.1.

Consider the inclusion of $V \hookrightarrow A$. From the universal property of the Albanese, we obtain a commutative diagram

$$
\begin{array}{ccc}
V & \lhook\joinrel\longrightarrow & A \\
{\scriptstyle \cong}\big\downarrow & {\scriptstyle \xi}\nearrow & \\
F & & \\
{\scriptstyle \mathrm{Alb}_F}\big\uparrow & & \\
\mathrm{Alb}(F) & = & JY
\end{array}
$$

In other words, since JY is isomorphic to the Albanese variety of $V \cong F$ [CG72], there is a morphism $\xi : JY \to A$, whose restriction to $F \subset JY$ yields an isomorphism

$$\xi|_F : F \xrightarrow{\sim} V.$$

Since V generates A, ξ is surjective, and hence an isogeny for dimension reasons. Our goal is to show that the isogeny ξ is an isomorphism.

To this end, observe that

$$\xi(\Theta_Y) = \xi(F - F) = V - V = \Theta.$$

In fact, $\xi|_{\Theta_Y} : \Theta_Y \to \Theta$ is generically finite of degree one, because $\xi|_F$ is an isomorphism and the difference maps $V \times V \to V - V$ and $F \times F \to F - F$ both have degree 6. It follows that $\xi|_{\Theta_Y} : \Theta_Y \to \Theta$ is birational. Since JY is the Albanese variety of Θ_Y and A is the Albanese variety of Θ, we obtain an induced morphism

$$\mathrm{alb}(\xi|_{\Theta_Y}) : JY \longrightarrow A$$

which coincides with ξ, since its restriction to Θ_Y has that property. On the other hand, $\mathrm{alb}(\xi|_{\Theta_Y})$ is an isomorphism, because $\xi|_{\Theta_Y}$ is birational. We conclude that

$$\xi : (JY, \Theta_Y) \longrightarrow (A, \Theta)$$

is an isomorphism of ppavs which identifies F with V.

Remark 11.3. If in the proof of Theorem 6.1 one assumes additionally that $\mathcal{I}_V(\Theta)$ is a GV-sheaf, then the conclusion of §11.3 follows immediately from the paper [LT15] of Lombardi and Tirabassi, where it is shown that the inclusion of a GV-subvariety does not factor through any nontrivial isogeny.

F. Reduction to the known cases

12. The purpose of this section is to prove the following result, Theorem 12.1, which reduces Theorem B to Theorem 4.1, Theorem 5.1 and Theorem 6.1 above.

Theorem 12.1. *Let (A, Θ) be an indecomposable ppav of dimension 5, and let $V, W \subset A$ be smooth surfaces such that $[V] = [W] = \frac{\theta^3}{3!}$ and $V + W = \Theta$. Let*

$$g : V \times V \longrightarrow V - V \subset A$$

denote the difference map. Then $\deg(g) \in \{1, 6\}$, and moreover,

  (1) *if $\deg(g) = 1$, then $V + V$ is a translate of Θ.*
  (2) *if $\deg(g) = 6$, then $V - V$ is a translate of Θ.*

The rest of the section is devoted to the proof of this theorem; we note that the case where V is a translate of $-W$ is immediate by Lemma 2.1, and that we can use Theorem 5.1 to deal with the cases where V is a translate of W.

13. We begin by recalling that by Lemma 2.1,

$$\deg(g) \in \{1, 2, 3, 6\}.$$

If $\deg(g) = 6$, then $[V - V] = [\Theta]$ and so that statement of Theorem 12.1 is immediate. The other cases are much more difficult, and will be treated in the following subsections respectively. Our notation will always be that of Theorem 12.1.

Before we turn to our case by case study, let us first prove the following general lemma, which we will use repeatedly.

Lemma 13.1. *Under the assumptions of Theorem 12.1, we have $V - V = W - W$.*

Proof. If (A, Θ) is isomorphic to the Jacobian of a smooth curve C, then V and W are translates of $\pm W_2(C)$ by [Deb95], and so the lemma holds. We may therefore assume in what follows that (A, Θ) is not isomorphic to a Jacobian.

