

\mathcal{D} -modules in birational geometry

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Introduction

These are notes written for a course I taught at Northwestern during the Winter and Spring quarters of 2020, as well as for a course I am currently (Spring semester 2021) teaching at Harvard. They cover various topics on \mathcal{D} -modules, Hodge modules, and especially their connections to birational geometry. At the moment they are limited to the material I was able to cover during those two quarters. My intention is however to gradually add more material, especially on various properties and applications of Hodge modules.

Among the **topics to be added**: vanishing and positivity package for Hodge modules; applications of Hodge modules to generic vanishing, holomorphic forms, families of varieties; Hodge ideals and applications; Hodge filtration on local cohomology.

What these notes are *not* meant to be is a careful introduction to \mathcal{D} -modules, or to mixed Hodge modules (or to birational geometry for that matter). For general \mathcal{D} -module theory there are several excellent references (as well as lots of online lecture notes). Among these, the main reference that covers essentially all the basic facts needed here is [HTT]. Nevertheless as these notes develop I will include more precise references, and also an introductory chapter with some background on \mathcal{D} -modules, as well as on constructions in birational geometry.

As for the theory of mixed Hodge modules, the fundamental references are Saito's papers [Sa1] and [Sa2]. There are also his more recent improvement [Sa7] and survey [Sa8]. Schnell's overview of the theory [Sch] is an excellent quick introduction to the topic, while the developing MHM project [MHM] of Sabbah and Schnell is meant to gradually become a comprehensive reference.

CHAPTER 1

Background on filtered \mathcal{D} -modules

This chapter contains a very brief review of basic definitions and facts from the theory of \mathcal{D} -modules. The main source for this material, including proofs and further details, is the book [HTT]. For more specialized facts regarding the theory of filtered \mathcal{D} -modules, the main source is [Sa1].

1.1. Generalities on \mathcal{D} -modules

We consider a smooth complex variety X of dimension n . We denote by \mathcal{D}_X the sheaf of differential operators on X ; this is the sheaf of \mathbf{C} -subalgebras of $\mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$ generated by \mathcal{O}_X and T_X , where \mathcal{O}_X acts by multiplication by functions, and we think of the tangent sheaf T_X as being the sheaf of derivations $\mathcal{D}er_{\mathbf{C}}(\mathcal{O}_X) \subset \mathcal{E}nd_{\mathbf{C}}(\mathcal{O}_X)$.

Locally in algebraic coordinates x_1, \dots, x_n on an affine neighborhood U around a point $x \in X$, it can be described as follows. Consider the corresponding local basis $\partial_1, \dots, \partial_n$ of vector fields, dual to dx_1, \dots, dx_n . They satisfy $[\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{i,j}$ for all i and j . If we denote $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, then we have

$$\mathcal{D}_X(U) = \bigoplus_{\alpha \in \mathbf{N}^n} \mathcal{O}_X(U) \partial^\alpha.$$

Therefore locally we will write sections of \mathcal{D}_X as finite sums $P = \sum_{\alpha} a_{\alpha} \partial^{\alpha}$, with $a_{\alpha} \in \mathcal{O}_X(U)$. The *order* of P is the maximal $|\alpha| = \alpha_1 + \cdots + \alpha_n$ appearing in this sum.

REMARK 1.1.1 (Affine space). When $X = \mathbf{A}^n$, we can take x_1, \dots, x_n to be global coordinates, and we have $\mathcal{D}_X(X) = A_n$, the n -th *Weyl algebra*. In this case the theory is essentially equivalent to the theory of modules over A_n ; an excellent introduction, including some special features in this case, can be found in [Co].

DEFINITION 1.1.2. The *order filtration* on \mathcal{D}_X is defined (locally) by

$$F_k \mathcal{D}_X := \left\{ \sum_{\alpha} a_{\alpha} \partial^{\alpha} \mid |\alpha| \leq k \right\}.$$

It can be easily checked that this induces a global filtration on X , and $F_k \mathcal{D}_X$ is called the sheaf of differential operators on X of order at most k . It satisfies the following properties:

- (1) $F_{\bullet} \mathcal{D}_X$ is an increasing, exhaustive filtration by locally free sheaves of finite rank.
- (2) $F_0 \mathcal{D}_X = \mathcal{O}_X$ and $F_1 \mathcal{D}_X \simeq \mathcal{O}_X \oplus T_X$.
- (3) $F_k \mathcal{D}_X \cdot F_{\ell} \mathcal{D}_X \subseteq F_{k+\ell} \mathcal{D}_X$ for all k and ℓ .
- (4) If $P \in F_k \mathcal{D}_X$ and $Q \in F_{\ell} \mathcal{D}_X$, then $[P, Q] \in F_{k+\ell-1} \mathcal{D}_X$.

We use the notation

$$\mathrm{gr}_k^F \mathcal{D}_X := F_k \mathcal{D}_X / F_{k-1} \mathcal{D}_X \quad \text{and} \quad \mathrm{gr}_\bullet^F \mathcal{D}_X := \bigoplus_{k \in \mathbf{N}} \mathrm{gr}_k^F \mathcal{D}_X$$

for the individual, respectively total, associated graded objects. The natural extension to differential operators of the assignment

$$T_X(U) \longrightarrow \mathcal{O}_X(U)[y_1, \dots, y_n], \quad \partial_i \mapsto y_i$$

induces isomorphisms

$$\mathrm{gr}_k^F \mathcal{D}_X \simeq \mathrm{Sym}^k T_X \quad \text{and} \quad \mathrm{gr}_\bullet^F \mathcal{D}_X \simeq \mathrm{Sym}^\bullet T_X,$$

where $\mathrm{Sym}^\bullet T_X$ is the symmetric algebra of the tangent sheaf of X ; see [HTT, §1.1]. In particular, if $\pi: T^*X \rightarrow X$ is the cotangent bundle of X , since by the standard correspondence between vector bundles locally free sheaves (see e.g. [Ha, Ch.II, Exer. 5.17, 5.18]) we have $T^*X \simeq \mathbf{Spec}(\mathrm{Sym}^\bullet T_X)$, it follows that

$$\mathrm{gr}_\bullet^F \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}.$$

DEFINITION 1.1.3. A sheaf of \mathcal{O}_X -modules \mathcal{M} is a *left \mathcal{D}_X -module* if for every open set $U \subseteq X$, $\mathcal{M}(U)$ has a left $\mathcal{D}_X(U)$ -module structure, compatible with restrictions. Informally, \mathcal{M} admits an action by differentiation. An analogous definition can be made for *right \mathcal{D}_X -modules*.

It is well known that the data of a \mathcal{D}_X -module structure on \mathcal{M} is equivalent to the data of a \mathbf{C} -linear map

$$\nabla: \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_X^1$$

satisfying the properties of an integrable connection. Concretely, for every open set $U \subseteq X$, and $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{M}(U)$ we have

$$(1.1.1) \quad \nabla(fs) = f\nabla(s) + s \otimes df,$$

and in addition $\nabla \circ \nabla = 0$. See for instance [HTT, Lemma 1.2.1]. This interpretation leads to one of the key objects associated to a \mathcal{D}_X -module:

DEFINITION 1.1.4. The *de Rham complex* of \mathcal{M} is the complex

$$\mathrm{DR}(\mathcal{M}) = \left[\mathcal{M} \rightarrow \Omega_X^1 \otimes \mathcal{M} \rightarrow \dots \rightarrow \Omega_X^n \otimes \mathcal{M} \right].$$

with \mathbf{C} -linear differentials induced by iterating ∇ . We consider it to be placed in degrees $-n, \dots, 0$.¹

We next give a few first examples of \mathcal{D}_X -modules of a geometric nature. They will play an important role throughout.

¹We will work with this convention, even though strictly speaking as such it is usually considered to be the de Rham complex associated to the corresponding right \mathcal{D} -module.

EXAMPLE 1.1.5. (1) The structure sheaf \mathcal{O}_X is a left \mathcal{D}_X -module, via the usual differentiation of functions. It is sometimes called the *trivial* \mathcal{D}_X -module.

(2) More generally, a vector bundle E on X endowed with an integrable (or flat) connection

$$\nabla: E \rightarrow E \otimes \Omega_X^1$$

is an example of a \mathcal{D}_X -module according to the discussion above. Example (1) is the special case corresponding to the standard differential $d: \mathcal{O}_X \rightarrow \Omega_X^1$.

It turns out that this is the only way in which a \mathcal{D}_X -module can be coherent as an \mathcal{O}_X -module; see [HTT, Theorem 1.4.10]:

THEOREM 1.1.6. *If \mathcal{M} is a \mathcal{D}_X -module which is coherent as an \mathcal{O}_X -module, then \mathcal{M} is locally free, hence an integrable connection.*

We now turn to examples that are not coherent as \mathcal{O}_X -modules.

(3) Let D be an effective divisor on X , and consider the quasi-coherent \mathcal{O}_X -module of rational functions which are regular away from D and have with poles of arbitrary order along D , i.e

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD).$$

This is sometimes called the *localization* of \mathcal{O}_X along D ; indeed if $U = \text{Spec}(R)$ is an affine open set in X in which $D = Z(f)$ with $f \in R$, then $\mathcal{O}_X(*D) = R_f$, the localization of R at f . This has an obvious action of differential operators by the quotient rule, hence $\mathcal{O}_X(*D)$ is a left \mathcal{D}_X -module.

(4) Combining the two examples above, the \mathcal{O}_X -module \mathcal{Q} defined by

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(*D) \longrightarrow \mathcal{Q} \longrightarrow 0$$

is again naturally a left \mathcal{D}_X -module, as the first inclusion is obviously a morphism of \mathcal{D}_X -modules. It is in fact well known that $\mathcal{Q} \simeq \mathcal{H}_D^1(\mathcal{O}_X)$, the first local cohomology sheaf of \mathcal{O}_X along D .

More generally, let $Z \subset X$ be an arbitrary closed subscheme. For an integer $q \geq 0$, we denote by $\mathcal{H}_Z^q(\mathcal{O}_X)$ the q -th local cohomology sheaf of \mathcal{O}_X , with support in Z ; see [LC, §1]. This is a quasi-coherent \mathcal{O}_X -module whose sections are annihilated by suitable powers of the ideal sheaf \mathcal{I}_Z ; it depends only on the reduced structure of Z .

For every affine open subset $U \subseteq X$, if $R = \mathcal{O}_X(U)$ and $I = \mathcal{I}_Z(U)$, then $\mathcal{H}_Z^q(\mathcal{O}_X)(U)$ is the local cohomology module $H_I^q(R)$. One of the equivalent descriptions of this module is as follows (see for instance [LC, Theorem 2.3] and the discussion thereafter): if $I = (f_1, \dots, f_r)$ and for a subset $J \subseteq \{1, \dots, r\}$, we denote $f_J := \prod_{i \in J} f_i$, then there is a Čech-type complex

$$C^\bullet: \quad 0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots \longrightarrow C^r \longrightarrow 0,$$

where

$$C^p = \bigoplus_{|J|=p} R_{f_J},$$

and we have

$$H_I^q(R) \simeq H^q(C^\bullet).$$

The differentials in C^\bullet respect the \mathcal{D} -module structure on each C^k , obtained from localization as in (2) above, and consequently $\mathcal{H}_Z^q(\mathcal{O}_X)$ is again a left \mathcal{D}_X -module.

It is often useful to consider *right* \mathcal{D}_X -modules. Just as with \mathcal{O}_X in the case of left \mathcal{D}_X -modules, there is also a “trivial” right \mathcal{D}_X -module, namely the *canonical bundle* $\omega_X := \bigwedge^n \Omega_X^1$. Its natural right \mathcal{D}_X -module structure is given as follows: if x_1, \dots, x_n are local algebraic coordinates on X , for any $f \in \mathcal{O}_X$ and any $P \in \mathcal{D}_X$ the action is

$$(f \cdot dx_1 \wedge \cdots \wedge dx_n) \cdot P = {}^tP(f) \cdot dx_1 \wedge \cdots \wedge dx_n.$$

Here, if $P = \sum_\alpha g_\alpha \partial^\alpha$, then ${}^tP = \sum_\alpha (-1)^{|\alpha|} \partial^\alpha g_\alpha$ is its formal adjoint.

This structure leads to an equivalence of categories between left and right \mathcal{D}_X -modules given by

$$\mathcal{M} \mapsto \mathcal{N} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \quad \text{and} \quad \mathcal{N} \mapsto \mathcal{M} = \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}).$$

See [HTT, Proposition 1.2.9 and 1.2.12] for details.

A few more words about the classes of \mathcal{D}_X -modules we are considering, and their relationship to good filtrations. First, we will essentially always work with \mathcal{D}_X -modules which are quasi-coherent as \mathcal{O}_X -modules. The sheaf \mathcal{D}_X itself is a first such example, as a union of locally free \mathcal{O}_X -modules of finite rank.

Moreover, we will usually restrict to *coherent* \mathcal{D}_X -modules. Recall that by definition \mathcal{M} is coherent if, locally, it is finitely generated over \mathcal{D}_X , and every submodule is locally finitely presented. However the following equivalent description is very helpful; see [HTT, Proposition 1.4.9]:

PROPOSITION 1.1.7. *A \mathcal{D}_X -module \mathcal{M} is coherent if and only if it is quasi-coherent as an \mathcal{O}_X -module and locally finitely generated as a \mathcal{D}_X -module.*

1.2. Filtered \mathcal{D} -modules

We say that a \mathcal{D}_X -module \mathcal{M} is *filtered* if there exists an increasing filtration $F = F_\bullet \mathcal{M}$ by coherent \mathcal{O}_X -modules, bounded from below and satisfying

$$F_k \mathcal{D}_X \cdot F_\ell \mathcal{M} \subseteq F_{k+\ell} \mathcal{M} \quad \text{for all } k, \ell \in \mathbf{Z}.$$

We use the notation (\mathcal{M}, F) for this data. The filtration is called *good* if the inclusions above are equalities for $\ell \gg 0$, which is in turn equivalent to the fact that the total associated graded object

$$\text{gr}_\bullet^F \mathcal{M} = \bigoplus_k \text{gr}_k^F \mathcal{M} = \bigoplus_k F_k \mathcal{M} / F_{k-1} \mathcal{M}$$

is finitely generated over $\text{gr}_\bullet^F \mathcal{D}_X \simeq \text{Sym}^\bullet T_X$; see [HTT, Proposition 2.1.1]. We can therefore also think of $\text{gr}_\bullet^F \mathcal{M}$ as a coherent sheaf on T^*X ; as such we sometimes write $\text{gr}^F \mathcal{M}$ when we forget about the grading.

A basic point is that working with good filtrations places us in the category of coherent \mathcal{D}_X -modules; see [HTT, Theorem 2.1.3]:

THEOREM 1.2.1. *A \mathcal{D}_X -module is coherent if and only if it admits a (globally defined) good filtration.*

Assume now that \mathcal{M} has a good filtration $F_\bullet \mathcal{M}$. The compatibility of this filtration with the standard filtration on \mathcal{D}_X , meaning in particular that

$$F_1 \mathcal{D}_X \cdot F_p \mathcal{M} \subseteq F_{p+1} \mathcal{M} \quad \text{for all } p$$

implies that this induces a filtration on the de Rham complex of \mathcal{M} by the formula

$$F_k \text{DR}(\mathcal{M}) = \left[F_k \mathcal{M} \rightarrow \Omega_X^1 \otimes F_{k+1} \mathcal{M} \rightarrow \cdots \rightarrow \Omega_X^n \otimes F_{k+n} \mathcal{M} \right].$$

We write $\text{DR}(\mathcal{M}, F)$ when we take the filtration into account. For any integer k , the associated graded complex for this filtration is

$$\text{gr}_k^F \text{DR}(\mathcal{M}) = \left[\text{gr}_k^F \mathcal{M} \rightarrow \Omega_X^1 \otimes \text{gr}_{k+1}^F \mathcal{M} \rightarrow \cdots \rightarrow \Omega_X^n \otimes \text{gr}_{k+n}^F \mathcal{M} \right].$$

When descending the differentials in $F_k \text{DR}(\mathcal{M})$ to the associated graded, the second term in (1.1.1) disappears, and therefore the differentials in $\text{gr}_k^F \text{DR}(\mathcal{M})$ become \mathcal{O}_X -linear. Hence this is now a complex of coherent \mathcal{O}_X -modules in degrees $-n, \dots, 0$, providing an object in $\mathbf{D}^b(\text{Coh}(X))$.

EXAMPLE 1.2.2 (The trivial filtered \mathcal{D} -module). Consider $\mathcal{M} = \mathcal{O}_X$ with the natural left \mathcal{D}_X -module structure, and $F_k \mathcal{O}_X = \mathcal{O}_X$ for $k \geq 0$, while $F_k \mathcal{O}_X = 0$ for $k < 0$. The de Rham complex of \mathcal{M} is

$$\text{DR}(\mathcal{O}_X) = \left[\mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow \cdots \rightarrow \Omega_X^n \right][n].$$

For the induced filtration $F_\bullet \text{DR}(\mathcal{O}_X)$, note that

$$\text{gr}_{-k}^F \text{DR}(\mathcal{O}_X) = \Omega_X^k[n-k] \quad \text{for all } k.$$

Note that in the holomorphic category we would have that $\text{DR}(\mathcal{O}_X)$ is quasi-isomorphic to $\mathbf{C}[n]$ by the holomorphic Poincaré Lemma. In the algebraic category we still have that the hypercohomology $\mathbb{H}^i(X, \text{DR}(\mathcal{O}_X))$, usually called the algebraic de Rham cohomology $H_{\text{dR}}^{i+n}(X)$, is isomorphic to the singular cohomology $H^{i+n}(X, \mathbf{C})$ by the Grothendieck comparison theorem.

EXAMPLE 1.2.3 (Variations of Hodge structure). This example is again better phrased in the holomorphic category. Note that the previous example corresponds to the trivial variation of Hodge structure (VHS) on X ; it can be extended to arbitrary such objects. Recall (see e.g. [Vo, Ch.10]) that a \mathbf{Q} -VHS of weight ℓ on X is the data

$$\mathbf{V} = (\mathcal{V}, F^\bullet, \mathbf{V}_{\mathbf{Q}})$$

where:

- $\mathbf{V}_{\mathbf{Q}}$ is a \mathbf{Q} -local system on X .