Since V and W have minimal class, the addition morphism $f : V \times W \longrightarrow \Theta$ has degree 6. For a general point $v \in V$, the reduced preimage of $v + W$ decomposes as

$$f^{-1}(v + W)^{\mathrm{red}} = (v \times W) \cup R \cup Q,$$

where $f(Q) \subsetneq v + W$ and each component of R dominates $v + W$ via f. For general $w \in W$, $f^{-1}(v + w)$ is a union of 6 distinct points and so $f|_R : R \longrightarrow v + W$ has degree 5. In particular, R is nonempty. By Theorem 3.3, V has property $(\mathcal{P})$ with respect to W, which means that $\mathrm{pr}_2(R) \subsetneq W$. If R contains a component R' such that $\mathrm{pr}_2(R')$ is a point, then $R' = V \times w$. Applying f shows that V is a translate of W. It then follows from Theorem 5.1 that (A, Θ) is isomorphic to the Jacobian of a smooth curve, which contradicts our assumption.

We may therefore assume that there is a component R' of R whose image under the second projection is a curve

$$\mathrm{pr}_2(R') = C_v \subset W.$$

If $\mathrm{pr}_1(R')$ is also a curve, then W (and hence Θ) has a curve summand and so (A, Θ) is the Jacobian of a smooth curve by [Sch14]. We may thus assume that $\mathrm{pr}_1(R') = V$. This means that for general $v_2 \in V$ there are points $w_1, w_2 \in W$ with

$$v_2 + w_2 = v + w_1.$$

In particular $v - v_2 \in W - W$. Since we can make this argument for each general v, and v_2 was chosen generally, the containment holds for general $(v, v_2) \in V \times V$, and it follows that $V - V \subseteq W - W$. The lemma follows now from the irreducibility, reducedness and non-degeneracy of V and W, which implies that $V - V$ and $W - W$ are both irreducible, reduced and of the same dimension; for the latter, see for instance Lemma 2.1. □

14. The case $\deg(g) = 1$**.** Our goal will be to show that either V has a curve summand, or that $\pm V$ is a translate of W. In the latter case, if V is a translate of W, we can use Theorem 5.1 to conclude, and if V is a translate of $-W$, the assertion is immediate by Lemma 2.1. In the former case, if we succeed in showing that V has a curve summand, then it will follow that Θ has a curve summand and so (A, Θ) is isomorphic to the Jacobian of a smooth curve C by [Sch14]. Since V has minimal class, one can by [Deb95] find an isomorphism $(A, \Theta) \cong (JC, \Theta_C)$ that identifies V with $W_2(C)$. In particular, $V + V = \Theta$ because $W_2(C) + W_2(C) = \Theta_C$.

Therefore, we now proceed to show that either V has a curve summand, or that $\pm V$ is a translate of W. If $\deg(g) = 1$, then g is birational. Consequently, for general $w \in W$, the reduced preimage

$$g^{-1}(W - w)^{\mathrm{red}} \subseteq V \times V$$

contains a component R_w that is birational to W. For $i = 1, 2$, this gives rise to a rational map

$$\varphi_{i,w} : W \overset{\sim}{\dashrightarrow} R_w \xrightarrow{\mathrm{pr}_i} V,$$

which depends on $w \in W$.

We want to study the maps $\varphi_{i,w}$ as w varies. To this end, for $i = 1, 2$ consider the diagram

$$V \times V \xleftarrow{g^{-1}} V - V = W - W \longleftarrow W \times W$$

$$\mathrm{pr}_i \downarrow \qquad\qquad\qquad \Phi_i$$

$$V \longleftarrow$$

where Φ_i is defined to be the rational map obtained via composition. Since W is smooth and $V \subseteq A$, we have that Φ_i is in fact a morphism. Moreover, Φ_i, viewed as a morphism to A, factors through the Albanese $\mathrm{Alb}(W) \times \mathrm{Alb}(W)$. Since, after translation, the induced map $\mathrm{Alb}(W) \times \mathrm{Alb}(W) \to A$ decomposes as a sum of homomorphisms, we may write

$$\Phi_i = \varphi_i + c_i : W \times W \longrightarrow V$$

with $\varphi_i, c_i : W \to A$. In particular,

$$(14.1) \qquad\qquad V = \mathrm{im}(\varphi_i) + \mathrm{im}(c_i)$$

for $i = 1, 2$. By construction, for general $w \in W$ we have $\varphi_{i,w}(w') = \varphi_i(w') + c_i(w)$ for all $w' \in W$.