- $\mathcal{V} = \mathbf{V}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathcal{O}_X$ is a vector bundle with integrable connection ∇ , endowed with a decreasing filtration with subbundles $F^p = F^p \mathcal{V}$ satisfying the following two properties:
- for all $x \in X$, the data $\mathbf{V}_x = (\mathcal{V}_x, F_x^\bullet, \mathbf{V}_{\mathbf{Q},x})$ is a Hodge structure of weight ℓ .
- Griffiths transversality: for each p , ∇ induces a morphism

$$\nabla : F^p \longrightarrow F^{p-1} \otimes \Omega_X^1.$$

Recall now that we can think of $\mathcal{M} = \mathcal{V}$ as a left \mathcal{D}_X -module. We reindex the filtration as $F_p \mathcal{M} = F^{-p} \mathcal{V}$; this is a good filtration on \mathcal{M} . It is also well known that

$$\mathrm{DR}(\mathcal{M}) \simeq \mathbf{V}_{\mathbf{C}}[n],$$

where $\mathbf{V}_{\mathbf{C}} = \mathbf{V}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$. By construction the graded pieces $\mathrm{gr}_F^k \mathcal{M}$ are locally free, and are sometimes known as the *Hodge bundles* of the VHS.

It is worth pointing out right away a well-known connection with birational geometry. The most common geometric example of a VHS is obtained as follows: let $f : Y \rightarrow X$ be a smooth projective morphism, of relative dimension k . Then $R^k f_* \mathbf{Q}_Y$ supports a VHS on X given by the Hodge structure on the singular cohomology H^k of the fibers. Using the \mathcal{D} -module notation above, it is not hard to see that the lowest non-zero piece in the Hodge filtration on \mathcal{V} is

$$F_{-k} \mathcal{V} = f_* \omega_{X/Y}.$$

A similar interpretation can be given to all $R^i f_* \omega_{X/Y}$. This, as well as its consequences, will be discussed more later on, and will be extended to arbitrary morphisms.

Here is also a non-example:

EXERCISE 1.2.4. If D is a hypersurface in X , show that unless D is smooth, the pole order filtration

$$P_k \mathcal{O}_X(*D) := \mathcal{O}_X((k+1)D)$$

is not a good filtration on $\mathcal{O}_X(*D)$.

REMARK 1.2.5 (Left-right rule for filtrations). Recall that the mapping

$$\mathcal{M} \rightarrow \mathcal{N} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

establishes an equivalence between the categories of left and right \mathcal{D}_X -modules. This can be extended to filtered \mathcal{D}_X -modules, according to the following convention:

$$F_k \mathcal{M} = F_{k-n} \mathcal{N} \otimes_{\mathcal{O}_X} \omega_X^{-1}.$$

For instance, the trivial filtration on the right \mathcal{D}_X -module ω_X , the right analogue of Example 1.2.2, is given by $F_k \omega_X = 0$ for $k < -n$ and $F_k \omega_X = \omega_X$ for $k \geq -n$.

Characteristic varieties and holonomic \mathcal{D} -modules. Let \mathcal{M} be a \mathcal{D}_X -module endowed with a good filtration $F_\bullet \mathcal{M}$. Recall that the associated graded object $\mathrm{gr}^F \mathcal{M}$ can be thought of as a coherent sheaf on T^*X . The *characteristic variety* (or *singular support*) of \mathcal{M} is the support of this sheaf:

$$\mathrm{Ch}(\mathcal{M}) := \mathrm{Supp}(\mathrm{gr}^F \mathcal{M}) \subseteq T^*X.$$

This invariant satisfies a number of basic properties; see [HTT, §2.2]:

- (1) $\text{Ch}(\mathcal{M})$ is independent of the choice of good filtration $F_\bullet \mathcal{M}$ (and therefore it is an invariant of any coherent \mathcal{D}_X -module).
- (2) If $\pi: T^*X \rightarrow X$ is the projection map, then $\pi(\text{Ch}(\mathcal{M})) = \text{Supp}(\mathcal{M})$.
- (3) $\text{Ch}(\mathcal{M})$ is *conical*, i.e. in each fiber over X it is a cone over a subvariety in \mathbf{P}^{n-1} .
- (4) If $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P} \rightarrow 0$ is a short exact sequence of coherent \mathcal{D}_X -modules, then

$$\text{Ch}(\mathcal{N}) = \text{Ch}(\mathcal{M}) \cup \text{Ch}(\mathcal{P}).$$

- (5) $\text{Ch}(\mathcal{M}) = T_X^*X$, i.e. the zero-section of T^*X , if and only if \mathcal{M} is an integrable connection.

The fundamental result about characteristic varieties is a dimension bound; see [HTT, §2.3], and also the references therein for its history.

THEOREM 1.2.6 (Bernstein inequality). *If $0 \neq \mathcal{M}$ is a coherent \mathcal{D}_X -module, then for each component Z of $\text{Ch}(\mathcal{M})$ we have*

$$\dim Z \geq n.$$

A stronger statement holds true in fact: $\text{Ch}(\mathcal{M})$ is *involutive* with respect to the standard symplectic structure on T^*X . In any event, imposing equality in this bound leads to a celebrated class of \mathcal{D} -modules.

DEFINITION 1.2.7. A coherent \mathcal{D}_X -module \mathcal{M} is *holonomic* if either $\mathcal{M} = 0$ or $\dim \text{Ch}(\mathcal{M}) = n$.

EXAMPLE 1.2.8. An integrable connection \mathcal{M} is holonomic, since $\text{Ch}(\mathcal{M}) = T_X^*X$.

It is not hard to see that holonomic \mathcal{D} -modules are generically as in the example above; see [HTT, Proposition 3.1.6].

PROPOSITION 1.2.9. *Let \mathcal{M} be a holonomic \mathcal{D}_X -module. Then there exists a dense open set $U \subseteq X$ such that $\mathcal{M}|_U$ is an integrable connection.*

Further examples of holonomic \mathcal{D} -modules are produced once the basic properties of functors on \mathcal{D} -modules are recorded. One such is:

EXAMPLE 1.2.10. The \mathcal{D} -modules in Example 1.1.5 (3), and therefore (4), are in fact also holonomic. In other words, the basic statement is that if D is an effective divisor on X , then the localization \mathcal{D}_X -module $\mathcal{O}_X(*D)$ is holonomic. This follows from the fact that direct images of holonomic \mathcal{D} -modules are holonomic; in this case $\mathcal{O}_X(*D)$ is the direct image of \mathcal{O}_U via the open embedding $U = X \setminus D \hookrightarrow \mathcal{O}_X$. The holonomicity of $\mathcal{O}_X(*D)$ is closely related to the existence of Bernstein-Sato polynomials.

1.3. Push-forward of \mathcal{D} -modules

A detailed exposition on the material in this section can be found in [HTT, §1.3 and §1.5].

Let $f: Y \rightarrow X$ be a morphism of smooth varieties. The *transfer module* of f is defined as

$$\mathcal{D}_{Y \rightarrow X} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X.$$

This is simply $f^*\mathcal{D}_X$ as an \mathcal{O}_Y -module, and we will use this notation when thinking of it as such; note in particular that it is filtered by $f^*F_k\mathcal{D}_X$.

However for our purposes it is endowed with the structure of a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$ -bimodule. Indeed, the right $f^{-1}\mathcal{D}_X$ -module structure comes simply from the right hand side of the tensor product. On the other hand, the left \mathcal{D}_Y -module structure is a general phenomenon:

REMARK 1.3.1 (Pullback of \mathcal{D} -modules). If \mathcal{M} is a left \mathcal{D}_X -module, then the \mathcal{O}_Y -module $f^*\mathcal{M}$ has a natural left \mathcal{D}_Y -module structure coming from the natural morphism $\mathcal{D}_Y \rightarrow f^*\mathcal{D}_X$, induced in turn by the natural morphism of vector bundles

$$T_Y \rightarrow f^*T_X.$$

EXAMPLE 1.3.2. Let $i: Y \hookrightarrow X$ be a closed embedding, with X of dimension n and Y of dimension r . If we choose local algebraic coordinates x_1, \dots, x_n on X such that $Y = (x_{r+1} = \dots = x_n = 0)$, then

$$\mathcal{D}_{Y \rightarrow X} \simeq \mathcal{D}_Y \otimes_{\mathbf{C}} \mathbf{C}[\partial_{r+1}, \dots, \partial_n].$$

The push-forward of \mathcal{D} -modules is a priori more naturally defined for right \mathcal{D} -modules; this is similar to the fact that we don't have a natural push-forward of functions, but rather of distributions. As a preliminary definition, for a right \mathcal{D}_Y -module \mathcal{N} , we take its push-forward to X to be

$$f_+\mathcal{N} := f_*(\mathcal{N} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}).$$

(We will see in a second that we should rather denote this by $\mathcal{H}^0 f_+\mathcal{N}$.) It has a right \mathcal{D}_X -module structure as follows: we use the natural right $f^{-1}\mathcal{D}_X$ -module structure on $\mathcal{D}_{Y \rightarrow X}$ to obtain a right $f_*f^{-1}\mathcal{D}_X$ -module structure on $f_+\mathcal{N}$. We then restrict scalars via the adjunction morphism $\mathcal{D}_X \rightarrow f_*f^{-1}\mathcal{D}_X$.

Due to the use of the left exact functor f_* in combination with the right exact functor \otimes , the definition above is not so well behaved, for instance with respect to the composition of morphism. This is remedied by working with derived functors. Note that in [HTT, §1.5] the functor below is denoted by \int_f , reminiscent of integration by fibers.

DEFINITION 1.3.3. The push-forward functor on right \mathcal{D}_Y modules is defined as

$$f_+ : \mathbf{D}^b(\mathcal{D}_Y^{\text{op}}) \longrightarrow \mathbf{D}^b(\mathcal{D}_X^{\text{op}}), \quad \mathcal{N}^\bullet \mapsto \mathbf{R}f_*(\mathcal{N}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}),$$

where $\mathbf{D}^b(\mathcal{D}_Y^{\text{op}})$ is the bounded derived category of right \mathcal{D}_Y -modules, and similarly for $\mathbf{D}^b(\mathcal{D}_X^{\text{op}})$.²

²See the beginning of [HTT, §1.5] for a discussion of these derived categories.

Note that with this definition it is not too hard to check that for a composition of morphisms we have $(g \circ f)_+ = g_+ \circ f_+$; see [HTT, Proposition 1.5.21].

We can of course also define the push-forward functor for left \mathcal{D}_Y -modules by applying the left-right rule at both ends of the above construction. This is packaged nicely by considering an analogue of the transfer module, namely

$$\mathcal{D}_{X \leftarrow Y} := \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\omega_X^{-1},$$

which is a $(f^{-1}\mathcal{D}_X, \mathcal{D}_Y)$ -bimodule. Push-forward on the bounded derived category of left \mathcal{D}_Y -modules is then defined as

$$f_+ : \mathbf{D}^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}^b(\mathcal{D}_X), \quad \mathcal{M}^\bullet \mapsto \mathbf{R}f_* (\mathcal{D}_{X \leftarrow Y} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{M}^\bullet).$$

EXAMPLE 1.3.4 (Open embeddings). Let $j : U \hookrightarrow X$ be the embedding of an open subset of X . Then $\mathcal{D}_{X \leftarrow U} \simeq \mathcal{D}_U$, and so

$$j_+ = \mathbf{R}j_*$$

meaning j_+ is the same as the usual direct image functor on \mathcal{O}_U -modules.

In particular, if $U = X \setminus D$, where D is an effective divisor, then

$$j_+ \mathcal{O}_U \simeq \mathcal{O}_X(*D).$$

(This contains the statement that $\mathcal{H}^i j_+ \mathcal{O}_U = 0$ for $i > 0$.)

More generally, if $Z \subseteq X$ is an arbitrary closed subset, this can be expressed in terms of local cohomology. A standard calculation (exercise!) shows that there is a short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow j_* \mathcal{O}_U \rightarrow \mathcal{H}_Z^1 \mathcal{O}_X \rightarrow 0,$$

while

$$R^k j_* \mathcal{O}_U \simeq \mathcal{H}_Z^{k+1} \mathcal{O}_X \quad \text{for } k \geq 1.$$

EXAMPLE 1.3.5 (Closed embeddings). Let $i : Y \hookrightarrow X$ be the embedding of a closed subset of X . With the notation of Example 1.3.2 (see also [HTT, Ex. 1.3.5] for more details) we have

$$\mathcal{D}_{X \leftarrow Y} \simeq \mathbf{C}[\partial_{r+1}, \dots, \partial_n] \otimes_{\mathbf{C}} \mathcal{D}_Y,$$

with the obvious bimodule actions.

Let now \mathcal{M} be a \mathcal{D}_Y -module. Since in this case i_* is an exact functor, we have $\mathcal{H}^k i_+ \mathcal{M} = 0$ for $k \neq 0$. Moreover, using the description of $\mathcal{D}_{X \leftarrow Y}$ above, we have

$$\mathcal{H}^0 i_+ \mathcal{M} \simeq \mathbf{C}[\partial_{r+1}, \dots, \partial_n] \otimes_{\mathbf{C}} \mathcal{M}.$$

The left \mathcal{D}_X -module action is given by:

- $\partial_{r+1}, \dots, \partial_n$ act by $\partial_j \cdot (P \otimes m) = \partial_j P \otimes m$.
- $\partial_1, \dots, \partial_r$ act by $\partial_j \cdot (P \otimes m) = P \otimes \partial_j m$.
- $f \in \mathcal{O}_X$ acts by $f \cdot (P \otimes m) = P \otimes (f_Y)m$.

The most important result about this functor is Kashiwara's Theorem, which we now state. First, let's introduce some notation:

- $\text{Mod}_{\text{qc}}(\mathcal{D}_Y)$ stands for the category of \mathcal{D}_Y -modules that are quasi-coherent as \mathcal{O}_Y -modules.
- $\text{Mod}_{\text{qc}}^Y(\mathcal{D}_X)$ stands for the category of \mathcal{D}_X -modules that are supported on Y , and are quasi-coherent as \mathcal{O}_Y -modules.

THEOREM 1.3.6 (Kashiwara's Equivalence). *The functor i_+ induces an equivalence of categories*

$$i_+ : \text{Mod}_{\text{qc}}(\mathcal{D}_Y) \rightarrow \text{Mod}_{\text{qc}}^Y(\mathcal{D}_X).$$

Its inverse is given by the functor $\mathcal{H}^{n-r}i^$, and moreover on the category $\text{Mod}_{\text{qc}}^Y(\mathcal{D}_X)$ we have $\mathcal{H}^k i^* = 0$ for $k \neq n - r$.*

For a proof of this theorem see [HTT, Theorem 1.6.1]. The equivalence also restricts to the categories of coherent \mathcal{D} -modules on both sides, due to general results described below.

EXAMPLE 1.3.7 (Projections). Assume that $Y = X \times Z$, with Z another smooth variety, and let $f = p_1 : Y \rightarrow X$ be the projection onto the first factor. In this case f_+ can be described as follows; see [HTT, Proposition 1.5.28].

We denote $d = \dim Z = \dim Y - \dim X$, and for each $0 \leq k \leq d$ we consider

$$\Omega_{Y/X}^k := p_2^* \Omega_Z^k.$$

The *relative de Rham complex* of a left \mathcal{D}_Y -module \mathcal{M} is the complex

$$\text{DR}_{Y/X}(\mathcal{M}) : 0 \rightarrow \mathcal{M} \rightarrow \Omega_{Y/X}^1 \otimes \mathcal{M} \rightarrow \cdots \rightarrow \Omega_{Y/X}^d \otimes \mathcal{M} \rightarrow 0$$

which we consider placed in degrees $-d, \dots, 0$, and with differentials given by

$$d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^d (dx_i \otimes \omega) \otimes \partial_i s,$$

where x_1, \dots, x_d are local coordinates on Z .

Then the following isomorphism holds

$$\mathcal{D}_{X \leftarrow Y} \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{M} \simeq \text{DR}_{Y/X}(\mathcal{M})$$

and therefore the push-forward can be computed as

$$f_+ \mathcal{M} \simeq \mathbf{R}f_* \text{DR}_{Y/X}(\mathcal{M}).$$

REMARK 1.3.8. It is worth noting that each entry $\text{DR}_{Y/X}(\mathcal{M})^k = \Omega_{Y/X}^{n+k} \otimes \mathcal{M}$ is a $f^{-1}\mathcal{D}_X$ -module thanks to the rule

$$P \cdot (\omega \otimes s) = \omega \otimes ((P \otimes 1) \cdot s)$$

induced by the mapping $f^{-1}\mathcal{D}_X \rightarrow \mathcal{D}_Y$, $P \rightarrow P \otimes 1$.

In order to establish various properties of push-forward functors, it often suffices to consider the concrete descriptions in the examples above. The reason is that every morphism $f: Y \rightarrow X$ can be written as the composition of the closed embedding $Y \hookrightarrow Y \times X$ given by the graph of f , followed by the second projection.

Using this approach, the following important properties of the push-forward functor can be established. The derived categories in the statement below are those corresponding to objects whose entries are quasi-coherent, coherent, or holonomic respectively; they can be shown to be equivalent to those of objects whose *cohomologies* are of this kind.

THEOREM 1.3.9. *Let $f: Y \rightarrow X$ be a morphism of smooth varieties. Then:*

(1) f_+ preserves quasi-coherence (over \mathcal{O}_X), in the sense that it induces a functor

$$f_+: \mathbf{D}_{\text{qc}}^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}_{\text{qc}}^b(\mathcal{D}_X).$$

(2) If f is projective, then f_+ preserves coherence, in the sense that it induces a functor

$$f_+: \mathbf{D}_c^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}_c^b(\mathcal{D}_X).$$

(3) f_+ preserves holonomicity, in the sense that it induces a functor

$$f_+: \mathbf{D}_h^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}_h^b(\mathcal{D}_X).$$

The proof of (1) can be found in [HTT, Proposition 1.5.29], that of (2) in [HTT, Theorem 2.5.1], and that of (3) in [HTT, Theorem 3.2.3].

REMARK 1.3.10. Note in particular that in (3) we are not imposing any conditions on the morphism f . It is also the case that the derived pullback functor $\mathbf{L}f^*$ preserves holonomicity; see [HTT, Theorem 3.2.3].

1.4. Induced \mathcal{D} -modules and filtered differential morphisms

In this section we show that complexes of filtered \mathcal{D}_X -modules are quasi-isomorphic to complexes whose entries are special types of \mathcal{D} -modules arising from plain old \mathcal{O}_X -modules. These can be further interpreted as complexes of \mathcal{O}_X -modules with a special type of \mathbf{C} -linear differentials. Since we will apply this in the next section to studying filtered push-forward, which we will do for *right* \mathcal{D}_X -modules, we will work in this setting.

Note once and for all that all types of filtrations appearing below are required to satisfy the property that $F_p = 0$ for $p \ll 0$.