The first claim is that either $\mathrm{im}(\varphi_1)$ or $\mathrm{im}(\varphi_2)$ has dimension ≤ 1. Indeed, for general $w \in W$,

$$(14.2) \qquad\qquad \varphi_i(W) + c_i(w) = \Phi_i(W, w) = \mathrm{pr}_i(R_w).$$

By Theorem 3.3, W has property $(\mathcal{P})$ with respect to V, and this implies by Lemma 3.2 that the only subvariety of $g^{-1}(W - w)$ that dominates both factors of $V \times V$ is the diagonal. In particular, R_w does not dominate both factors, so that by (14.2) one of the images $\varphi_i(W)$ must not be a surface, establishing the claim.

Now we are ready to conclude. If $\mathrm{im}(\varphi_1)$ or $\mathrm{im}(\varphi_2)$ is a curve, then V has a curve summand by (14.1). If $\mathrm{im}(\varphi_1)$ or $\mathrm{im}(\varphi_2)$ is a point, then by the construction of φ_i, we have that $g^{-1}(W - w)^{\mathrm{red}}$ is of the form $v \times V$ or $V \times v$ for some point $v \in V$. Applying g to this shows that $\pm V$ is a translate of W. This completes the proof of Theorem 12.1 in the case where $\deg(g) = 1$.

15. Ruling out $\deg(g) = 2$. In this section we will derive a contradiction from the assumption that $\deg(g) = 2$. The proof is somewhat lengthy, and we break it up into several parts.

15.1. *Some preliminary observations.* We start by explaining why we may assume:

- (A, Θ) is not isomorphic to the Jacobian of a smooth curve.
- V and W do not admit curve summands.
- V is not a translate of $\pm W$.

Indeed, if (A, Θ) were isomorphic to the Jacobian of a smooth curve C, then C would have to be nonhyperelliptic since by [Deb95], $V = W_2(C)$ is assumed to be smooth, and then the difference map $g : W_2(C) \times W_2(C) \to W_2(C) - W_2(C)$ would have degree 1. Consequently, since (A, Θ) can be assumed not to be a Jacobian, and $\Theta = V + W$, by [Sch14] we may further assume that V and W have no curve summand. Finally, as mentioned previously, we may by Lemma 2.1 and Theorem 5.1 assume that V is not a translate of $\pm W$.

Going forward, the main observation is the following. For general $v \in V$,

$$g^{-1}(v - V)^{\text{red}} = (v \times V) \cup R_v \cup Q,$$

where $g(Q) \subsetneq v - V$ and each component of R_v dominates $v - V$ via g.[1] Since $g^{-1}(v - v_2)$ is given by two distinct points if $v_2 \in V$ is general, R_v is irreducible and birational to V. This allows us to consider the rational map

(15.1) $$\psi_{i,v} : V \dashrightarrow^{\sim} R_v \xrightarrow{\text{pr}_i} V.$$

15.2. *The global description of the $\psi_{i,v}$.* We want to study the maps $\psi_{i,v}$ as v varries. We will use this to arrive at the main goal of this subsection, the description of V as the image of the sum of maps in (15.2), below.

To this end, consider the map

$$(g \times 1_V) : V \times V \times V \longrightarrow (V - V) \times V$$

$$(v_1, v_2, v) \mapsto (v_1 - v_2, v)$$

and let $\mathcal{V} \subseteq (V - V) \times V$ be given by the image $(g \times 1_V)(\text{pr}_{13}^{-1}(\Delta_V))$, where Δ_V denotes the diagonal in $V \times V$. The fiber of $\mathcal{V}$ over a point $v \in V$ via the last projection is given by $\mathcal{V}_v^{\text{red}} = (v - V) \times v$. In other words, $\mathcal{V}$ is the family of $v - V$.

The points of $\mathcal{G} := (g \times 1_V)^{-1}(\mathcal{V})$ can be described set theoretically as

$$\mathcal{G} = (g \times 1_V)^{-1}(\mathcal{V}) = \{(v_1, v_2, v) \in V \times V \times V : v_1 - v_2 \in v - V\}.$$

By construction, the fiber of $\mathcal{G}$ over a point $v \in V$ via the last projection is given by $\mathcal{G}_v^{\text{red}} = g^{-1}(v - V)^{\text{red}} \times v$. In other words, $\mathcal{G}$ is the family of $g^{-1}(v - V)^{\text{red}}$.

Clearly $\text{pr}_{13}^{-1}(\Delta_V)$ is one component of $\mathcal{G}$. We also know that $\mathcal{G} \to \mathcal{V}$ has degree two generically and so there is a unique second component

$$\mathcal{R} \subseteq \mathcal{G} \subseteq V \times V \times V$$

with the property that $g \times 1_V$ restricted to $\mathcal{R}$ yields a dominant map $\mathcal{R} \to \mathcal{V}$. By construction, the fiber of $\mathcal{R}$ over a point $v \in V$ via the last projection satisfies

$$(\mathcal{R}_v)^{\text{red}} = R_v \times v;$$

i.e., $\mathcal{R}$ is the family of R_v.

We next claim that $\mathcal{R}$ is birational to $V \times V$. Indeed, consider the map over V

$$F : \mathcal{R} \longrightarrow A \times V,$$

$$F(v_1, v_2, v) = (v - (v_1 - v_2), v).$$

From the definition of $\mathcal{G}$, the image of this map lies in $V \times V$. Restricted to general fibers, this map F is the birational map between $R_v \times v$ and $V \times v$ mentioned above, and therefore F is generically one-to-one, and so $\mathcal{R}$ is birational onto $V \times V$.

Finally, for $i = 1, 2$, define the map Ψ_i as the composition

$$\Psi_i : V \times V \xrightarrow{F^{-1}} \mathcal{R} \xrightarrow{\text{pr}_i} V$$

where pr_i is the projection onto the i-th factor.

As in §14, we may use the Albanese to conclude that

$$\Psi_i = \psi_i + d_i : V \times V \longrightarrow V$$

_____

[1] For instance, the diagonal Δ_V is contained in Q.

with $\psi_i, d_i : V \to A$. In particular,

$$(15.2) \qquad V = \operatorname{im}(\psi_i) + \operatorname{im}(d_i)$$

for $i = 1, 2$. By construction, for general $v \in V$ we have $\psi_{i,v}(v') = \psi_i(v') + d_i(v)$ for all $v' \in V$.

15.3. *Reducing to the case $V = -V$*. The purpose of this subsection is to use (15.2) to reduce to the case

$$(15.3) \qquad V = -V.$$

The first claim in this direction is that either $\operatorname{im}(\psi_1)$ or $\operatorname{im}(\psi_2)$ has dimension ≤ 1. Indeed, since V is geometrically nondegenerate, we know that the addition map of V with any subvariety has the expected dimension. Consequently, from (15.2), if $\psi_i(V)$ were a surface, and hence a translate of V, then d_i would have to be constant. If both the maps d_i were constant, then we would have that

$$R_v = \{(\psi_{v,1}(v'), \psi_{v,2}(v')) \mid v' \in V\} = \{(\psi_1(v') + d_1, \psi_2(v') + d_2) \mid v' \in V\}$$

would be independent of v. But this is a contradiction since $g(R_v) = v - V$, and the $v - V$ move (for instance since $V - V$ is of dimension 4 by the geometric nondegeneracy of V). This establishes the claim that either $\operatorname{im}(\psi_1)$ or $\operatorname{im}(\psi_2)$ has dimension ≤ 1.

Now, if the dimension of the image of ψ_i is one, then by (15.2), V would have a curve summand, which we are assuming is not the case. Therefore, one of the ψ_i is constant.

If ψ_1 is constant, then (as at the end of §14) $R_v = v' \times V$ for some $v' \neq v$ (here we take $v' \neq v$ since the case $v' = v$ corresponds to the component of $g^{-1}(v-V)^{\mathrm{red}}$ that we specifically left out of R_v in the definition). This cannot happen because $v - V = v' - V$ implies $v - V - W = v' - V - W$, and hence $v = v'$, since $-V - W$ is a translate of Θ. On the other hand, if ψ_2 is constant, then $R_v = V \times v'$. This implies $v - V = V + v'$ for some $v' \in V$. Up to replacing V by a translate, we may therefore assume $V = -V$.

15.4. *The case $V = -V$*. We have now reduced to the case $V = -V$. This subsection is devoted to arriving at a contradiction under this additional assumption.