We begin by denoting by $\text{FM}(\mathcal{D}_X)$ the category of filtered right \mathcal{D}_X -modules. A morphism in this category is a \mathcal{D}_X -module morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ such that $f(F_k \mathcal{M}) \subseteq F_k \mathcal{N}$ for all k . Special objects in this category are those induced by filtered \mathcal{O}_X -modules. Concretely, consider a filtered \mathcal{O}_X -module $(\mathcal{G}, F_\bullet \mathcal{G})$, with respect to the trivial filtration on \mathcal{O}_X given by $F_k \mathcal{O}_X = 0$ for $k < 0$ and $F_k \mathcal{O}_X = \mathcal{O}_X$ for $k \geq 0$. We can associate to it an object in $\text{FM}(\mathcal{D}_X)$ given by

$$\mathcal{M} := \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \quad \text{and} \quad F_k \mathcal{M} := \sum_{i=0}^k F_{k-i} \mathcal{G} \otimes F_i \mathcal{D}_X.$$

DEFINITION 1.4.1. An object in $\text{FM}(\mathcal{D}_X)$ is an *induced filtered \mathcal{D}_X -module* if it is isomorphic to one defined as above. We use the notation $\text{FM}_i(\mathcal{D}_X)$ for the full subcategory of $\text{FM}(\mathcal{D}_X)$ whose objects are induced filtered \mathcal{D}_X -modules.

REMARK 1.4.2. Saito [Sa1, §2.2] calls the functor

$$\mathcal{M} := \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \mapsto \text{DR}(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{O}_X \in \text{Mod}(\mathbf{C}_X).$$

the *de Rham functor* on induced \mathcal{D} -modules. We have of course a canonical isomorphism $\text{DR}(\mathcal{M}) \simeq \mathcal{G}$.

LEMMA 1.4.3. *Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules. Then there is a natural homomorphism*

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \rightarrow \text{Hom}_{\mathbf{C}}(\mathcal{F}, \mathcal{G})$$

given by $\bullet \otimes_{\mathcal{D}_X} \mathcal{O}_X$, and this homomorphism is injective.

PROOF. By adjunction we have an isomorphism

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X),$$

and therefore the homomorphism in the statement corresponds to sending φ to ψ in each diagram of the form

$$\begin{array}{ccc} & & \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ & \nearrow \varphi & \downarrow \\ \mathcal{F} & \xrightarrow{\psi} & \mathcal{G} \end{array}$$

where the \mathcal{O}_X -module structure on the top sheaf is obtained by restriction of scalars from its \mathcal{D}_X -module structure (hence in particular ψ is not \mathcal{O}_X -linear), and the vertical arrow is obtained by sending $s \otimes P \mapsto P(1)s$, for a section s of \mathcal{G} and a differential operator P .

Assume now that $\varphi \neq 0$. Hence there is an open set U with local coordinates x_1, \dots, x_n , and a section $s \in \Gamma(U, \mathcal{F})$ such that $\varphi(s) \neq 0$. We write

$$\varphi(s) = \sum_{\alpha} t_{\alpha} \otimes \partial^{\alpha},$$

where the sum is finite, over $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, and $t_{\alpha} \in \Gamma(U, \mathcal{G})$. We use the standard notation $\partial^{\alpha} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, and we will also use $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. Let

$$k_0 := \min \{|\alpha| \mid t_{\alpha} \neq 0\},$$

and consider $\beta \in \mathbf{N}^n$ such that $|\beta| = k_0$ and $t_{\beta} \neq 0$. We then have

$$\psi(s \cdot x^{\beta}) = \sum_{\alpha} t_{\alpha} \otimes \partial^{\alpha}(x^{\beta}) = \beta! \cdot t_{\beta} \neq 0,$$

hence $\psi \neq 0$. □

DEFINITION 1.4.4. Let \mathcal{F} and \mathcal{G} be \mathcal{O}_X -modules. The group of *differential morphisms* from \mathcal{F} to \mathcal{G} is defined as the image in $\text{Hom}_{\mathbf{C}}(\mathcal{F}, \mathcal{G})$ of the homomorphism in the Lemma above, and is denoted by $\text{Hom}_{\text{Diff}}(\mathcal{F}, \mathcal{G})$.

Note that $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ admits a filtration whose p -th term is

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_X} F_p \mathcal{D}_X),$$

and therefore we can define a filtration on differential morphisms where $F_p \mathrm{Hom}_{\mathrm{Diff}}(\mathcal{F}, \mathcal{G})$ is (the image of) this subgroup. We call these differential morphisms *of order $\leq p$* .

We now give a filtered version of this construction.

DEFINITION 1.4.5. Let (\mathcal{F}, F) and (\mathcal{G}, F) be filtered \mathcal{O}_X -modules. The group of *filtered differential morphisms*

$$\mathrm{Hom}_{\mathrm{Diff}}((\mathcal{F}, F), (\mathcal{G}, F))$$

is the subgroup of $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{F}, \mathcal{G})$ consisting of morphisms f satisfying, for every p and q , the fact that the composition

$$F_p \mathcal{F} \hookrightarrow \mathcal{F} \xrightarrow{f} \mathcal{G} \rightarrow \mathcal{G}/F_{p-q-1} \mathcal{G}$$

(which itself is a differential morphism) has order $\leq q$.

EXERCISE 1.4.6. Restricting to filtered morphisms, the homomorphism in Lemma 1.4.3 induces an isomorphism

$$\mathrm{Hom}_{\mathrm{FM}(\mathcal{D}_X)}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \simeq \mathrm{Hom}_{\mathrm{Diff}}((\mathcal{F}, F), (\mathcal{G}, F)).$$

DEFINITION 1.4.7. We denote by $\mathrm{FM}(\mathcal{O}_X, \mathrm{Diff})$ the additive category whose objects are filtered \mathcal{O}_X -modules, and whose morphisms are filtered differential morphisms.

Putting together all of the above, we obtain the following interpretation of the category of induced filtered \mathcal{D} -modules.

PROPOSITION 1.4.8. *The functor*

$$\mathrm{DR}^{-1}: \mathcal{G} \mapsto \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

induces an equivalence of categories

$$\mathrm{DR}^{-1}: \mathrm{FM}(\mathcal{O}_X, \mathrm{Diff}) \xrightarrow{\simeq} \mathrm{FM}_i(\mathcal{D}_X).$$

It is not hard to see that this equivalence extends to an equivalence of triangulated categories

$$(1.4.1) \quad \mathrm{DR}^{-1}: \mathbf{D}(\mathrm{FM}(\mathcal{O}_X, \mathrm{Diff})) \xrightarrow{\simeq} \mathbf{D}(\mathrm{FM}_i(\mathcal{D}_X)).$$

However, we need a brief discussion of these and other derived categories that will be used from now on, which is done in the next remark.

REMARK 1.4.9 (Definition of derived categories). By $\mathbf{D}(\mathrm{FM}(\mathcal{D}_X))$, $\mathbf{D}(\mathrm{FM}_i(\mathcal{D}_X))$ and all the others, we mean \mathbf{D}^* , where $*$ can be either absent or any of $*$ = $-$, $+$, b . However the definition of these derived categories needs some explanation. I will only do this for $\mathbf{D}(\mathrm{FM}(\mathcal{D}_X))$, as all the others are similar. First, note that $\mathrm{FM}(\mathcal{D}_X)$ is an additive category which has (co)kernels and (co)images, but it is not in general an abelian category.³

³Given a morphism $\varphi: \mathcal{M} \rightarrow \mathcal{N}$ in $\mathrm{FM}(\mathcal{D}_X)$, it is not necessarily the case that the induced morphism $\mathrm{Coim}(\varphi) \rightarrow \mathrm{Im}(\varphi)$ is an isomorphism.

Hence we are not looking at the derived category associated to an abelian category. We form $C^*(\mathrm{FM}(\mathcal{D}_X))$, the category of complexes of objects in $\mathrm{FM}(\mathcal{D}_X)$, and then the homotopy category $K^*(\mathrm{FM}(\mathcal{D}_X))$, where the homotopies are required to preserve the filtrations. It is not hard to see that $K^*(\mathrm{FM}(\mathcal{D}_X))$ has a natural structure of triangulated category.⁴ Finally, the *filtered derived category* $\mathbf{D}^*(\mathrm{FM}(\mathcal{D}_X))$ is the localization of $K^*(\mathrm{FM}(\mathcal{D}_X))$ at the class of filtered quasi-isomorphisms. As with the derived category of an abelian category, one can show that there is a unique triangulated structure on $\mathbf{D}^*(\mathrm{FM}(\mathcal{D}_X))$ such that the canonical localization functor $K^*(\mathrm{FM}(\mathcal{D}_X)) \rightarrow \mathbf{D}^*(\mathrm{FM}(\mathcal{D}_X))$ is exact.

On the other hand, the derived category $\mathbf{D}(\mathrm{FM}(\mathcal{O}_X, \mathrm{Diff}))$ is obtained by inverting D -quasi-isomorphisms in $\mathrm{FM}(\mathcal{O}_X, \mathrm{Diff})$, meaning those morphisms that are mapped to (filtered) quasi-isomorphisms via the functor DR^{-1} .

We next observe that every object in $\mathrm{FM}(\mathcal{D}_X)$ admits a finite resolution by induced filtered \mathcal{D}_X -modules, and use this to find a quasi-inverse for the equivalence above. To this end, recall the Spencer complex

$$0 \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n T_X \rightarrow \cdots \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} T_X \rightarrow \mathcal{D}_X \rightarrow 0,$$

placed in degrees $-n, \dots, 0$. The differentials are such that this complex is isomorphic in local coordinates x_1, \dots, x_n to the Koszul complex $K(\mathcal{D}_X; \partial_1, \dots, \partial_n)[n]$ associated to the (right) action of $\partial_1, \dots, \partial_n$ on \mathcal{D}_X . We consider this to be a complex of filtered left \mathcal{D}_X -modules, where the filtration on $\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X$ is given by

$$F_k(\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X) := F_{k+i} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X.$$

This complex is filtered quasi-isomorphic to the left \mathcal{D}_X -module \mathcal{O}_X with the trivial filtration; see [HTT, Lemma 1.5.27].

Consider now an arbitrary $(\mathcal{M}, F) \in \mathrm{FM}(\mathcal{D}_X)$. Recall that $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ has a natural right \mathcal{D}_X -module structure (see e.g. [HTT, Proposition 1.2.9(ii)]). Applying $\mathcal{M} \otimes_{\mathcal{O}_X} \bullet$ to the complex above corresponds to the Spencer complex of $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$:

$$(1.4.2) \quad 0 \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n T_X \rightarrow \cdots \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} T_X \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow 0.$$

PROPOSITION 1.4.10. *The complex in (1.4.2) is a complex of filtered induced right \mathcal{D}_X -modules, quasi-isomorphic to (\mathcal{M}, F) .*

PROOF. By the same [HTT, Proposition 1.2.9(ii)], all the terms $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X$ have a natural (filtered) right \mathcal{D}_X -module structure, and it is not hard to check that there is a filtered isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^i T_X \simeq \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X \otimes_{\mathcal{O}_X} \mathcal{D}_X,$$

with the obvious right \mathcal{D}_X -module structure on the right hand side (exercise!). This realizes our complex as a complex of induced \mathcal{D}_X -modules, using the natural \mathcal{O}_X -module structure on $\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X$.

⁴Essentially one has to note that the cone of a morphism in $\mathrm{FM}(\mathcal{D}_X)$ carries a natural filtration, such that all the morphisms in the associated exact triangle are compatible with the filtrations.

On the other hand, there is a natural map

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \mathcal{M}, \quad m \otimes P \mapsto P(1)m,^5$$

and this is a surjective filtered right \mathcal{D}_X -module morphism. Placing this at the right end of the complex in (1.4.2), the claim is that it induces a filtered quasi-isomorphism between this complex and (\mathcal{M}, F) (which then finishes the proof). This is of course equivalent to saying that the complex in (1.4.2) has no cohomology except at the right-most term.

Let's reinterpret this for convenience in terms of the corresponding complex in $\text{FM}(\mathcal{O}_X, \text{Diff})$. Concretely, our complex is obtained by applying DR^{-1} to the complex in $\text{FM}(\mathcal{O}_X, \text{Diff})$

$$\widetilde{\text{DR}}(\mathcal{M}, F) : \quad 0 \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^n T_X \rightarrow \cdots \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} T_X \rightarrow \mathcal{M} \rightarrow 0.$$

(we are using the notation introduced in [Sa1, §2]), where the filtration on this complex obtained by setting

$$F_p(\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X) = F_{p+i} \mathcal{M} \otimes_{\mathcal{O}_X} \wedge^i T_X.$$

Note that $\widetilde{\text{DR}}(\mathcal{M}, F)$ is obtained by applying $(\mathcal{M}, F) \otimes_{\mathcal{D}_X} \bullet$ to the Spencer complex resolving \mathcal{O}_X , and therefore in local coordinates x_1, \dots, x_n it is isomorphic to the Koszul complex $K(\mathcal{M}; \partial_1, \dots, \partial_n)[n]$ associated to the elements $\partial_1, \dots, \partial_n$ acting on \mathcal{M} , hence the assertion. \square

Restating the last part of the proof of the Proposition above, we have the following:

COROLLARY 1.4.11. *For every $(\mathcal{M}, F) \in \text{FM}(\mathcal{D}_X)$ there is a natural quasi-isomorphism of filtered complexes of right \mathcal{D}_X -modules*

$$\text{DR}^{-1} \widetilde{\text{DR}}(\mathcal{M}, F) \rightarrow (\mathcal{M}, F).$$

Via standard homological algebra, this discussion leads to the following equivalence of filtered derived categories:

PROPOSITION 1.4.12. *The natural functor $\mathbf{D}(\text{FM}_i(\mathcal{D}_X)) \rightarrow \mathbf{D}(\text{FM}(\mathcal{D}_X))$ is an equivalence of categories.*

Moreover, restricted to filtered induced \mathcal{D} -modules, the functor $\widetilde{\text{DR}}$ provides a quasi-inverse for the functor DR^{-1} in (1.4.1).

Together with the equivalence in (1.4.1), the Proposition above shows that in order to study operations on $\mathbf{D}(\text{FM}(\mathcal{D}_X))$ we may restrict to complexes of induced \mathcal{D} -modules, or to filtered differential complexes. We will take advantage of this below, when defining the push-forward functor for filtered \mathcal{D} -modules.

Proposition 1.4.10 gives us a canonical approach to finding resolutions by induced \mathcal{D} -modules. Other explicit resolutions may however be more meaningful and easier to work with. The following extended example is very important for applications.

⁵Note that this map is induced by tensoring with \mathcal{M} over \mathcal{O}_X the natural map $\mathcal{D}_X \rightarrow \mathcal{O}_X$, taking an operator P to $P(1)$ (and realizing the quasi-isomorphism between the Spencer complex and \mathcal{O}_X).

EXAMPLE 1.4.13 (Localization along an SNC divisor). Let E be a reduced simple normal crossing (SNC) divisor on a smooth n -dimensional variety Y .⁶ Recall that $\omega_Y(*E) \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*E)$ stands for the right \mathcal{D} -module version of the localization along the divisor E , in other words the sheaf of n -forms with arbitrary poles along E .

We endow $\omega_Y(*E)$ with what we will call the *Hodge filtration*, namely

$$F_k \omega_Y(*E) := \omega_Y(E) \cdot F_{k+n} \mathcal{D}_Y \quad \text{for } k \geq -n.$$

For instance, the first two nonzero terms are

$$F_{-n} \omega_Y(*E) = \omega_Y(E) \quad \text{and} \quad F_{-n+1} \omega_Y(*E) = \omega_Y(2E) \cdot \text{Jac}(E),$$

where $\text{Jac}(E)$ is the Jacobian ideal of E , i.e. $F_1 \mathcal{D}_Y \cdot \mathcal{O}_Y(-E)$, whose zero locus is the singular locus of E .

To gain intuition for what comes next, recall that the right \mathcal{D} -module ω_Y has a standard filtered resolution

$$0 \rightarrow \mathcal{D}_Y \rightarrow \Omega_Y^1 \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \cdots \rightarrow \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \omega_Y \rightarrow 0$$

by induced \mathcal{D}_Y -modules. This is simply the resolution of ω_Y , with the trivial filtration $F_k \omega_X = \omega_X$ for $k \geq -n$ and 0 otherwise, as in Remark 1.2.5, described by the procedure in Proposition 1.4.10; see also [HTT, Lemma 1.2.57]. It is a simple check that $\widehat{\text{DR}}(\omega_X)$, i.e. the associated complex in $\text{FM}(\mathcal{O}_X, \text{Diff})$, is the standard de Rham complex

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \cdots \xrightarrow{d} \omega_Y \longrightarrow 0.$$

A similar type of resolution by right induced \mathcal{D}_Y -modules can be found for $\omega_Y(*E)$, only this time it will correspond to the de Rham complex with log poles along E .

PROPOSITION 1.4.14. *The right \mathcal{D}_Y -module $\omega_Y(*E)$ has a filtered resolution by induced \mathcal{D}_Y -modules, given by*

$$0 \rightarrow \mathcal{D}_Y \rightarrow \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \cdots \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \rightarrow \omega_Y(*E) \rightarrow 0.$$

Here the morphism

$$\omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \omega_Y(*E)$$

is given by $\omega \otimes P \rightarrow \omega \cdot P$ (the \mathcal{D} -module operation), and for each p the morphism

$$\Omega_Y^p(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \Omega_Y^{p+1}(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$$

is given by $\omega \otimes P \mapsto d\omega \otimes P + \sum_{i=1}^n (dx_i \wedge \omega) \otimes \partial_i P$, in local coordinates x_1, \dots, x_n .

PROOF. It is not hard to check that the expression in the statement is indeed a complex, which we call A^\bullet . We consider on $\Omega_Y^p(\log E)$ the filtration

$$F_i \Omega_Y^p(\log E) = \begin{cases} \Omega_Y^p(\log E) & \text{if } i \geq -p \\ 0 & \text{if } i < -p, \end{cases}$$

⁶We use this notation since in practice we will consider this setting on a log resolution $f: Y \rightarrow X$ of a pair (X, D) , with $E = f^{-1}(D)_{\text{red}}$.

and on $\Omega_Y^p(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ the tensor product filtration. This filters A^\bullet by subcomplexes $F_{k-n}A^\bullet$ given by

$$\cdots \rightarrow \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} F_{k-1}\mathcal{D}_Y \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \rightarrow F_k\omega_Y(*E) \rightarrow 0$$

for each $k \geq 0$. Note that they can be rewritten as

$$\cdots \rightarrow \omega_Y(E) \otimes T_Y(-\log E) \otimes_{\mathcal{O}_Y} F_{k-1}\mathcal{D}_Y \xrightarrow{\beta_k} \omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \rightarrow F_k\omega_Y(*E) \rightarrow 0,$$

where $T_Y(-\log E)$ is the dual of $\Omega_Y^1(\log E)$, and we use the isomorphisms $\omega_Y(E) \otimes \wedge^i T_Y(-\log E) \simeq \Omega_Y^{n-i}(\log E)$.