To do this, we turn our attention to the structure of the $g^{-1}(W - w)$ for $w \in W$. By Theorem 3.3, W has property $(\mathcal{P})$ with respect to V. Moreover, by Lemma 13.1,

$$V - V = W - W.$$

By Lemma 3.2, it follows that for $w \in W$ general,

$$g^{-1}(W - w)^{\mathrm{red}} = R_w \cup Q_w$$

with R_w and Q_w satisfying: $g(Q_w) \subsetneq W - w$, every component of R_w dominates $W - w$ via g, and no component of R_w dominates both factors of $V \times V$. The restriction of g to R_w has degree 2. Moreover, R_w is anti-symmetric in the following sense: if we let $\sigma : V \times V \to V \times V$ be the map interchanging the factors, then since $V = -V$, we have that $R_w = -\sigma(R_w) \subseteq V \times V$.

The main claim, proven in Lemma 15.4 below, is that R_w is irreducible. We assume this for the moment, and complete the argument. We immediately obtain from irreducibility that R_w is not mapped via either projection surjectively onto V, since then by anti-symmetry, it would be mapped surjectively onto both factors,

which we have shown is not possible. Similarly, we obtain that R_w is not mapped via either projection to a point. Indeed, without loss of generality, if $\mathrm{pr}_1(R_w) = v$, then $R_w = v \times V$, and so by anti-symmetry we have $R_w = (V \times (-v)) \cup (v \times V)$, contradicting the irreducibility of R_w (and also the fact that we have ruled out the possibility that either projection of R_w maps surjectively to V).

So we may assume that $\mathrm{pr}_i(R_w) = C_i \subsetneq V$ is a curve, $i = 1, 2$. But then we obtain an isomorphism $(\mathrm{pr}_1 \times \mathrm{pr}_2) : R_w \to C_1 \times C_2 \subseteq V \times V$, and composing with g we would arrive at the conclusion that W has a curve summand, which contradicts our assumption. Therefore, up to showing that R_w is irreducible, we have succeeded in showing that $\deg(g) = 2$ is not possible.

Lemma 15.4. *R_w is irreducible.*

Proof. The basic idea is similar to the arguments we have been using. If R_w is reducible for general w, then the family $\mathscr{R}_W \to W$ of R_w has geometrically reducible generic fiber (with two geometric components). Therefore, after a generically finite (degree 1 or 2) base change by say $W' \to W$, we may assume that the base change $\mathscr{R}_{W'}$ of the family $\mathscr{R}_W$ has two components $\mathscr{R}_{W',j} \subseteq V \times V \times W'$, $j = 1, 2$. By construction each component is birational to $W \times W'$. We may thus consider for $i = 1, 2$ the induced rational maps

$$W \times W' \dashrightarrow \mathscr{R}_{W',j} \xrightarrow{\mathrm{pr}_i} V,$$

then using the universal property of the Albanese and the property $(\mathcal{P})$ one obtains that V has a curve summand, giving a contradiction.

The details are as follows. Let $w \in W$ be general, and let us assume that

$$R_w = R_{w,1} \cup R_{w,2}$$

is reducible. Since $g|_{R_w} : R_w \to W - w$ has degree two, $R_{w,j}$ is birational via g to $W - w$ for $j = 1, 2$. Since W has property $(\mathcal{P})$ with respect to V by Theorem 3.3, Lemma 3.2 implies that for some $i_j \in \{1, 2\}$,

$$\mathrm{pr}_{i_j} : R_{w,j} \longrightarrow V$$

is not surjective. By our assumptions W is not a translate of $\pm V$ (which would be the case if the projection were a point), and so the above projection has image a curve $C_{w,i_j,j} \subsetneq V$.

As usual, we are interested in the maps

$$\varphi_{w,i,j} : W \overset{\sim}{\dashrightarrow} R_{w,j} \xrightarrow{\mathrm{pr}_i} V$$

as w varies in W. The above discussion implies that for general $w \in W$, and $j = 1, 2$, then for some $i_j \in \{1, 2\}$ we have $\mathrm{im}(\varphi_{w,i_j,j}) = C_{w,i_j,j}$ is a curve.