It is clear directly from the definition that every such complex is exact at the term $F_k\omega_Y(*E)$. We now check that they are exact at the term $\omega_Y(E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y$. Let us assume that, in the local coordinates x_1, \dots, x_n , the divisor E is given by $x_1 \cdots x_r = 0$. Using the notation $\omega = dx_1 \wedge \cdots \wedge dx_n$, we consider an element

$$u = \frac{\omega}{x_1 \cdots x_r} \otimes \sum_{|\alpha| \leq k} g_\alpha \partial^\alpha$$

mapping to 0 in $F_k\omega_Y(*E) = \omega_Y(E) \cdot F_k\mathcal{D}_Y$. This means that

$$\sum_{|\alpha| \leq k, \alpha_i = 0 \text{ if } i > r} \alpha_1! \cdots \alpha_r! \cdot g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r} = 0.$$

We show that u is in the image of the morphism β_k by using a descending induction on $|\alpha|$. What we need to prove is the following claim: for each α in the sum above, with $|\alpha| = k$, there exists some i with $\alpha_i > 0$ such that x_i divides g_α . If so, an easy calculation shows that the term $u_\alpha = \frac{\omega}{x_1 \cdots x_r} \otimes g_\alpha \partial^\alpha$ is in the image of β_k , and hence it is enough to prove the statement for $u - u_\alpha$. Repeating this a finite number of times, we can reduce to the case when all $|\alpha| \leq k - 1$. But the claim is clear: if x_i did not divide g_α for all i with $\alpha_i > 0$, then the Laurent monomial $x_1^{-\alpha_1} \cdots x_r^{-\alpha_r}$ would appear in the term $g_\alpha \cdot x_1^{-\alpha_1} \cdots x_r^{-\alpha_r}$ of the sum above, but in none of the other terms.

To check the rest of the statement, note that after discarding the term on the right, the associated graded complexes

$$\begin{aligned} \cdots &\longrightarrow \omega_Y(E) \otimes \bigwedge^2 T_Y(-\log E) \otimes_{\mathcal{O}_Y} S^{k-2}T_Y \longrightarrow \\ &\longrightarrow \omega_Y(E) \otimes T_Y(-\log E) \otimes_{\mathcal{O}_Y} S^{k-1}T_Y \longrightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} S^k T_Y \longrightarrow 0 \end{aligned}$$

are acyclic. Indeed, each such complex is, up to a twist, an Eagon-Northcott complex associated to the inclusion of vector bundles of the same rank

$$\varphi: T_Y(-\log E) \rightarrow T_Y.$$

Concretely, in the notation on [La, p.323], the complex above is (EN_k) tensored by $\omega_Y(E)$. According to [La, Theorem B.2.2(iii)], (EN_k) is acyclic provided that

$$\text{codim } D_{n-\ell}(\varphi) \geq \ell \quad \text{for all } 1 \leq \ell \leq \min\{k, n\},$$

where

$$D_s(\varphi) = \{y \in Y \mid \text{rk}(\varphi_y) \leq s\}$$

are the degeneracy loci of φ . But locally φ is given by the diagonal matrix

$$\text{Diag}(x_1, \dots, x_r, 1, \dots, 1)$$

so this condition is verified by a simple calculation. \square

REMARK 1.4.15. It is again an immediate check that the associated filtered differential complex $\widehat{\text{DR}}(\omega_Y(*E))$ is precisely the well-known de Rham complex of holomorphic forms with log poles along E , namely

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^1(\log E) \xrightarrow{d} \cdots \xrightarrow{d} \omega_Y(E) \longrightarrow 0.$$

1.5. Push-forward of filtered \mathcal{D} -modules

We want to enhance the definition of push-forward of \mathcal{D} -modules to the filtered setting, following a construction due to Saito [Sa1, §2.1-2.3]. We do this in the setting of right \mathcal{D} -modules. The usual left-right transformation allows us to recover the corresponding construction for left \mathcal{D} -modules.

Let $f: Y \rightarrow X$ be a morphism of smooth complex varieties. Recall that the associated transfer module

$$\mathcal{D}_{Y \rightarrow X} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$$

has the structure of a $(\mathcal{D}_Y, f^{-1}\mathcal{D}_X)$ -bimodule, and is used to define the push-forward functor at the level of derived categories by the formula

$$f_+ : \mathbf{D}(\mathcal{D}_Y) \longrightarrow \mathbf{D}(\mathcal{D}_X), \quad \mathcal{M}^\bullet \mapsto \mathbf{R}f_* (\mathcal{M}^\bullet \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}).$$

Here we loosely use the symbol $\mathbf{D}(\mathcal{D}_X)$ to stand for $\mathbf{D}_{\text{qcoh}}^*(\mathcal{D}_X)$, where $*$ can be any either absent, or any of $-$, $+$ or b for instance; recall that all the \mathcal{D} -modules we work with are assumed to be quasi-coherent. If f is proper, which is often our focus, this induces a functor

$$f_+ : \mathbf{D}_{\text{coh}}^b(\mathcal{D}_Y) \longrightarrow \mathbf{D}_{\text{coh}}^b(\mathcal{D}_X)$$

between the bounded derived categories of coherent \mathcal{D} -modules.

Note furthermore that $\mathcal{D}_{Y \rightarrow X}$ has a natural filtration given by $f^*F_k\mathcal{D}_X$. More precisely, the sheaf $f^{-1}\mathcal{D}_X$ carries a filtration induced by the standard filtration on \mathcal{D}_X .

By analogy with the previous section, we considered the categories $\text{FM}(f^{-1}\mathcal{D}_X)$ and $\text{FM}_i(f^{-1}\mathcal{D}_X)$ of filtered $f^{-1}\mathcal{D}_X$ -modules and filtered *induced* $f^{-1}\mathcal{D}_X$ -modules respectively, where the latter are isomorphic to filtered $f^{-1}\mathcal{D}_X$ -modules of the form $\mathcal{G} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X$, with \mathcal{G} a filtered $f^{-1}\mathcal{O}_X$ -module.

We define the functor

$$\text{DR}_{Y/X} : \text{FM}_i(\mathcal{D}_Y) \rightarrow \text{FM}_i(f^{-1}\mathcal{D}_X), \quad (\mathcal{M}, F) \mapsto (\mathcal{M}, F) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}.$$

This is indeed well defined, since if $\mathcal{M} = \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$, with \mathcal{G} a filtered \mathcal{O}_Y -module, then we have

$$\text{DR}_{Y/X}(\mathcal{M}) \simeq \mathcal{G} \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}_X.$$

with filtration given, as for \mathcal{M} , by the tensor product filtration

$$F_k \mathrm{DR}_{Y/X}(\mathcal{M}) := \sum_{i=0}^k F_{k-i} \mathcal{G} \otimes_{f^{-1}\mathcal{O}_X} f^{-1} F_i \mathcal{D}_X.$$

EXERCISE 1.5.1. Show that $\mathrm{DR}_{Y/X}$ takes filtered quasi-isomorphisms of complexes in $\mathrm{FM}_i(\mathcal{D}_Y)$ to filtered quasi-isomorphisms of complexes in $\mathrm{FM}_i(f^{-1}\mathcal{D}_X)$.

We next use the definitions and notation on derived categories discussed in Remark 1.4.9. Given the exercise above, we have an induced functor

$$\mathrm{DR}_{Y/X}: \mathbf{D}(\mathrm{FM}_i(\mathcal{D}_Y)) \rightarrow \mathbf{D}(\mathrm{FM}_i(f^{-1}\mathcal{D}_X)).$$

In combination with Proposition 1.4.12 and its obvious analogue, we can in turn see this as a functor

$$\mathrm{DR}_{Y/X}: \mathbf{D}(\mathrm{FM}(\mathcal{D}_Y)) \rightarrow \mathbf{D}(\mathrm{FM}(f^{-1}\mathcal{D}_X)).$$

We next define a direct image functor

$$\mathbf{D}(\mathrm{FM}(f^{-1}\mathcal{D}_X)) \rightarrow \mathbf{D}(\mathrm{FM}(\mathcal{D}_X)),$$

which composed with $\mathrm{DR}_{Y/X}$ will give rise to our desired filtered direct image functor

$$f_+: \mathbf{D}(\mathrm{FM}(\mathcal{D}_Y)) \rightarrow \mathbf{D}(\mathrm{FM}(\mathcal{D}_X)).$$

DEFINITION 1.5.2. Let (\mathcal{M}, F) be a filtered $f^{-1}\mathcal{D}_X$ -module. Its *topological direct image* is defined as $f_*(\mathcal{M}, F) = (\mathcal{N}, F)$, where

$$\mathcal{N} := \bigcup_{k \in \mathbf{Z}} f_* F_k \mathcal{M} \subseteq f_* \mathcal{M} \quad \text{and} \quad F_k \mathcal{N} := f_* F_k \mathcal{M}.$$

Here on the right hand side we use the standard sheaf-theoretic direct image.

REMARK 1.5.3. This definition can be made in great generality, and usually it is not necessarily the case that $\mathcal{N} = f_* \mathcal{M}$. However this is always true in the case we are interested in, namely the case of algebraic varieties (since every open set is quasi-compact), and also in the case of complex analytic varieties if f is proper.

We therefore obtain a functor

$$f_*: \mathrm{FM}(f^{-1}\mathcal{D}_X) \rightarrow \mathrm{FM}(\mathcal{D}_X).$$

We would like to extend this functor to the derived category $\mathbf{D}(\mathrm{FM}(f^{-1}\mathcal{D}_X))$, in order to finish our construction.

First recall that to every module \mathcal{M} over a sheaf of rings on Y , in particular over $f^{-1}\mathcal{D}_X$, we can associate the flasque sheaf of discontinuous sections $\mathcal{S}^0(\mathcal{M})$ defined on every open set $U \subseteq Y$ by

$$\Gamma(U, \mathcal{S}^0(\mathcal{M})) = \prod_{x \in U} \mathcal{M}_x,$$

and we have a functorial inclusion $\mathcal{M} \hookrightarrow \mathcal{S}^0(\mathcal{M})$.

Let's now consider a filtered version. To $(\mathcal{M}, F) \in \text{FM}(f^{-1}\mathcal{D}_X)$ we associate $\mathcal{I}^0(\mathcal{M}, F)$, namely the filtered $f^{-1}\mathcal{D}_X$ -module (\mathcal{N}, F) given by

$$\mathcal{N} := \bigcup_{k \in \mathbf{Z}} \mathcal{I}^0(F_k \mathcal{M}) \subseteq \mathcal{I}^0(\mathcal{M}) \quad \text{and} \quad F_k \mathcal{N} := \mathcal{I}^0(F_k \mathcal{M}).$$

We have a filtered inclusion $(\mathcal{M}, F) \xrightarrow{i} \mathcal{I}^0(\mathcal{M}, F)$, and we define

$$\mathcal{I}^1(\mathcal{M}, F) := \text{Coker}(i).$$

Continuing in this fashion, we obtain a complex

$$0 \rightarrow \mathcal{I}^0(\mathcal{M}, F) \rightarrow \mathcal{I}^1(\mathcal{M}, F) \rightarrow \dots$$

in $\text{C}^+(\text{FM}(f^{-1}\mathcal{D}_X))$ which is filtered quasi-isomorphic to (\mathcal{M}, F) , and consequently a functor

$$\mathcal{I}^\bullet: \text{FM}(f^{-1}\mathcal{D}_X) \rightarrow \text{C}^+(\text{FM}(f^{-1}\mathcal{D}_X)).$$

Note now that in the context we are considering, basic properties of higher direct images tell us that there exists an integer $N > 0$ such that for every sheaf \mathcal{F} of abelian groups on Y we have $R^i f_* \mathcal{F} = 0$ for $i > N$. (In our setting of algebraic varieties we can in fact take $N = \dim Y$.) We modify our resolution by taking

$$\widetilde{\mathcal{I}}^j(\mathcal{M}, F) = \mathcal{I}^j(\mathcal{M}, F) \quad \text{for } j \leq N, \quad \widetilde{\mathcal{I}}^j(\mathcal{M}, F) = 0 \quad \text{for } j > N + 1,$$

and

$$\widetilde{\mathcal{I}}^{N+1}(\mathcal{M}, F) = \text{Coker}(\mathcal{I}^{N-1}(\mathcal{M}, F) \rightarrow \mathcal{I}^N(\mathcal{M}, F)).$$

We thus obtain a *finite* resolution

$$\widetilde{\mathcal{I}}^\bullet(\mathcal{M}, F): \quad 0 \rightarrow \widetilde{\mathcal{I}}^0(\mathcal{M}, F) \rightarrow \widetilde{\mathcal{I}}^1(\mathcal{M}, F) \rightarrow \dots \rightarrow \widetilde{\mathcal{I}}^{N+1}(\mathcal{M}, F) \rightarrow 0$$

of (\mathcal{M}, F) with filtered sheaves having the same properties as those in $\mathcal{I}^\bullet(\mathcal{M}, F)$. Moreover, all the entries $\widetilde{\mathcal{I}}^j(\mathcal{M}, F)$ are *filtered f -acyclic* in the sense that

$$R^i f_*(F_k \widetilde{\mathcal{I}}^j(\mathcal{M}, F)) = 0 \quad \text{for all } k \in \mathbf{Z}, \quad i > 0.$$

This follows by construction and the assumption on N (exercise!).

We can extend this construction to complexes. If $C^\bullet = (\mathcal{M}^\bullet, F_\bullet \mathcal{M}^\bullet)$ is an object in $\text{C}(\text{FM}(f^{-1}\mathcal{D}_X))$, we can form the double complex $(\widetilde{\mathcal{I}}^p(\mathcal{M}^q, F_\bullet \mathcal{M}^q))_{p,q}$, and define $\widetilde{\mathcal{I}}^\bullet(C^\bullet)$ to be the total complex of this double complex. We thus have a functor

$$\widetilde{\mathcal{I}}^\bullet: \text{C}(\text{FM}(f^{-1}\mathcal{D}_X)) \rightarrow \text{C}(\text{FM}(f^{-1}\mathcal{D}_X))$$

such that C^\bullet is filtered quasi-isomorphic to $\widetilde{\mathcal{I}}^\bullet(C^\bullet)$, with $-$, $+$ and bounded versions.

Finally, this allows us to define the exact functor of triangulated categories we are interested in, as

$$\mathbf{R}f_*: \mathbf{D}(\text{FM}(f^{-1}\mathcal{D}_X)) \rightarrow \mathbf{D}(\text{FM}(\mathcal{D}_X)), \quad \mathbf{R}f_* C^\bullet := f_*(\widetilde{\mathcal{I}}^\bullet(C^\bullet)).$$

This functor is well defined thanks to the following

EXERCISE 1.5.4. Show that if $A^\bullet \rightarrow B^\bullet$ is a filtered quasi-isomorphism in the category $\mathbf{C}(\mathrm{FM}(f^{-1}\mathcal{D}_X))$, then the induced $f_*(\widetilde{\mathcal{F}}^\bullet(A^\bullet)) \rightarrow f_*(\widetilde{\mathcal{F}}^\bullet(B^\bullet))$ is a filtered quasi-isomorphism as well.

The following property is a direct consequence of the definition and of filtered f -acyclicity:

COROLLARY 1.5.5. *If C^\bullet represents an object in $\mathbf{D}(\mathrm{FM}(f^{-1}\mathcal{D}_X))$, then*

$$\mathcal{H}^i F_k(\mathbf{R}f_* C^\bullet) \simeq R^i f_*(F_k C^\bullet).$$

Note that this allows us to obtain the filtration on each $R^i f_* C^\bullet$ as follows:

$$F_k R^i f_* C^\bullet = \mathrm{Im}[\mathcal{H}^i F_k(\mathbf{R}f_* C^\bullet) \rightarrow \mathcal{H}^i(\mathbf{R}f_* C^\bullet)] = \mathrm{Im}[R^i f_*(F_k C^\bullet) \rightarrow R^i f_* C^\bullet].$$

It is however not necessarily the case that this last map is injective, and therefore the filtration is in general not simply given by $R^i f_*(F_k C^\bullet)$. That this is actually true for those filtered \mathcal{D} -modules that underlie Hodge modules is a deep property of Hodge-theoretic flavor that we will discuss in the next section.

Finally, as mentioned above, composing $\mathbf{R}f_*$ with $\mathrm{DR}_{Y/X}$, we obtain the desired filtered direct image functor

$$f_+ : \mathbf{D}(\mathrm{FM}(\mathcal{D}_Y)) \rightarrow \mathbf{D}(\mathrm{FM}(\mathcal{D}_X)).$$

If f is proper, this induces a functor

$$f_+ : \mathbf{D}_{\mathrm{coh}}^b(\mathrm{FM}(\mathcal{D}_Y)) \rightarrow \mathbf{D}_{\mathrm{coh}}^b(\mathrm{FM}(\mathcal{D}_X)).$$

REMARK 1.5.6. It is immediate from the definitions that if we forget the filtration, this functor coincides with the usual direct image functor f_+ on the derived category of \mathcal{D}_X -modules recalled at the beginning of this section.

EXAMPLE 1.5.7. A case when filtered push-forward is quite simple is that of closed embeddings, where the functor $\mathbf{R}f_*$ described above is acyclic. Let's assume for simplicity that $i: Y \hookrightarrow X$ is the embedding of a smooth hypersurface given locally by $(t = 0)$, and let $(\mathcal{M}, F) \in \mathrm{FM}(\mathcal{D}_Y)$. (Similar formulas hold for an arbitrary closed embedding.) Analogously to Example 1.3.5 in the left \mathcal{D} -module setting, we have that $\mathcal{D}_{Y \rightarrow X} \simeq \mathcal{D}_Y \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$, so

$$i_+ \mathcal{M} \simeq \mathcal{M} \otimes_{\mathbf{C}} \mathbf{C}[\partial_t],$$

while the filtration is simply given by the convolution filtration, which is easily seen to be expressed as

$$F_k i_+ \mathcal{M} = \sum_{i \geq 0} F_{k-i} \mathcal{M} \otimes \partial_t^i.$$

Note that according to the left-right rule for \mathcal{D} -modules, if we start with a filtered left \mathcal{D}_Y -module (\mathcal{N}, F) , then the analogous formula should be

$$F_k i_+ \mathcal{N} = \sum_{i \geq 0} F_{k-i-1} \mathcal{N} \otimes \partial_t^i,$$

as on the left hand side we are shifting the filtration by $n = \dim X$, while on the right hand side by $n - 1 = \dim Y$.

1.6. Strictness

A special property that is crucial in the theory of filtered \mathcal{D} -modules underlying Hodge modules is the strictness of the filtration.

DEFINITION 1.6.1. Let $f : (\mathcal{M}, F) \rightarrow (\mathcal{N}, F)$ be a morphism of filtered \mathcal{D}_X -modules. Then f is called *strict* if

$$f(F_k \mathcal{M}) = F_k \mathcal{N} \cap f(\mathcal{M}) \quad \text{for all } k.$$

Similarly, a complex of filtered \mathcal{D}_X -modules $(\mathcal{M}^\bullet, F_\bullet \mathcal{M}^\bullet)$ is called strict if all of its differentials are strict. Via a standard argument, the notion of strictness makes sense more generally for objects in the derived category $\mathbf{D}(\text{FM}(\mathcal{D}_X))$ of filtered \mathcal{D}_X -modules.

Note that it is only in the case of a strict complex that the cohomologies of \mathcal{M}^\bullet can also be seen as filtered \mathcal{D}_X -modules with the induced filtration. An equivalent interpretation is given by the following:

EXERCISE 1.6.2. The complex $(\mathcal{M}^\bullet, F_\bullet \mathcal{M}^\bullet)$ is strict if and only if, for every $i, k \in \mathbf{Z}$, the induced morphism

$$\mathcal{H}^i F_k \mathcal{M}^\bullet \longrightarrow \mathcal{H}^i \mathcal{M}^\bullet$$

is injective.

As a preview, a crucial property of the filtered \mathcal{D} -modules underlying Hodge modules will be the following. If $f : Y \rightarrow X$ is a proper morphism of smooth varieties, and (\mathcal{M}, F) is one such filtered \mathcal{D}_Y -module, then $f_+(\mathcal{M}, F)$ is strict as an object in $\mathbf{D}(\text{FM}(\mathcal{D}_X))$; here f_+ is the filtered direct image functor discussed in the previous section. By the Exercise above, this means that

$$\mathcal{H}^i F_k f_+(\mathcal{M}, F) \rightarrow \mathcal{H}^i f_+(\mathcal{M}, F)$$

is injective for all integers i and k . By Corollary 1.5.5 and the discussion right after, this is equivalent to the injectivity of the natural morphism

$$(1.6.1) \quad R^i f_* (F_k(\mathcal{M} \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X})) \rightarrow R^i f_* (\mathcal{M} \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}).$$

Moreover, the image of this morphism, isomorphic to the term on the left hand side, is the term $F_k \mathcal{H}^i f_+(\mathcal{M}, F)$.

In conclusion, in the strict case the cohomologies of direct images of filtered \mathcal{D} -modules are themselves filtered \mathcal{D} -modules with the induced filtration, and it will sometimes be possible to have a reasonably good grasp of the filtration on such direct images.

EXAMPLE 1.6.3 (Absolute case). The absolute case gives a good idea of the meaning of strictness, and how it is natural in Hodge theory. In this context it can be seen as a generalization of the degeneration at E_1 of the classical Hodge-to-de Rham spectral sequence. Concretely, let Y be a smooth variety, and (\mathcal{M}, F) a filtered (say regular, holonomic) \mathcal{D} -module on Y . The natural inclusion of complexes $F_k \text{DR}(\mathcal{M}) \hookrightarrow \text{DR}(\mathcal{M})$ induces, after passing to cohomology, a morphism

$$\varphi_{k,i} : H^i(Y, F_k \text{DR}(\mathcal{M})) \longrightarrow H^i(Y, \text{DR}(\mathcal{M})).$$

Now for the constant map $f: Y \rightarrow \text{pt}$, the definition of pushforward gives

$$f_+ \mathcal{M} \simeq \mathbf{R}\Gamma(Y, \text{DR}(\mathcal{M})),$$

and by the discussion above, the image of $\varphi_{k,i}$ is $F_k H^i(Y, \text{DR}(\mathcal{M}))$. The strictness of $f_+(\mathcal{M}, F)$ is therefore equivalent to the injectivity of $\varphi_{k,i}$ for all k and i , which is in turn equivalent to

$$\text{gr}_k^F H^i(Y, \text{DR}(\mathcal{M})) \simeq \mathbf{H}^i(Y, \text{gr}_k^F \text{DR}(\mathcal{M})).$$

Note that on the right hand side we have the hypercohomology of a complex of coherent \mathcal{O}_Y -modules, while on the left hand side the (associated graded of the) cohomology of the perverse sheaf $\text{DR}(\mathcal{M})$.

On the other hand, in our setting the spectral sequence of a filtered complex takes the form

$$E_1^{p,q} = \mathbf{H}^{p+q}(Y, \text{gr}_{-q}^F \text{DR}(\mathcal{M})) \implies H^{p+q}(Y, \text{DR}(\mathcal{M})).$$

It is then standard to check that the translated strictness condition above is equivalent to the E_1 -degeneration of this spectral sequence.

For instance, when Y is projective and $\mathcal{M} = \mathcal{O}_Y$, corresponding to the trivial VHS \mathbf{Q}_Y , then this is the degeneration of the classical Hodge-to-de Rham spectral sequence; see Example 1.2.2. As mentioned above, this property extends to the filtered \mathcal{D} -modules associated to (polarized) Hodge modules on Y , and to their push-forwards via proper morphisms.

CHAPTER 2

The Bernstein-Sato polynomial

The aim of this chapter is to introduce and study the Bernstein-Sato polynomial of a regular function, and explain its connection to invariants in birational geometry. We use freely the basics of the theory of \mathcal{D} -modules, for instance as in [HTT].

2.1. Push-forward via a graph embedding

Let X be a smooth complex variety of dimension n , and $f \in \mathcal{O}_X$ an arbitrary nontrivial function. Many \mathcal{D} -module constructions that depend on f and will be studied below (like the V -filtration along f , or the associated nearby and vanishing cycles) are much easier to perform when the zero locus of f is smooth. For arbitrary f , one usually reduces to this case using the following standard construction. Let

$$\iota: X \hookrightarrow X \times \mathbf{C}, \quad x \mapsto (x, f(x))$$

be the closed embedding given by the graph of f . Denote $Y = X \times \mathbf{C}$, and let t be the coordinate on the second factor \mathbf{C} , so that $(t = 0)$ is the smooth hypersurface $X \times \{0\}$ in Y . For a \mathcal{D}_X -module \mathcal{M} , we consider the \mathcal{D} -module theoretic direct image

$$\iota_+ \mathcal{M} := \mathcal{M} \otimes_{\mathbf{C}} \mathbf{C}[\partial_t].$$

Recall in particular that we think of the structure sheaf \mathcal{O}_X as the (trivial) left \mathcal{D}_X -module obtained from the standard action of differential operators on functions. Furthermore, to the effective divisor $D = (f = 0)$ on X we can associate the left \mathcal{D}_X -module $\mathcal{O}_X(*D)$ of functions with poles of arbitrary order along D ; this is of course nothing but the localization $\mathcal{O}_X[\frac{1}{f}]$, with the obvious action of differential operators.

It will be especially important to have a good understanding of the \mathcal{D}_Y -modules $\iota_+ \mathcal{O}_X$ and $\iota_+ \mathcal{O}_X(*D)$, including a useful construction due to Malgrange. We focus on this next.

EXERCISE 2.1.1. (1) Show that the action of t on $\iota_+ \mathcal{O}_X$ is injective, so that we have an embedding

$$\iota_+ \mathcal{O}_X \hookrightarrow (\iota_+ \mathcal{O}_X)_t.$$

(2) Show that $(\iota_+ \mathcal{O}_X)_t = \iota_+ \mathcal{O}_X(*D)$. (Hence $\iota_+ \mathcal{O}_X$ naturally embeds in $\iota_+ \mathcal{O}_X(*D)$.)

LEMMA 2.1.2. *There is an isomorphism*

$$\iota_+ \mathcal{O}_X \simeq \mathcal{O}_X[t]_{f-t} / \mathcal{O}_X[t].$$

PROOF. Recall that by definition we have $\iota_+ \mathcal{O}_X \simeq \mathcal{O}_X \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$. If δ denotes the class of $\frac{1}{f-t}$ in $\iota_+ \mathcal{O}_X$, the claim is that every element in $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]$ can be written uniquely as

$$\sum_{j \geq 0} h_j \partial_t^j \delta,$$

with $h_j \in \mathcal{O}_X$, only finitely many of these being nonzero. Then the asserted isomorphism is clear, as such an element can be identified with $\sum_{j \geq 0} h_j \otimes \partial_t^j \in \iota_+ \mathcal{O}_X$.

Uniqueness follows from the fact that for any $k \geq 0$ the elements

$$1, \frac{1}{f-t}, \dots, \frac{1}{(f-t)^k}$$

are linearly independent over \mathcal{O}_X . On the other hand, for existence note that

$$\partial_t^j \delta = j! \cdot \frac{1}{(f-t)^{j+1}},$$

while every element in $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]$ is the class of

$$\frac{g_1}{f-t} + \dots + \frac{g_k}{(f-t)^k}$$

for some $k \geq 1$, and $g_1, \dots, g_k \in \mathcal{O}_X$. □

REMARK 2.1.3. In $\iota_+ \mathcal{O}_X$ we have the useful formula

$$(2.1.1) \quad f\delta = t\delta.$$

EXERCISE 2.1.4. In the interpretation given by Lemma 2.1.2, the \mathcal{D}_Y -module structure on $\iota_+ \mathcal{O}_X$ is given by:

- (1) $g \cdot (h\partial_t^j \delta) = (gh)\partial_t^j \delta$, for $g \in \mathcal{O}_X$.
- (2) $t \cdot (h\partial_t^j \delta) = (fh)\partial_t^j \delta - jh\partial_t^{j-1} \delta$.
- (3) $\partial_t \cdot (h\partial_t^j \delta) = h\partial_t^{j+1} \delta$.
- (4) $D(h\partial_t^j \delta) = D(h)\partial_t^j \delta - (D(f)h)\partial_t^{j+1} \delta$.

Even when studying constructions on $\iota_+ \mathcal{O}_X$, it will be convenient to work in the larger \mathcal{D} -module $\iota_+ \mathcal{O}_X(*D)$. The advantage it provides is that multiplication by t on it is bijective.

More generally let's consider for the next few paragraphs a \mathcal{D}_X -module \mathcal{M} on which *multiplication by f is bijective*; in other words, \mathcal{M} has a natural structure of $\mathcal{O}_X(*D)$ -module.

LEMMA 2.1.5. *Under this hypothesis, multiplication by t is bijective on $\iota_+ \mathcal{M}$.*

PROOF. We consider on $\iota_+ \mathcal{M}$ the filtration given by

$$G_p = G_p \iota_+ \mathcal{M} := \bigoplus_{j=0}^p \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X \partial_t^j \delta.$$

By Exercise 2.1.4(2), multiplication by t preserves this filtration; moreover, for every $p \geq 0$, via the obvious isomorphism $G_p/G_{p-1} \simeq \mathcal{M}$, multiplication by t gets identified with multiplication by f . We thus obtain by induction on p the fact that multiplication by t on G_p is an isomorphism, which gives the conclusion as G_\bullet is exhaustive. \square

We now give the main construction. Let $\mathcal{D}\langle t, s \rangle$ be the subsheaf of $\mathcal{D}_{X \times \mathbf{C}}$ generated by \mathcal{D}_X , t , and $s = -\partial_t t$. Note that t and s satisfy $st = t(s-1)$ and more generally

$$(2.1.2) \quad P(s)t = tP(s-1) \quad \text{for all } P \in \mathbf{C}[s].$$

We also consider the localization $\mathcal{D}_X\langle t, t^{-1}, s \rangle = \mathcal{D}_X\langle t, t^{-1}, \partial_t \rangle$ of $\mathcal{D}\langle t, s \rangle$. (This is the push-forward of the sheaf of differential operators from $X \times \mathbf{C}^*$ to $X \times \mathbf{C}$.) Note that in this ring we have $\partial_t = -st^{-1}$ and from (2.1.2) we obtain

$$(2.1.3) \quad t^{-1}P(s) = P(s-1)t^{-1} \quad \text{for all } P \in \mathbf{C}[s].$$

A $\mathcal{D}\langle t, t^{-1}, s \rangle$ -module is simply a $\mathcal{D}_{X \times \mathbf{C}}$ -module on which t acts bijectively.

We consider the $\mathcal{D}_X\langle t, t^{-1}, s \rangle$ -module $\mathcal{M}[s]f^s$ defined as follows. As an \mathcal{O}_X -module, we have an isomorphism

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[s] \simeq \mathcal{M}[s]f^s, \quad u \otimes s^j \rightarrow us^j f^s.$$

The symbol f^s is formal, and motivates the \mathcal{D}_X -action: a derivation D in $\text{Der}_{\mathbf{C}}(\mathcal{O}_X)$ acts by

$$D \cdot us^j f^s = \frac{D(f)}{f} us^{j+1} f^s + D(u)s^j f^s.$$

The action of s on $\mathcal{M}[s]f^s$ is the obvious one, while the action of t is given by the automorphism “ $s \rightarrow s+1$ ”, that is

$$us^j f^s \rightarrow fu(s+1)^j f^s.$$

Recall that in this language s corresponds to $-\partial_t t$; note also that f^s corresponds to δ .

Next comes the main technical point, observed in [Ma]. To state it, for every $i \geq 0$, we set

$$Q_i(x) = i! \cdot \binom{x+i-1}{i} := \prod_{j=0}^{i-1} (x+j) \in \mathbf{Z}[x]$$

(with the convention $Q_0 = 1$).

PROPOSITION 2.1.6. *If \mathcal{M} is a \mathcal{D}_X -module on which f acts bijectively, then we have an isomorphism of $\mathcal{D}_X\langle t, t^{-1}, s \rangle$ -modules*

$$\varphi: \mathcal{M}[s]f^s \xrightarrow{\simeq} \iota_+ \mathcal{M}, \quad us^j f^s \rightarrow u \otimes (-\partial_t t)^j \delta.$$

The inverse isomorphism ψ is given by

$$u \otimes \partial_t^j \delta \rightarrow \frac{u}{f^j} Q_j(-s) f^s.$$

PROOF. It is straightforward to check that φ and ψ are $\mathcal{D}_X\langle t, t^{-1}, s \rangle$ -linear. In order to see that they are isomorphisms, consider on $\mathcal{M}[s]f^s$ and $\iota_+\mathcal{M}$ the filtrations given by

$$G_p\mathcal{M}[s]f^s = \bigoplus_{i=0}^p \mathcal{M}s^i f^s \quad \text{and} \quad G_p\iota_+\mathcal{M} = \bigoplus_{j=0}^p \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X \partial_t^j \delta.$$

It is clear that φ and ψ preserve the filtrations. Note now that $(\partial_t t)^p \delta$ is equal to $f^p \partial_t^p \delta$ plus a sum of monomials which have lower degree in ∂_t . Moreover, we have canonical isomorphisms

$$G_p\mathcal{M}[s]f^s / G_{p-1}\mathcal{M}[s]f^s \simeq \mathcal{M} \simeq G_p\iota_+\mathcal{M} / G_{p-1}\iota_+\mathcal{M}.$$

and so the corresponding endomorphism of \mathcal{M} induced by φ is given by multiplication with $(-1)^p f^p$. Since this is an isomorphism, we conclude by induction on p that each induced map $G_p\mathcal{M}[s]f^s \rightarrow G_p\iota_+\mathcal{M}$ is an isomorphism, and hence so is φ .

The formula for the inverse isomorphism ψ follows if we show that in $\iota_+\mathcal{O}_X(*D)$ we have

$$Q_j(\partial_t t)\delta = f^j \partial_t^j \delta \quad \text{for all } j \geq 0.$$

We argue by induction on j , the case $j = 0$ being obvious. Assuming the formula for some j , we apply (2.1.3) and the fact that $\partial_t = -st^{-1}$ to write

$$f^{j+1} \partial_t^{j+1} \delta = f \partial_t Q_j(-s)\delta = -fst^{-1} Q_j(-s)\delta = f(-s) Q_j(-s-1)t^{-1} \delta = Q_{j+1}(-s)\delta.$$

This completes the proof of the proposition. \square

2.2. Definition, existence, and examples

In this section X is a smooth variety over \mathbf{C} (or more generally over a field of characteristic 0), and f is a non-invertible regular function on X . The following theorem was proved by Bernstein when f is a polynomial, and by Björk and Kashiwara in general.¹

THEOREM 2.2.1. *There exists a polynomial $b(s) \in \mathbf{C}[s]$, and a polynomial $P(s) \in \mathcal{D}_X[s]$ whose coefficients are differential operators on X , such that the relation*

$$P(s)f^{s+1} = b(s) \cdot f^s$$

holds formally in the \mathcal{D}_X -module $\mathcal{O}_X[\frac{1}{f}, s] \cdot f^s$. (Here f^{s+1} stands for $f \cdot f^s$.)

REMARK 2.2.2. Recall that we have discussed the \mathcal{D} -module in the statement of the Theorem in §2.1 above; the action of derivations on X on its elements is given by

$$D(wf^s) = \left(D(w) + sw \frac{D(f)}{f} \right) f^s.$$

It carries an obvious action of s , as part of its $\mathcal{D}_X\langle t, t^{-1}, s \rangle$ -module structure.

In the case of polynomials $f \in \mathbf{C}[X_1, \dots, X_n]$, the proof of this theorem is a simple application of the fact that $\mathcal{O}_{\mathbf{C}^n}[\frac{1}{f}](s) \cdot f^s$ is holonomic as a module over the Weyl algebra $A_n(\mathbf{C}(s)) \simeq A_n(\mathbf{C})(s)$ associated to the field $\mathbf{C}(s)$. This is proved using the Bernstein filtration on A_n ; see [Co, Ch.10, §3]. In the general case this filtration is not available, but we follow a similar approach replacing it by general properties of holonomic \mathcal{D} -modules.

¹They also proved it in the case of germs of holomorphic functions on complex manifolds.