Now, as in §15.2, we can construct

$$\mathscr{R}_W \subseteq V \times V \times W$$

with general fiber of the last projection given by $R_w \times w$, as well as a map

$$F : \mathscr{R}_W \longrightarrow A \times W$$

over W given by $F(v_1, v_2, w) = (w - (v_1 - v_2), w)$, which has image in $W \times W$, and is generically $2 : 1$. Since the general fiber R_w is reducible, the geometric generic fiber of $\mathscr{R}_W$ is reducible, and there is a smooth projective surface W' and a generically finite map $W' \to W$ so that the base change $\mathscr{R}_{W'} = \mathscr{R}_{W',1} \cup \mathscr{R}_{W',2}$ is reducible (in

fact, using the monodromy of the fibers, one can see that this can be taken to be degree 1 or degree 2). Via degree considerations on the fibers, the morphisms

$$F_j : \mathscr{R}_{W',j} \longrightarrow W \times W'$$

obtained from F via base change, are birational.

Now, for $i, j = 1, 2$, define the map $\Phi_{i,j}$ as the composition

$$\Phi_{i,j} : W \times W' \xdashrightarrow{F_j^{-1}} \mathscr{R}_{W',j} \xrightarrow{\mathrm{pr}_i} V$$

where pr_i is the projection onto the i-th factor. As in §14, we may use the Albanese to conclude that

$$\Phi_{i,j} = \varphi_{i,j} + c_{i,j} : W \times W' \longrightarrow V$$

with $\varphi_{i,j} : W \to A$ and $c_{i,j} : W' \to A$. In particular,

$$(15.5) \qquad\qquad V = \mathrm{im}(\varphi_{i,j}) + \mathrm{im}(c_{i,j})$$

for $i, j = 1, 2$. By construction, for general $w \in W$ we have $\varphi_{w,i,j}(\tilde{w}) = \varphi_{i,j}(\tilde{w}) + c_{i,j}(w')$ for all $\tilde{w} \in W$, where $w' \in W'$ maps to w under $W' \to W$. Since we have shown that for $j = 1, 2$ there is some $i_j \in \{1, 2\}$ so that $\mathrm{im}(\varphi_{w,i_j,j}) = C_{w,i_j,j}$ is a curve, (15.5) establishes that V has a curve summand, providing the desired contradiction. $\qquad\square$

16. Ruling out $\deg(g) = 3$. In this section we will derive a contradiction from the assumption that $\deg(g) = 3$. By the same argument as at the beginning of §15, $\deg(g) = 3$ implies (A, Θ) is not isomorphic to the Jacobian of a smooth curve, and that V does not admit a curve summand.

We can further assume that V is not a translate of $-V$. Indeed, if V were a translate of $-V$, then up to translation we may assume $V = -V$. But then the difference map g would factor through the involution

$$V \times V \longrightarrow V \times V, \quad (v_1, v_2) \mapsto (-v_2, -v_1),$$

and so g could not have degree 3.

In order to present the main argument in the case $\deg(g) = 3$, we introduce some notation. For general $(v_1, v_2) \in V \times V$, the preimage $g^{-1}(v_1 - v_2)$ consists of three points, which we denote by

$$g^{-1}(v_1 - v_2) = \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}.$$

That is,

$$v_1 - v_2 = v_3 - v_4 = v_5 - v_6.$$

Also, for a general point $v_1 \in V$ we will use the notation

$$g^{-1}(v_1 - V)^{\mathrm{red}} = (v_1 \times V) \cup R_{v_1} \cup Q$$

with $g(Q) \subsetneq v_1 - V$, and where each component of R_{v_1} dominates $v_1 - V$ via g.

Lemma 16.1. *In the above notation,*

(1) *If we fix a general point $v_1 \in V$ and move $v_2 \in V$, then the points v_4 and v_6 stay in an irreducible curve $C(v_1) \subseteq V$, whereas v_3 and v_5 sweep out V; i.e., $\mathrm{pr}_2(R_{v_1}) = C(v_1)$ and $\mathrm{pr}_1(R_{v_1}) = V$.*

(2) *For general v_1, we have that R_{v_1} is irreducible.*

(3) *There is a dense open susbset $U \subseteq V$ such that $\overline{\bigcup_{v_1 \in U} C(v_1)} = V$.*

Proof. We fix a general point $v_1 \in V$ and use the notation introduced above. Since for general $v_2 \in V$, the preimage $g^{-1}(v_1 - v_2)$ is given by three distinct points, the restriction

$$g|_{R_{v_1}} : R_{v_1} \longrightarrow v_1 - V$$

has degree two. In particular, R_{v_1} is nonempty.