PROOF OF THEOREM 2.2.1. We denote by $\mathbf{C}(s)$ the field of rational functions in the variable s . We also denote by $j: U \hookrightarrow X$ the natural inclusion of $U = X \setminus Z(f)$. Via base field extension we can consider X and U as being defined over $\mathbf{C}(s)$; we use the notation

$$X_s := X \times_{\text{Spec } \mathbf{C}} \text{Spec } \mathbf{C}(s) \quad \text{and} \quad U_s := U \times_{\text{Spec } \mathbf{C}} \text{Spec } \mathbf{C}(s),$$

with the corresponding inclusion $j_s: U_s \hookrightarrow X_s$. We then have

$$\mathcal{D}_{U_s} \simeq \mathcal{D}_U \times_{\mathbf{C}} \mathbf{C}(s) \quad \text{and} \quad \mathcal{D}_{X_s} \simeq \mathcal{D}_X \times_{\mathbf{C}} \mathbf{C}(s).$$

Thinking of f^s as a formal symbol as before, we now consider the \mathcal{D}_{U_s} -module $\mathcal{M} := \mathcal{O}_{U_s} \cdot f^s$, where the action of a derivation on X is given by

$$D(gf^s) = \left(D(g) + sg \frac{D(f)}{f} \right) f^s.$$

(Note that f is invertible on U_s .) Note that \mathcal{M} is a holonomic \mathcal{D}_{U_s} -module. Indeed, by analogy with the trivial filtration on \mathcal{O}_{U_s} , on \mathcal{M} we can consider the filtration given by

$$F_k \mathcal{M} = \mathcal{O}_{U_s} \cdot f^s \quad \text{for } k \geq 0 \quad \text{and} \quad F_k \mathcal{M} = 0 \quad \text{for } k < 0.$$

This is a good filtration such that

$$\text{Ch}(\mathcal{M}) = \text{Ch}(\mathcal{O}_U)_s = (T_U^* U)_s,$$

the scalar extension of the zero section of T^*U , hence holonomicity is clear. Now the main claim is:

Claim. The \mathcal{D}_{X_s} -submodule

$$\mathcal{N} := \mathcal{D}_{X_s} f^s \subseteq j_{s+} \mathcal{M}$$

is *holonomic*.

In order to show this, we note that the construction of a maximal holonomic submodule of a finitely generated \mathcal{D} -module works over arbitrary base fields, and is functorial (GIVE REFERENCE). Hence there exists a maximal holonomic submodule $\mathcal{N}' \subseteq \mathcal{N}$, compatible with restriction to open sets. Note now that

$$\mathcal{N}|_{U_s} \subseteq \mathcal{M} = \mathcal{O}_{U_s} \cdot f^s.$$

By the observation above, we conclude that $\mathcal{N}|_{U_s}$ is holonomic, hence

$$\mathcal{N}'|_{U_s} = \mathcal{N}|_{U_s}.$$

In other words, in the short exact sequence of \mathcal{D}_{X_s} -modules

$$0 \longrightarrow \mathcal{N}' \longrightarrow \mathcal{N} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

the quotient \mathcal{Q} is supported on the zero locus of f in X_s . Consequently, if we look at the section f^s of \mathcal{N} , there exists an integer $k_0 \geq 0$ such that $f^{k_0} f^s \in \mathcal{N}'$. Hence

$$\mathcal{D}_{X_s} f^{k_0} f^s \subseteq \mathcal{N}'$$

and so $\mathcal{D}_{X_s} f^{k_0} f^s$ is holonomic. Finally note that we have an isomorphism of \mathcal{D}_{X_s} -modules

$$\mathcal{D}_{X_s} f^s \xrightarrow{\simeq} \mathcal{D}_{X_s} f^{k_0} f^s, \quad P(s) f^s \mapsto P(s + k_0) f^{k_0} f^s$$

induced by the automorphism $s \mapsto s + k$ of \mathcal{D}_{X_s} .

Having established the claim, let us conclude the proof of the main statement. Consider the chain of submodules

$$\cdots \subseteq \mathcal{D}_{X_s} f^2 f^s \subseteq \mathcal{D}_{X_s} f f^s \subseteq \mathcal{D}_{X_s} f^s.$$

Since $\mathcal{D}_{X_s} f^s$ is holonomic, hence of finite length, this chain stabilizes. There exists therefore an integer $m \geq 0$ such that

$$f^m f^s \in \mathcal{D}_{X_s} f^{m+1} f^s.$$

Applying now the similar automorphism $s \mapsto s - m$, we conclude that $f^s \in \mathcal{D}_{X_s} f f^s$, and so there exists $Q(s) \in \mathcal{D}_{X_s} = \mathcal{D}_X \otimes_{\mathbf{C}} \mathbf{C}(s)$ such that

$$f^s = Q(s) f f^s.$$

Clearing denominators, this operator can be rewritten as $Q(s) = P(s)/b(s)$, where $P(s) \in \mathcal{D}_X[s]$ and $b(s) \in \mathbf{C}(s)$, so the identity becomes

$$b(s) \cdot f^s = P(s) f f^s,$$

which is what we were after. □

DEFINITION 2.2.3. The set of all polynomials $b(s)$ satisfying an identity as in Theorem 2.2.1 clearly forms an ideal in the polynomial ring $\mathbf{C}[s]$. The monic generator of this ideal is called the *Bernstein-Sato polynomial* of f , and is denoted $b_f(s)$.

There is also a local version of the Bernstein-Sato polynomial. We discuss this next, together with its relationship with the global version above.

LEMMA 2.2.4. *If x is a point in X , then there exists an open neighborhood U of x such that for any other open neighborhood V we have*

$$b_{f|U}(s) \mid b_{f|V}(s).$$

PROOF. For simplicity, let's denote $b_U(s) = b_{f|U}(s)$. Start with any open neighborhood $x \in U_0$ and assume that it does not satisfy the property we want. There exists then another neighborhood $x \in U_1$ such that b_{U_0} does not divide b_{U_1} . As $b_{U_0 \cap U_1} \mid b_{U_1}$ since we can restrict the Bernstein-Sato identity on U_1 to $U_0 \cap U_1$, and similarly for U_0 , it follows that $b_{U_0 \cap U_1}$ is a proper factor of b_{U_0} .

If the neighborhood $U_0 \cap U_1$ again does not satisfy the property in the statement, then by a similar argument there exists a neighborhood $x \in U_2$ such that $b_{U_0 \cap U_1 \cap U_2}$ is a proper factor of $b_{U_0 \cap U_1}$. Continuing this way, since b_{U_0} has finitely many factors at some point the polynomial has to stabilize, and we obtain an open set $U_0 \cap U_1 \cap \dots \cap U_r$ satisfying the assertion. □

DEFINITION 2.2.5. The *local Bernstein-Sato polynomial* of f at x is

$$b_{f,x}(s) := b_{f|U}(s),$$

where U is an open neighborhood of x as in Lemma 2.2.4.

PROPOSITION 2.2.6. *If X is affine, then the global and local Bernstein-Sato polynomials are related by the formula*

$$b_f(s) = \operatorname{lcm}_{x \in X} b_{f,x}(s).$$

In fact, let $\{U_i\}_{i \in I}$ be any open cover of X . Then

$$b_f(s) = \operatorname{lcm}_{i \in I} b_{f|_{U_i}}(s).$$

PROOF. Denoting as in the previous proof $b_{U_i} = b_{f|_{U_i}}$, since clearly $b_{U_i} \mid b_f$ for all i we have

$$b'(s) := \operatorname{lcm}_{i \in I} b_{U_i}(s) \mid b_f(s).$$

Consider now the Bernstein-Sato identity on each U_i , namely

$$b_{U_i}(s)f|_{U_i}^s = P_i(s)f|_{U_i}^{s+1} \quad \text{with} \quad P_i(s) \in \mathcal{D}_{U_i}(s).$$

Since U_i is an open set in the affine variety X , there exist for each i and operator $Q_i(s) \in \mathcal{D}_X[s]$, and $g_i \in \mathcal{O}_X(X)$, such that $g_i P_i(s) = Q_i(s)$. We then have the identity

$$g_i b_{U_i}(s)f^s = Q_i(s)f^{s+1}$$

on X , which implies that $g_i b'(s)f^s \in \mathcal{D}_X[s]f^{s+1}$. Define now

$$I = \{g \in \mathcal{O}_X(X) \mid gb'(s)f^s \in \mathcal{D}_X[s]f^{s+1}\}.$$

This is clearly an ideal in $\mathcal{O}_X(X)$, and for each $i \in I$ we have that $g_i \in I \setminus \mathfrak{m}_x$ for every $x \in U_i$. It follows that $I = (1)$, hence $b_f(s) \mid b'(s)$. \square

REMARK 2.2.7. On an arbitrary X , the result of Proposition 2.2.6 continues to hold if we think of the Bernstein-Sato polynomial as being the monic polynomial of minimal degree such that

$$b(s)f^s \in \mathcal{D}_X[s]f^{s+1}$$

in a sheaf-theoretic sense.

EXERCISE 2.2.8. If g is an invertible function on X , then $b_{gf}(s) = b_f(s)$.

REMARK 2.2.9 (Bernstein-Sato polynomials of divisors). Let D be an arbitrary effective divisor on X . For any two functions f_1, f_2 defining D on an open set $U \subset X$, there exists $g \in \mathcal{O}_X^*(U)$ such that $f_1 = f_2 g$. The results and exercise above imply then that it makes sense to define a Bernstein-Sato polynomial $b_D(s)$ associated to D , and

$$b_D(s) = \operatorname{lcm}_{x \in D} b_{f,x}(s),$$

where f is any locally defining equation for D in a neighborhood of X .

We next list a few basic facts regarding Bernstein-Sato polynomials.

REMARK 2.2.10. (1) For f invertible we could simply take $b_f(s) = 1$. This is why we restrict to f non-invertible.

(2) If f is arbitrary, then we have

$$(s+1) \mid b_f(s).$$

Indeed, take $s = -1$ in the identity in Theorem 2.2.1, to obtain

$$b_f(-1) = P(-1) \frac{1}{f}.$$

Since f is not constant, this is only possible when $b_f(-1) = 0$.

(3) If f is smooth, by a simple reduction we can assume that $f = x_1$ in local algebraic coordinates x_1, \dots, x_n . It follows that $b_f(s) = s + 1$, due to (2) and the formula

$$\partial_{x_1} x_1^{s+1} = (s+1)x_1^s.$$

The converse is also true, meaning that if $b_f(s) = (s+1)$, then f is smooth; see ??.

In the rest of the section we discuss a few standard examples.

EXAMPLE 2.2.11. (1) Let $f = x_1^2 + \dots + x_n^2$, and consider the Laplace operator $\Delta = \partial_1^2 + \dots + \partial_n^2$. A simple calculation gives

$$\Delta f^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right)f^s.$$

According to the last comment in the Remark above, we obtain that

$$b_f(s) = (s+1)\left(s + \frac{n}{2}\right).$$

We will also obtain this as a special case of the general calculation for all weighted homogeneous singularities, Theorem 2.3.4 below.

(2) Let $f = x^2 + y^3$ be a cusp in the plane. A well-known, though tedious and not easily motivated, calculation is that

$$= (s+1)\left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right)f^s,$$

and in fact

$$b_f(s) = (s+1)\left(s + \frac{5}{6}\right)\left(s + \frac{7}{6}\right).$$

This again will be implied by the general result for weighted homogeneous singularities.

(3) Let $f = \det(x_{ij})$ be the determinant of a generic matrix in $n \times n$ variables. A formula attributed to Cayley is

$$\det(\partial/\partial x_{ij}) f^{s+1} = (s+1)(s+2) \cdots (s+n) f^s.$$

We actually have $b_f(s) = (s+1)(s+2) \cdots (s+n)$, and this will also be a special case of the general result below.

(4) The case of a divisor with SNC support can also be computed explicitly. Let

$$f = x_1^{a_1} \cdots x_n^{a_n}, \quad a_i \in \mathbf{N}.$$

Note that in one variable x , and $a \geq 1$, we have the formula

$$\partial_x^a x^{as+a} = a^a \left(s+1\right) \left(s+1 - \frac{1}{a}\right) \cdots \left(s+1 - \frac{a-1}{a}\right) x^{as}.$$

Therefore using the operator

$$P = \frac{1}{\prod_{i=1}^n a_i^{a_i}} \cdot \partial_1^{a_1} \cdots \partial_n^{a_n},$$

a straightforward calculation gives

$$P f^{s+1} = b_{a_1, \dots, a_n}(s) f^s, \quad b_{a_1, \dots, a_n}(s) := \prod_{i=1}^n \left(\prod_{k=0}^{a_i-1} \left(s + 1 - \frac{k}{a_i} \right) \right)$$

and so $b_{a_1, \dots, a_n} \mid b_f$. We can in fact see that $b_f = b_{a_1, \dots, a_n}$ as follows. Starting with an expression

$$\left(\sum_{\alpha, \beta, j} a_{\alpha\beta}^j s^j \underline{x}^\alpha \underline{\partial}^\beta \right) \underline{x}^{a(s+1)} = c(s) \cdot \underline{x}^{as},$$

where we use the notation $\underline{x}^a = x_1^{a_1} \cdots x_n^{a_n}$, etc., by comparing the terms containing \underline{x}^{as} on both sides we see that the only contribution coming from the left hand side is from terms of the form $a_{\alpha\beta}^j \underline{x}^\alpha \underline{\partial}^\beta$ with $a_{\alpha\beta}^j \neq 0$ and $\beta_i = \alpha_i + a_i$.² We then have that $\beta_i \geq a_i$ for all i , hence employing the one variable formula above repeatedly we see that each $\underline{\partial}^\beta \underline{x}^{a(s+1)}$ contributes a polynomial (in s) term divisible by b_{a_1, \dots, a_n} . It follows that $b_{a_1, \dots, a_n} \mid c(s)$.

(5) If $f(x_1, \dots, x_n) = g(x_1, \dots, x_m) \cdot h(x_{m+1}, \dots, x_n)$, it is immediate to see that

$$b_f \mid b_g \cdot b_h.$$

Whether equality holds seems to be an open problem. Note that for arbitrary $f = g \cdot h$ it is very easy to produce examples where this divisibility does not hold (and there is no reason for it to do so). Consider for instance the triple point $f = xy(x+y)$ in \mathbf{C}^2 , and take $g = xy$ and $h = x+y$. Then

$$b_g(s) \cdot b_h(s) = (s+1)^3 \quad \text{while} \quad \left(s + \frac{2}{3} \right) \mid b_f(s).$$

For the last statement one can for instance use the general formula for weighted homogeneous singularities in Theorem 2.3.4, or the fact that the log canonical threshold of f is $2/3$ combined with Theorem 2.7.2 below.

(6) Hyperplane arrangements. TO ADD.

2.3. Quasi-homogeneous singularities

In this section we study an extended example, proving a general formula for the Bernstein-Sato polynomial of a quasi-homogeneous isolated singularity.³

We start by recalling a few basic notions from singularity theory. For a polynomial $f \in \mathbf{C}[X] = \mathbf{C}[X_1, \dots, X_n]$, inside the ring of convergent power series $\mathbf{C}\{X\}$ we consider

²A priori there may be terms of other type in the differential operator, but after differentiation their contributions must cancel each other.

³I thank Mingyi Zhang for giving lectures at Northwestern on this topic, which I am following here.

the associated *Jacobian ideal*

$$J(f) := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \subseteq \mathbf{C}\{X\}.$$

Similarly, the *Tjurina ideal* is

$$(f, J(f)) = \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \subseteq \mathbf{C}\{X\}.$$

The *Milnor* and *Tjurina algebras* associated to f are

$$M_f := \mathbf{C}\{X\}/J(f) \quad \text{and} \quad T_f := \mathbf{C}\{X\}/(f, J(f)),$$

while the corresponding Milnor and Tjurina numbers are

$$\mu_f := \dim_{\mathbf{C}} M_f \quad \text{and} \quad \tau_f := \dim_{\mathbf{C}} T_f.$$

If $w = (w_1, \dots, w_n) \in \mathbf{Q}_{>0}^n$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, we denote

$$|w| = w_1 + \dots + w_n \quad \text{and} \quad \langle w, \alpha \rangle = w_1 \alpha_1 + \dots + w_n \alpha_n \in \mathbf{Q}_{>0}.$$

DEFINITION 2.3.1. We say that a polynomial $f = \sum_{\alpha \in \mathbf{N}^n} a_{\alpha} X^{\alpha} \in \mathbf{C}[X]$ is *quasi-homogeneous* (or *weighted homogeneous*) of type $(w; \rho)$ if for every $\alpha \in \mathbf{N}^n$ such that $a_{\alpha} \neq 0$ we have

$$\rho(X^{\alpha}) := \langle w, \alpha \rangle = \rho.$$

We simply say that f is quasi-homogeneous or weighted-homogeneous (with respect to the weights w_1, \dots, w_n) if $\rho = 1$.

REMARK 2.3.2. Note that the weights are not invariant under linear change of coordinates.

In what follows we assume that $f(0) = 0$, and 0 is an isolated singular point of $Z(f)$. It is a well-known fact that this condition is equivalent to $\mu_f < \infty$, and also to $\tau_f < \infty$; see e.g. [GLS, Lemma 2.3]. Under this assumption, a theorem of K. Saito [KSa] states that a polynomial f is quasi-homogeneous (of any weight) after some biholomorphic change of coordinates if and only if the following equivalent conditions hold:

$$f \in J(f) \iff \mu_f = \tau_f.$$

EXAMPLE 2.3.3. (1) The most common examples of quasi-homogeneous polynomials are the diagonal hypersurfaces

$$f = X_1^{a_1} + \dots + X_n^{a_n} \in \mathbf{C}[X],$$

with weights $w_i = \frac{1}{a_i}$ for $i = 1, \dots, n$.

(2) Let $f = x^3 + xy^3 \in \mathbf{C}[X, Y]$. This is quasi-homogeneous, with weights $w_1 = \frac{1}{3}$ and $w_2 = \frac{2}{9}$. We have $\tau_f = \mu_f = 7$, and a monomial basis of the Milnor algebra seen as a \mathbf{C} -vector space is $1, x, x^2, y, xy, x^2y, y^2$.

(3) Let $f = x^5 + y^5 + x^2y^2 \in \mathbf{C}[X, Y]$. Then f is not a quasi-homogeneous polynomial. Indeed, it is not hard to compute that

$$10 = \tau_f < \mu_f = 11.$$

Let's now fix a quasi-homogeneous polynomial $f \in \mathbf{C}[X]$ with an isolated singularity, of weights w_1, \dots, w_n , and denote $\mu = \mu_f$. We write

$$M_f = \bigoplus_{i=1}^{\mu} \mathbf{C} \cdot e_i,$$

where e_i are a monomial basis for the Milnor algebra as a \mathbf{C} -vector space. We also consider the set of rational numbers

$$\Sigma = \{\rho(e_1), \dots, \rho(e_n)\},$$

where each number appears without repetitions. With this notation, the main result is the following theorem obtained in [BGM]; see also [BGMM] for an extension to the case of polynomials which are non-degenerate with respect to their Newton polygon.

THEOREM 2.3.4. *The Bernstein-Sato polynomial of f is*

$$b_f(s) = (s + 1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho).$$

REMARK 2.3.5. The statement of the theorem implies that all of the roots of $b_f(s)$ different from -1 are simple, while -1 appears with multiplicity 1 or 2.

Before proving the theorem, we discuss some preliminaries. First, with respect to this set of weights, we define the *Euler vector field* as

$$\chi := \sum_{i=1}^n w_i x_i \partial_{x_i}.$$

Quasi-homogeneity immediately implies that we have the identity

$$\chi(f) = f.$$

LEMMA 2.3.6. *For every $u \in \mathbf{C}\{X\}$ and every $\rho \in \mathbf{Q}$ we have*

$$(s + |w| + \rho)u f^s = \left(\sum_{i=1}^n w_i \partial_{x_i}(x_i u) + \rho u - \chi(u) \right) f^s.$$

PROOF. This is a simple exercise, using the identities:

$$\sum_{i=1}^n w_i \partial_{x_i} x_i = \chi + |w| \quad (\text{as operators})$$

and

$$\chi(u f^s) = s u f^s + \chi(u) f^s.$$

□

PROOF OF THEOREM 2.3.4. *Step 1.* In this first step we show that

$$b_f(s) \mid (s + 1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho).$$

By definition, this follows if we show

$$(2.3.1) \quad (s+1) \cdot \prod_{\rho \in \Sigma} (s + |w| + \rho) \in \mathcal{D}_{\mathbf{C}^n}[s]f^s.$$

This in turn follows by setting $u = 1$ in the following more general:

Claim: For a quasi-homogeneous representative u of an element in M_f , we have

$$(s+1) \prod_{\rho \in \Sigma, \rho \geq \rho(u)} (s + |w| + \rho) u f^s = \sum_{i=1}^n A_i \partial_{x_i} \cdot f^{s+1}, \quad A_i \in A_n.$$

This can be proven by descending induction on the weight $\rho(u)$. First, if $\rho(u)$ is the maximal value in Σ , then $x_i u \in J(f)$ for all i . Since $\chi(u) = \rho(u)u$, by Lemma 2.3.6 we have

$$(s + |w| + \rho) u f^s = \left(\sum_{i=1}^n w_i \partial_{x_i} (x_i u) \right) f^s.$$

Modulo $A_n \cdot J(f) \cdot f^s$, this last term is equal to

$$\sum_{i=1}^n \partial_{x_i} \cdot v_i,$$

with v_i a quasi-homogeneous element in M_f of weight $\rho(u) + w_i$.

We now assume that the Claim is true for any u as above with $\rho(u) > \nu \in \Sigma$, and take a quasi-homogeneous representative u' of an element in M_f , such that $\rho(u') = \nu$. By the inductive assumption, for each $1 \leq i \leq n$ we have

$$\prod_{\rho \geq \rho(x_i u')} (s + |w| + \rho) (x_i u') f^s \in A_n \cdot J(f) \cdot f^s.$$

Notice that the action of any polynomial in s on $A_n \cdot J(f) \cdot f^s$ stays in $A_n \cdot J(f) \cdot f^s$, because

$$s \cdot \frac{\partial f}{\partial x_i} \cdot f^s = -(1 - w_i) \partial_i f f^s + \chi \cdot \frac{\partial f}{\partial x_i} \cdot f^s.$$

We can then multiply the products above by suitable factors of the form $(s + |w| + \rho')$, to get

$$\prod_{\rho \in \Sigma, \rho > \nu} (s + |w| + \rho) (x_i u') f^s \in A_n \cdot J(f) \cdot f^s$$

for all i . Using now Lemma 2.3.6, a straightforward calculation (acting $w_i \partial_i$ on these products and taking the sum) leads to

$$\prod_{\rho \in \Sigma, \rho \geq \nu} (s + |w| + \rho) u' f^s = \sum_{i=1}^n A_i \partial_{x_i} \cdot f^s,$$

with $A_i \in A_n$. Finally, multiplying both sides by $(s+1)$ and applying the formula

$$(s+1) \partial_{x_i} f^s = \partial_{x_i} f^{s+1},$$

we obtain the Claim.

Step 2. According to Step 1, we are left with proving that for every $\rho \in \Sigma$ we have

$$\tilde{b}_f(-|w| - \rho) = 0,$$

where $\tilde{b}_f = b_f(s)/(s+1)$ is the reduced Bernstein-Sato polynomial of f .

Note first that for the operator P in the functional equation

$$P(s)f^{s+1} = b_f(s)f^s$$

we have $P(-1) \cdot 1 = 0$, hence it is easy to check that we can write

$$P(s) = (s+1)Q(s) + \sum_{i=1}^n A_i \partial_{x_i}, \quad \text{with } Q(s) \in \mathbf{C}[s] \text{ and } A_i \in A_n[s].$$

Consequently, we obtain

$$\tilde{b}_f(s)f^s = (Q(s)f + \sum_{i=1}^n A_i \partial_{x_i}) \cdot f^s.$$

Let now u be a monomial representing an element in M_f , of weight $\rho = \rho(u) \in \Sigma$. Multiplying the identity above on the left by u , we obtain

$$(2.3.2) \quad \tilde{b}_f(s) \cdot u f^s = (Q'(s)f + \sum_{i=1}^n A'_i \partial_{x_i}) \cdot f^s,$$

where $Q' = uQ$, and similarly for A_i . On the other hand, we use Lemma 2.3.6 to get

$$(2.3.3) \quad (s + |w| + \rho) \cdot u f^s = \left(\sum_{i=1}^n w_i \partial_{x_i}(x_i u) \right) f^s.$$

Let's assume now that the conclusion is false, so that the polynomials $(s + |w| + \rho)$ and $\tilde{b}_f(s)$ are coprime. In this case there exist polynomials $p(s), q(s) \in \mathbf{C}[s]$ such that

$$p(s)(s + |w| + \rho) + q(s)\tilde{b}_f(s) = 1.$$

Using (2.3.2) and (2.3.3), we then obtain

$$u f^s = \left(p(s) \left(\sum_{i=1}^n w_i \partial_{x_i}(x_i u) \right) + q(s) \left(Q'(s)f + \sum_{i=1}^n A'_i \partial_{x_i} \right) \right) f^s.$$

Recalling that $\chi(f^s) = s f^s$ (see the proof of Lemma 2.3.6) and $f \in J(f)$, a straightforward calculation shows that an operator of the form

$$R = u - \sum_{i=1}^n B_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^n \partial_{x_i} C_i, \quad B_i, C_i \in A_n$$

is in the annihilator of f^s in A_n . However the Lemma below says that R belongs to $\sum_{\alpha \in \mathbf{N}} \partial^\alpha J(f)$, and so it follows that $u \in J(f)$, which is a contradiction. \square

LEMMA 2.3.7. *For any polynomial f with an isolated singularity at the origin, the annihilator of f^s in the Weil algebra A_n is generated by the operators*

$$\frac{\partial f}{\partial x_i} \cdot \partial_{x_j} - \frac{\partial f}{\partial x_j} \cdot \partial_{x_i}, \quad 1 \leq i < j \leq n.$$

PROOF. This is [Ya, Theorem 2.19]; see also [Gr, Appendix B]. It is a consequence of the fact that $\{\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_1}\}$ form a regular sequence in the case of an isolated singularity. \square

2.4. Analytic continuation of the archimedean zeta function

We present here one of the original motivations for the the introduction of Bernstein-Sato polynomials, namely Bernstein's approach to Gel'fand's problem on the analytic continuation of complex powers of polynomials. The original problem was stated in the context of real polynomials; we will discuss its complex version.

For an introduction to what comes next, we recall the following statement from complex analysis:

EXERCISE 2.4.1. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \in \mathbb{R}^n$, and let $\varphi \in C_c^0(\mathbb{R}^n)$ be a continuous complex \mathbf{C} -valued function with compact support. Then the function

$$Z_\varphi: \mathbf{C} \rightarrow \mathbf{C}, \quad s \mapsto \int_{\mathbb{R}^n} f(x)^s \varphi(x) dx$$

is an analytic function, where the meaning of $f(x)^s$ is $e^{s \log f}$.

Let now $f \in \mathbf{C}[x_1, \dots, x_n]$ be a nonconstant polynomial, and $s \in \mathbf{C}$. We will consider $|f|^{2s}$ as a distribution depending on the complex parameter s , in the sense described below. Note first that this time f has zeros, and therefore the function $|f|^2$ does not satisfy the strict positivity condition in Exercise 2.4.1 any more.

On the other hand, it is easy to see that the function $|f(x)|^{2s}$ is continuous in x if we restrict to the case when

$$s \in \mathbb{H}_0 := \{s \in \mathbf{C} \mid \operatorname{Re}(s) > 0\}.$$

For any continuous \mathbf{C} -valued function with compact support $\varphi \in C_c^0(\mathbf{C}^n)$, we then still have a well-defined function

$$Z_\varphi: \mathbb{H}_0 \rightarrow \mathbf{C}, \quad s \mapsto \int_{\mathbf{C}^n} |f(x)|^{2s} \varphi(x) dx.$$

PROPOSITION 2.4.2. Z_φ is an analytic function on \mathbb{H} .

PROOF. *First proof.* The statement is an immediate application of the complex version of the theorem on differentiation under the integral sign; see e.g. [?]. Indeed, the function $|f(x)|^{2s} \varphi(x)$ is integrable for each $s \in \mathbb{H}_0$, and is analytic as a function of s , hence with bounded partial derivatives on the support of φ . Therefore applying $\partial/\partial \bar{z}$ commutes with the integral sign, and the statement follows.

Second proof. Let me also include a slightly more tedious second proof, which has the potentially useful advantage of providing the coefficients of a power series expansion in a neighborhood of a point in \mathbb{H}_0 .

Fix a point $s_0 \in \mathbb{H}_0$. We show that there exists $\varepsilon > 0$ such that for all $s \in \mathbb{H}$ with $|s - s_0| < \varepsilon$ we have

$$Z_\varphi(s) = \sum_{k=0}^{\infty} a_k (s - s_0)^k,$$

a convergent power series with $a_k \in \mathbf{C}$.

For $s \in \mathbb{H}_0$ close enough to s_0 , using the usual expansion of the exponential function we have

$$(2.4.1) \quad |f(x)|^{2s} \varphi(x) = \sum_{k=0}^{\infty} \frac{(2\log|f(x)|)^k |f(x)|^{2s_0} \varphi(x)}{k!} \cdot (s - s_0)^k.$$

Since φ has compact support, we can find $C > 0$ such that

$$\text{Supp}(\varphi) \subseteq K := [-C, C]^n.$$

Claim. There exist real numbers $\varepsilon > 0$ and $M > 0$ such that

$$a_k := \sup_{x \in K} \left| \frac{(2\log|f(x)|)^k |f(x)|^{2s_0}}{k!} \right| \leq M/\varepsilon^k, \quad \forall k \in \mathbf{N},$$

with the convention that the quantity we are measuring is 0 at $x \in K$ such that $f(x) = 0$. Assuming the Claim, we obtain that the series in (2.4.1) converges absolutely and uniformly on $K \times \{|s - s_0| < \delta\}$ for any $0 < \delta < \varepsilon$, hence

$$Z_\varphi(s) = \sum_{k=0}^{\infty} \left(\int_{\mathbf{C}^n} \frac{(2\log|f|)^k |f|^{2s_0} \varphi}{k!} \right) (s - s_0)^k \quad \text{for } |s - s_0| < \varepsilon,$$

which concludes the proof.

We are left with proving the Claim. Denoting $r_0 = \text{Re}(s_0) \in \mathbb{R}_{>0}$, we note that

$$k! \cdot a_k = \sup_{x \in K} |(2\log|f(x)|)^k |f(x)|^{2s_0}| = |(2\log|f(x)|)^k |f(x)|^{r_0}| = \frac{2^k}{r_0^k} |(\log|f(x)|)^k |f(x)|^{r_0}|.$$

We consider $M \in \mathbb{R}_{>0}$ such that $|f(x)|^{r_0} \leq M$ for all $x \in K$, so that

$$a_k \leq \frac{\sup_{x \in (0, M]} 2^k |(\log x)^k x|}{k! r_0^k} \leq \frac{2^k \cdot \max\{M(\log M)^k, k^k e^{-k}\}}{k! r_0^k},$$

where the second inequality uses the easily checked fact that the $x(\log x)^k$ takes its minimum on $(0, 1]$ at $x = e^{-k}$. Finally, Stirling's formula says that

$$k! \sim \sqrt{2\pi k} \cdot k^k e^{-k},$$

which implies that, choosing a suitable $M > 0$, any $0 < \varepsilon < r_0$ will do for the Claim. \square

The problem proposed by Gel'fand was whether one can analytically continue Z_φ to a meromorphic function on \mathbf{C} ; he also asked whether its poles lie in a finite number of arithmetic progressions. The existence of the Bernstein-Sato polynomial leads to a positive answer.

THEOREM 2.4.3. *With the notation above, for every smooth complex-valued function with compact support $\varphi \in C_c^\infty(\mathbf{C}^n)$, Z_φ admits an analytic continuation to \mathbf{C} as a meromorphic function whose poles are of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_f(s)$ and $m \in \mathbf{N}$. In particular all the poles are negative rational numbers.*

Before giving the proof of the theorem, we record a technical point that will be useful later on as well. We start with the defining Bernstein-Sato formula

$$b(s)f^s = P(s)f^{s+1}, \quad \text{with } P \in A_n[s],$$

where A_n is the Weyl algebra, and for simplicity we denote $b(s) = b_f(s)$. We also consider the conjugate $\bar{P}(s)$, where we replace x_i by \bar{x}_i and ∂_{x_i} by $\partial_{\bar{x}_i}$.

LEMMA 2.4.4. *For every $s \in \mathbf{C}$ we have*

$$b(s)^2|f|^{2s} = P(s)\bar{P}(s)|f|^{2(s+1)}.$$

PROOF. Conjugating the Bernstein-Sato formula above gives

$$b(s)\bar{f}^s = \bar{P}(s)\bar{f}^{s+1}.$$

(Note that by Kashiwara's theorem, $b(s)$ has rational coefficients.) Hence multiplying the two formulas leads to

$$b(s)^2|f|^{2s} = (P(s)f^{s+1}) \cdot (\bar{P}(s)\bar{f}^{s+1}).$$

The fact that the coefficients of $\bar{P}(s)$ are antiholomorphic differential operators implies that

$$\bar{P}(s)(f^{s+1} \cdot \bar{f}^{s+1}) = f^{s+1} \cdot \bar{P}(s)\bar{f}^{s+1},^4$$

and therefore

$$P(s)\bar{P}(s)|f|^{2(s+1)} = P(s)\bar{P}(s)(f^{s+1} \cdot \bar{f}^{s+1}) = (P(s)f^{s+1}) \cdot (\bar{P}(s)\bar{f}^{s+1})$$

as similarly since P is holomorphic we have $P(s)(\bar{P}(s)\bar{f}^{s+1}) = 0$. \square

PROOF. (of Theorem 2.4.3). For every $m \in \mathbf{N}$, denote

$$\mathbb{H}_m := \{s \in \mathbf{C} \mid \operatorname{Re}(s) > -m\}.$$

We have seen that all such Z_φ are holomorphic on \mathbb{H}_0 , and we show by induction on m that they can be extended to meromorphic functions on \mathbb{H}_m with poles as stated.

Going back to the identity

$$b(s) \cdot f^s = P(s) \cdot f^{s+1},$$

we write

$$P(s) = \sum_{i=0}^p Q_i \cdot s^i, \quad Q_i \in A_n.$$

We then have

$$P(s)\bar{P}(s) = \sum_{i=0}^{2p} R_i \cdot s^i,$$

⁴Apply one $\partial/\partial \bar{z}_i$ at a time, and use the product rule.

for operators $R_i = \overline{R_i}$. For $s \in \mathbb{H}_0$ we can then write

$$b(s)^2 \int_{\mathbf{C}^n} |f(x)|^{2s} \varphi(x) = \sum_{i=0}^{2p} s^i \int_{\mathbf{C}^n} (R_i |f(x)|^{2(s+1)}) \cdot \varphi(x).$$

Now a simple application of integration by parts (see Exercise 2.4.5 below) shows that

$$\int_{\mathbf{C}^n} (R_i |f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbf{C}^n} |f(x)|^{2(s+1)} \cdot (R_i \varphi)(x).$$

As a consequence, for $s \in \mathbb{H}_0$ we obtain

$$Z_\varphi(s) = \frac{\sum_{i=0}^{2p} s^i Z_{R_i \varphi}(s+1)}{b(s)^2}.$$

But we are assuming by induction that $Z_{R_i \varphi}(s)$ can be extended to a meromorphic function with poles as prescribed on \mathbb{H}_m . Therefore the identity shows that $Z_\varphi(s)$ can be extended to a meromorphic function \mathbb{H}_{m+1} , whose poles are among the roots of $b(s)$ and the poles of the extension of $Z_{R_i \varphi}(s+1)$ to \mathbb{H}_{m+1} , again of the prescribed form. The last statement follows from Kashiwara's result, Theorem 2.5.1. \square

EXERCISE 2.4.5. Let $Q \in A_n$, and define Q' to be its image under the (anti-)automorphism of A_n sending x_i to x_i and ∂_i to $-\partial_i$. For every complex-valued $\varphi \in C_c^\infty(\mathbf{C}^n)$ we have

$$\int_{\mathbf{C}^n} (Q |f(x)|^{2(s+1)}) \cdot \varphi(x) = \int_{\mathbf{C}^n} |f(x)|^{2(s+1)} \cdot (Q' \varphi)(x),$$

and similarly for \overline{Q} . (Hint: apply the change of variables formula when $Q = \partial_i$, for each i , and then use induction on the degree of Q .)

REMARK 2.4.6 (The distribution $|f|^{2s}$). Usually the quantity $|f|^{2s}$ is thought of as a distribution, which on any $\varphi \in C_c^\infty(\mathbf{C}^n)$ is defined by

$$(|f|^{2s}, \varphi) := \int_{\mathbf{C}^n} |f(x)|^{2s} \varphi(x).$$

This depends holomorphically on s for $\operatorname{Re}(s) > 0$, and the theorem above says that it can be continued meromorphically to \mathbf{C} , with poles of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_f(s)$ and $m \in \mathbf{N}$. Here if $P(\cdot)$ denotes the set of poles, by definition we have

$$P(|f|^{2s}) := \bigcup_{\varphi} P(Z_\varphi(s)).$$

Note. Bernstein-Gel'fand and Atiyah originally answered Gel'fand's question making use of resolution of singularities. The proof presented here, making use instead of the existence of the Bernstein-Sato polynomial, was given later by Bernstein in [Be]. The exposition draws also on the lecture notes [Ay], [Gr].