Let now $R'_{v_1} \subseteq R_{v_1}$ be an irreducible component. Since V is not a translate of $-V$, $\mathrm{pr}_2(R'_{v_1})$ is not a point. On the other hand, suppose that $\mathrm{pr}_2(R'_{v_1})$ is equal to V. Then for general $v_2 \in V$ with $v_2 \neq v_1$, there are $v_3, v_4 \in V$ with $v_1 - v_2 = v_3 - v_4$ and $v_1 \neq v_3$, such that v_4 is a general point of V. Hence,

$$v_1 + v_4 = v_2 + v_3,$$

with $[v_1, v_4] \neq [v_2, v_3]$, and where $v_1 + v_4$ is a general point of $V + V$. Since $V \times V \longrightarrow V + V$ factors through $V^{(2)}$, it has by Lemma 2.1 either degree 2 or 6 and so the above observation shows that it has degree 6. Thus $V + V$ is a translate of Θ, which contradicts $\deg(g) = 3$ by Theorem 5.1.

We have thus shown that $\mathrm{pr}_2(R'_{v_1})$ is a curve for each irreducible component R'_{v_1} of R_{v_1}. In the next step, we aim to prove that $\mathrm{pr}_2(R_{v_1})$ is irreducible. For this it suffices to see that R_{v_1} is irreducible. If it is not, then as $g|_{R_{v_1}} : R_{v_1} \longrightarrow v_1 - V$ has degree two, each component of R_{v_1} is birational to V. Since we know that for each component R'_{v_1} of R_{v_1}, the projection $\mathrm{pr}_2(R'_{v_1})$ is an irreducible curve, the same argument as in the proof of Lemma 15.4 then shows that V has a curve summand, which contradicts our assumptions.

It remains to see that v_3 (resp. v_5) sweeps out V as v_2 moves. If this were not true, $\mathrm{pr}_i(R_{v_1})$ would be a curve not only for $i = 2$ but also for $i = 1$. Since we already know that R_{v_1} is irreducible, it would follow that the natural inclusion $R_{v_1} \subseteq \mathrm{pr}_1(R_{v_1}) \times \mathrm{pr}_2(R_{v_1})$ is an equality so that R_{v_1} is a product of curves. Applying g then shows that V has a curve summand, which contradicts our assumptions. This proves (1) of the lemma.

We have already shown (2), and (3) follows directly from the definition of $C(v_1)$, as for general $v \in V$, one can find v_1 with $v \in C(v_1)$. □

For any general point $v_1 \in V$, there is by Lemma 16.1 a rational map

$$\phi_{v_1} : V \dashrightarrow C(v_1)^{(2)}, \qquad v_2 \mapsto [v_4, v_6].$$

Lemma 16.2. *For general $v_1 \in V$, the rational map ϕ_{v_1} is dominant.*

Proof. We denote by $\mathrm{im}(\phi_{v_1})$ the (closure of the) image of ϕ_{v_1}. That is,

$$\mathrm{im}(\phi_{v_1}) = \overline{\phi_{v_1}(U)} \subseteq C(v_1)^{(2)}$$

for some Zariski open and dense subset $U \subset V$ on which ϕ_{v_1} is defined.

Let us first prove that $\mathrm{im}(\phi_{v_1})$ is not a point. That is, we want to prove that for $v_2 \in V$ general, $\phi_{v_1}(v_2) = [v_4, v_6]$ is not constant as v_2 moves. If it were, then v_4 and v_6 would not move, although v_2 does. This would imply that R_{v_1} is reducible (and also that V is a translate of $-V$), which contradicts our assumptions.

It remains to prove that $\mathrm{im}(\phi_{v_1})$ is not a curve. For a contradiction, we assume that $\mathrm{im}(\phi_{v_1})$ is a curve for general $v_1 \in V$.

For a general point $(v_1, v_2) \in V \times V$, let us again look at

$$(16.3) \qquad g^{-1}(v_1 - v_2) = \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}.$$

Since $\operatorname{im}(\phi_{v_1}) \subset C(v_1)^{(2)}$ is an irreducible curve, it follows that the intersection

$$[v_4 \times C(v_1)] \cap \operatorname{im}(\phi_{v_1})$$

is given by a finite set of points. Indeed, if this was not the case, then $\operatorname{im}(\phi_{v_1}) = [v_4 \times C(v_1)]$ which implies that if we fix v_1 and move v_2, then v_4 does not move and so V is a translate of $-V$, which contradicts our assumptions. We have thus proven that the above intersection is finite, which means that v_6 is determined by v_1 and v_4 up to finite ambiguity.

On the other hand, $v_1 - v_2 = v_3 - v_4$ is a general point of $V - V$. Therefore, $v_4 - v_3 \in V - V$ is a general point and so we can compute its preimage via g as

$$(16.4) \qquad g^{-1}(v_4 - v_3) = \{(v_4, v_3), (v_2, v_1), (v_6, v_5)\}.$$

Since $v_4 - v_3$ is a general point of $V - V$, it follows that v_4 is a general point of V.[2] Let us now fix the general point v_4 and move $v_3 \in V$. It then follows from Lemma 16.1 that v_1, v_5 stay in the irreducible curve $C(v_4)$, whereas v_2 and v_6 sweep out V.

Note that locally analytically, the labeling of points in (16.3) and (16.4) is consistent, and the discussion above shows that fixing v_4, then for small changes in v_3, the point v_1 moves in a one-dimensional family along the curve $C(v_4)$ while v_6 moves in a two-dimensional family. This is a contradiction, because we have seen earlier that for fixed v_4, the point v_6 is determined up to finite ambiguity by v_1 and so, for fixed v_4, the locus of all possible v_6's is one-dimensional, since $v_1 \in C(v_4)$ can only move along a curve. This finishes the proof of the lemma. $\qquad \square$

Let $\widetilde{C}(v_1)$ denote the normalization of $C(v_1)$. By Lemma 16.2, ϕ_{v_1} induces a dominant rational map

$$V \dashrightarrow \widetilde{C}(v_1)^{(2)}.$$

Since H^1 is a birational invariant of smooth varieties, this rational map induces an injective morphism of integral Hodge structures

$$H^1(\widetilde{C}(v_1), \mathbf{Z}) = H^1(\widetilde{C}(v_1)^{(2)}, \mathbf{Z}) \longrightarrow H^1(V, \mathbf{Z}),$$

which induces an inclusion of the family of Jacobians $J(\widetilde{C}(v_1))$ in the fixed abelian variety $\operatorname{Pic}^0(V)$. This implies that the abelian varieties $J(\widetilde{C}(v_1))$ are all isomorphic, as v_1 moves; since polarizations of a fixed abelian variety are discrete, we can then use the Torelli theorem to conclude that the family of curves $(\widetilde{C}(v_1))_{v_1 \in U}$, where v_1 runs through the points of some Zariski open and dense subset $U \subseteq V$, is isotrivial. Let us denote this isotrivial family of curves $\mathscr{C} \to U$, and fix $\widetilde{C} \cong \widetilde{C}(v_1)$ for some (any) $v_1 \in U$.

There exists a generically finite morphism $V' \to V$ such that the pullback of $\mathscr{C} \to U$ to some dense open subset $U' \subseteq V'$ is trivial. In other words, $\mathscr{C} \times_U U' \cong U' \times \widetilde{C}$. We restrict the trivial family $V' \times C'$ to a general curve $B \subseteq V'$ to obtain

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[2]The conclusion that for general $(v_1, v_2) \in V \times V$, the point v_4 is a general point of V is equivalent to saying that v_4 sweeps out V as we move (v_1, v_2) in $V \times V$. This should not be confused with Lemma 16.1, where we have seen that v_4 stays in a curve if we fix v_1 and move v_2.

the family $B \times \widetilde{C} \to B$, and then consider the rational map $B \times \widetilde{C} \dashrightarrow V$ which is fiberwise over $v_1 \in B \cap U'$ induced by

$$\widetilde{C} \cong \widetilde{C}(v_1) \longrightarrow C(v_1) \subseteq V.$$

As B is general, $\bigcup_{v_1 \in B \cap U'} C(v_1)$ is dense in V by Lemma 16.1. Consequently, $B \times \widetilde{C} \dashrightarrow V$ is a dominant rational map from a product of curves to V, and so V has a curve summand; see for instance [Sch14, Corollary 22]. This contradicts our assumptions, which finishes the proof of Theorem 12.1, and hence also the proof of Theorem B.

Remark 16.5. The smoothness assumption on V and W in Theorem 12.1 is not necessary. This follows from Theorem 4.1, but also from the proof of Theorem 12.1 given above, which can easily be adapted to the case where V and W are allowed to be singular.

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