2.5. Kashiwara's rationality theorem

In this section we prove the main theorem in [Ka1]. It is worth noting that Kashiwara's proof also works when f is the germ of an analytic function on a complex manifold.

THEOREM 2.5.1. *For every non-invertible regular function f on a smooth complex variety X , the roots of $b_f(s)$ are negative rational numbers.*

The module \mathcal{N}_f . We start by enhancing the constructions described in §2.1. We denote as always by D the effective divisor on X corresponding to f .

We consider the \mathcal{D}_X -submodule

$$\mathcal{N}_f := \mathcal{D}_X[s]f^s \subseteq \mathcal{O}_X\left[\frac{1}{f}, s\right]f^s \simeq \iota_+ \mathcal{O}_X(*D).$$

Since $\partial_t t \delta \in \iota_+ \mathcal{O}_X$, we have in fact $\mathcal{N}_f \subseteq \iota_+ \mathcal{O}_X$.

LEMMA 2.5.2. *Multiplication by t leaves \mathcal{N}_f invariant, so we have an endomorphism $t: \mathcal{N}_f \rightarrow \mathcal{N}_f$.*

PROOF. Recall that the action of t on $\mathcal{O}_X\left[\frac{1}{f}, s\right]f^s$ is given by the automorphism “ $s \rightarrow s + 1$ ”. Hence if $Q(s)f^s \in \mathcal{N}_f$, using (2.1.2) we have

$$t \cdot Q(s)f^s = (Q(s+1) \cdot f) \cdot f^s \in \mathcal{N}_f.$$

□

Observe that the image $t\mathcal{N}_f$ of the endomorphism in the Lemma is $\mathcal{D}_X[s]f \cdot f^s = \mathcal{D}_X[\partial_t t]t\delta$. Another remark is that the action of s clearly leaves \mathcal{N}_f invariant as well, so using the notation above \mathcal{N}_f is a $\mathcal{D}_X\langle t, s \rangle$ -module.

PROPOSITION 2.5.3. *The action of s induces an endomorphism*

$$s: \mathcal{N}_f/t\mathcal{N}_f \longrightarrow \mathcal{N}_f/t\mathcal{N}_f,$$

which has a minimal polynomial equal to the Bernstein-Sato polynomial $b_f(s)$.

PROOF. For the first assertion we only need to check that $s(t\mathcal{N}_f) \subseteq t\mathcal{N}_f$. But this is clear since by (2.1.2) we have

$$s \cdot t \cdot \mathcal{N}_f = t \cdot (s-1) \cdot \mathcal{N}_f.$$

For the second assertion, we first show that $b_f(s)$ is identically zero on $\mathcal{N}_f/t\mathcal{N}_f$. Let $Q(s)f^s \in \mathcal{N}_f$. Then, since $b_f(s)$ has constant coefficients, we have

$$b_f(s)Q(s)f^s = Q(s)b_f(s)f^s = Q(s)P(s)f \cdot f^s \in t\mathcal{N}_f.$$

On the other hand, say that $h(s) \in \mathbf{C}[s]$ is identically zero on $\mathcal{N}_f/t\mathcal{N}_f$. This means in particular that

$$h(s)f^s \in t\mathcal{N}_f,$$

which means that there exists $P(s) \in \mathcal{D}_X[s]$ such that $h(s)f^s = P(s)f \cdot f^s$. By the definition of the Bernstein-Sato polynomial we then have that $b_f(s)|h(s)$. □

REMARK 2.5.4. The previous result implies in particular that $\mathcal{N}_f/t\mathcal{N}_f$ is finitely generated over \mathcal{D}_X . More precisely, if $\deg b_f(s) = d$, then it is generated by the classes of $\delta, s\delta, \dots, s^{d-1}\delta$. Kashiwara showed in fact the stronger statement that $\mathcal{N}_f/t\mathcal{N}_f$ is holonomic; more below.

We record one more basic fact about \mathcal{N}_f :

LEMMA 2.5.5. *We have*

$$(\mathcal{N}_f)_t = (\iota_+ \mathcal{O}_X)_t = \iota_+ \mathcal{O}_X(*D).$$

PROOF. We need to show that for every $w \in \iota_+ \mathcal{O}_X$, there exists $m \in \mathbf{N}$ such that $t^m w \in \mathcal{N}_f$. Write

$$w = \sum_{j=0}^p h_j \partial_t^j \delta, \quad h_j \in \mathcal{O}_X,$$

and note that $t(h_j \partial_t^j \delta) = h_j (t \partial_t^j \delta)$. Hence it suffices to show that

$$t^j \partial_t^j \delta \in \mathcal{N}_f = \mathcal{D}_X[\partial_t t] \delta.$$

But this follows from the next Exercise, applied with $i = j$. \square

EXERCISE 2.5.6. For all integers $i \geq j \geq 0$, we have the operator identity

$$t^i \partial_t^j = \prod_{k=1}^j (\partial_t t - (i - k + 1)) \cdot t^{i-j}.$$

Main construction. We now describe the main technical construction of Kashiwara. We consider a log resolution $\mu: Y \rightarrow X$ of D , which is an isomorphism away from D , see §2.6, and we denote $f' = f \circ \mu$.

As in the discussion above, we denote

$$\mathcal{N}_f := \mathcal{D}_X[s] \cdot f^s \quad \text{and} \quad \mathcal{N}_{f'} := \mathcal{D}_{X'}[s] \cdot f'^s.$$

We also define

$$\mathcal{N} := \mathcal{H}^0 \mu_+ \mathcal{N}_{f'}.$$

We will see below that we can attach to \mathcal{N} a Bernstein-Sato polynomial $b_{\mathcal{N}}(s)$, and that $b_{\mathcal{N}}(s) \mid b_{f'}(s)$. Hence one of the key points in the proof of Kashiwara's theorem will be to compare $b_f(s) = b_{\mathcal{N}_f}(s)$ with $b_{\mathcal{N}}(s)$. This is however not immediate; it turns out that \mathcal{N}_f is a special type of sub-quotient of \mathcal{N} , but this requires a bit of work.

To this end, we construct a distinguished section $u \in \Gamma(X, \mathcal{N})$. First, since μ is birational, there is a natural nontrivial (hence injective) map $p^* \omega_X \rightarrow \omega_Y$, hence a natural section $s: \mathcal{O}_Y \rightarrow \omega_{Y/X}$. On the other hand, recall that by definition

$$\mathcal{N} = R^0 \mu_* (\mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{N}_{f'}),$$

where

$$\mathcal{D}_{X \leftarrow Y} = \mu^{-1} \mathcal{D}_X \otimes_{\mu^{-1} \mathcal{O}_X} \omega_{Y/X}$$

is the transfer $(\mu^{-1}\mathcal{D}_X, \mathcal{D}_Y)$ -bimodule. Note that $1 \in \mathcal{D}_X$ and s give rise to a homomorphism

$$\mathcal{O}_Y \rightarrow \mathcal{D}_{X \leftarrow Y}$$

of $(\mu^{-1}\mathcal{O}_X, \mathcal{O}_Y)$ -bimodules. On the other hand the section f'^s of $\mathcal{N}_{f'}$ gives rise to a \mathcal{O}_Y -module homomorphism $\mathcal{O}_Y \rightarrow \mathcal{N}_{f'}$. Taking the tensor product of these two we obtain a morphism

$$\mathcal{O}_Y \rightarrow \mathcal{D}_{X \leftarrow Y} \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{N}_{f'}$$

in the derived category of $\mu^{-1}\mathcal{O}_X$ -modules, and applying $R^0\mu_*$ to this we obtain our desired section $u \in \Gamma(X, \mathcal{N})$. We define

$$\mathcal{N}' := \mathcal{D}_X[s] \cdot u \subseteq \mathcal{N},$$

a $\mathcal{D}_X[s]$ -submodule of \mathcal{N} .

LEMMA 2.5.7. *The module \mathcal{N}' satisfies the following properties:*

(1) *It sits in a digram of $\mathcal{D}_X\langle s, t \rangle$ -modules*

$$\begin{array}{ccc} \mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\ \downarrow g & & \\ \mathcal{N}_f & & \end{array}$$

where i is the inclusion map, and g is surjective.

(2) *It is a coherent \mathcal{D}_X -module.*

PROOF. To see that \mathcal{N}' is a $\mathcal{D}_X\langle s, t \rangle$ -module, we only need to check that it is preserved by the action of t . Denoting by $1_{X \leftarrow Y}$ the section of $\mathcal{D}_{X \leftarrow Y}$ considered above, we have

$$t \cdot (1_{X \leftarrow Y} \otimes f'^s) = 1_{X \leftarrow Y} \otimes t f'^s = 1_{X \leftarrow Y} \otimes f f'^s = f(1_{X \leftarrow Y} \otimes f'^s),$$

which indeed shows that $t \cdot \mathcal{N}' \subseteq \mathcal{N}'$.

To complete the proof of (1), it suffices to check that the mapping

$$\mathcal{D}_X[s] \cdot u \rightarrow \mathcal{D}_X[s] \cdot f^s, \quad P(s)u \mapsto P(s)f^s$$

is well defined. But assuming that $P(s)u = 0$, and recalling that u and f^s coincide on the open set U , it follows that $P(s)f^s$ is zero on U and therefore annihilated by a power of f . This means that in fact $P(s)f^s = 0$, since $\mathcal{D}_X[s] \cdot f^s$ is torsion-free as an \mathcal{O}_X -module; indeed, it is a submodule of $\mathcal{O}_X[\frac{1}{f}, s]f^s$, which is isomorphic to $\mathcal{O}_X[\frac{1}{f}, s]$ as an \mathcal{O}_X -module, and therefore torsion-free.

The assertion in (2) follows immediately from (1), since \mathcal{N} is a coherent \mathcal{D}_X -module by the behavior of direct images under projective morphisms. \square

Bernstein-sato polynomials of special \mathcal{D} -modules. We will make use of the following general theorem; see [Ka2, Theorem 4.45].

THEOREM 2.5.8. *Let \mathcal{M} and \mathcal{N} be holonomic \mathcal{D}_X -modules. Then $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})_x$ is a finite dimensional \mathbf{C} -vector space for every $x \in X$.*

This allows us to deduce:

PROPOSITION 2.5.9. *Let \mathcal{M} be a $\mathcal{D}_X[s]$ -module, which is holonomic as \mathcal{D}_X -module. Then there exists a polynomial $b(s) \in \mathbf{C}[s]$ such that $b(s) \cdot \mathcal{M} = 0$. (Therefore the action of s on \mathcal{M} has a minimal polynomial.)*

PROOF. By Theorem 2.5.8, we know that $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{M})_x$ is a finite dimensional \mathbf{C} -vector space for all $x \in X$. It follows that there exists a polynomial $b_x(s)$ such that $b_x(s) \cdot \mathcal{M}_x = 0$, and this continues to hold on a neighborhood of x . We can now cover X with finitely many such neighborhoods, and take $b(s)$ to be the least common multiple of the respective $b_x(s)$. \square

LEMMA 2.5.10. *Let \mathcal{N} be a $\mathcal{D}_X\langle s, t \rangle$ -module which is holonomic as a \mathcal{D}_X -module. Then there exists an integer $N \geq 0$ such that $t^N \cdot \mathcal{N} = 0$.*

PROOF. Let $b(s)$ be a polynomial as in Proposition 2.5.9, so that $b(s) \cdot \mathcal{N} = 0$. If

$$b(s) = \prod_{i=1}^r (s + \alpha_i)^{m_i},$$

then every \mathcal{N}_x has a decomposition into subspaces on which some $s + \alpha_i$ acts nilpotently. Therefore for $N \gg 0$, the action of $b(s + N)$ on \mathcal{N} is bijective. Recalling that $P(s)t = tP(s - 1)$ for every $P \in \mathbf{C}[s]$, we have

$$b(s + N) \cdot t^N \cdot \mathcal{N} = t^N \cdot b(s) \cdot \mathcal{N} = 0,$$

hence $t^N \cdot \mathcal{N} = 0$. \square

Let now \mathcal{N} be a module $\mathcal{D}_X\langle s, t \rangle$ -module, assumed moreover to be coherent over \mathcal{D}_X , and such that $\mathcal{N}/t\mathcal{N}$ is holonomic. By Proposition 2.5.9, the action of s on $\mathcal{N}/t\mathcal{N}$ has a minimal polynomial, which we denote $b_{\mathcal{N}}(s)$. In other words, this is the monic polynomial of minimal degree such that

$$b_{\mathcal{N}}(s) \cdot \mathcal{N} \subseteq t\mathcal{N}.$$

EXAMPLE 2.5.11. We have then seen in Proposition 2.5.3 that $b_f(s) = b_{\mathcal{N}_f}(s)$. In fact Kashiwara shows in [Kal], in the general setting of germs of holomorphic functions, that $\mathcal{N}_f/t\mathcal{N}_f$ is holonomic, which by the discussion above leads to another proof of the existence of the Bernstein-Sato polynomial.

LEMMA 2.5.12. *Let \mathcal{N} be as above, and \mathcal{N}' a $\mathcal{D}_X\langle s, t \rangle$ -submodule of \mathcal{N} . Assume that \mathcal{N}/\mathcal{N}' is a holonomic \mathcal{D}_X -module. Then there exists an integer $N \geq 0$ such that*

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s+1) \cdots b_{\mathcal{N}}(s+N).$$

PROOF. Note that $b_{\mathcal{N}'}(s)$ makes sense, as $\mathcal{N}'/t\mathcal{N}'$ is holonomic as well; see the Exercise below.

Now since \mathcal{N}/\mathcal{N}' is holonomic, Lemma 2.5.10 implies that there exists $N \geq 0$ such that $t^N \cdot (\mathcal{N}/\mathcal{N}') = 0$, or equivalently

$$t^N \cdot \mathcal{N} \subseteq \mathcal{N}'.$$

As above, for every $j \geq 0$ we have

$$b_{\mathcal{N}}(s+j)t^j \cdot \mathcal{N} = t^j b_{\mathcal{N}}(s) \cdot \mathcal{N} \subseteq t^{j+1} \cdot \mathcal{N}.$$

Applying this repeatedly, we obtain

$$b_{\mathcal{N}}(s+N) \cdots b_{\mathcal{N}}(s) \cdot \mathcal{N}' \subseteq b_{\mathcal{N}}(s+N) \cdots b_{\mathcal{N}}(s) \cdot \mathcal{N} \subseteq t^{N+1} \cdot \mathcal{N} \subseteq t \cdot \mathcal{N}',$$

which implies the conclusion by the definition of $b_{\mathcal{N}'}(s)$. \square

EXERCISE 2.5.13. Show that in the situation above, i.e. \mathcal{N}' is a $\mathcal{D}_X\langle s, t \rangle$ -submodule of a module \mathcal{N} having the property that $\mathcal{N}/t\mathcal{N}$ is holonomic, we have that $\mathcal{N}'/t\mathcal{N}'$ is holonomic as well.

Proof of the Theorem. We go back to the main construction at the beginning of the section. A crucial technical result of Kashiwara, see [Ka1, Lemma 5.7] or [Ka2, p.113], is the following. Strictly speaking, in the theorem below one needs to assume that the zero locus of the Jacobian ideal $\text{Jac}(f)$ (generated locally by $\partial f/\partial x_1, \dots, \partial f/\partial x_n$) is contained in the zero locus of f . This however can be accomplished after passing to an open neighborhood of the zero locus of f , which is of course all we care about if we want to study $b_f(s)$. Recall that $\mathcal{N} = \mathcal{H}^0 \mu_+ \mathcal{N}_{f'}$.

THEOREM 2.5.14. *The \mathcal{D}_X -module \mathcal{N} is subholonomic. More precisely we have*

$$\text{Ch}(\mathcal{N}) = W_f \cup \Lambda,$$

where Λ is a Lagrangian subvariety of T^*X , and W_f is the closure of the subset

$$\{(x, \text{sdl}og f(x) \mid f(x) \neq 0, s \in \mathbf{C}\} \subset T^*X,$$

which is involutive and $(n+1)$ -dimensional.

COROLLARY 2.5.15. *The \mathcal{D}_X -module $\mathcal{N}/t\mathcal{N}$ is holonomic.*

PROOF. This is true for an arbitrary subholonomic \mathcal{D} -module preserved by the action of t . Indeed, consider the short exact sequence

$$0 \longrightarrow \mathcal{N} \xrightarrow{t} \mathcal{N} \longrightarrow \mathcal{N}/t\mathcal{N} \longrightarrow 0.$$

It implies that the characteristic variety of $\mathcal{N}/t\mathcal{N}$ satisfies

$$\text{Char}(\mathcal{N}/t\mathcal{N}) \subseteq \text{Char}(\mathcal{N}),$$

but also that its multiplicity along each irreducible component of $\text{Char}(\mathcal{N})$ is equal to zero. Hence its dimension is strictly smaller than that of $\text{Char}(\mathcal{N})$, so it can only be equal to $n = \dim X$. \square

By the discussion above, the Corollary implies that there exists a Bernstein-Sato polynomial $b_{\mathcal{N}}(s)$; it satisfies the following:

LEMMA 2.5.16. *We have the divisibility*

$$b_{\mathcal{N}}(s) \mid b_{f'}(s).$$

PROOF. By definition we have $b_{f'}(s) \cdot \mathcal{N}_{f'} \subseteq t\mathcal{N}_{f'}$, so there exists a $\mathcal{D}_{X'}$ -module endomorphism $\varphi: \mathcal{N}_{f'} \rightarrow \mathcal{N}_{f'}$ such that

$$b_{f'}(s) = t\varphi.$$

We can apply the functor $\mathcal{H}^0\mu_+$ to this identity; since the push-forward is in the sense of \mathcal{D}_Y -modules, it commutes with the s and t action, hence we obtain an induced endomorphism

$$b_{f'}(s): \mathcal{N} \rightarrow \mathcal{N}, \quad b_{f'}(s) = t\psi,$$

where $\psi = \mathcal{H}^0\mu_+(\varphi)$. This immediately gives the conclusion. \square

We can now address the main result of the section.

PROOF OF THEOREM 2.5.1. With the notation above, by Example 2.2.11(4) the statement holds for $b_{f'}(s)$. The result follows from the following stronger statement: there exists an integer $N \geq 0$ such that

$$(2.5.1) \quad b_f(s) \mid b_{f'}(s)b_{f'}(s+1) \cdots b_{f'}(s+N).$$

To show (2.5.1), we start by considering the $\mathcal{D}_X\langle s, t \rangle$ -module

$$\mathcal{M} := \mathcal{N}/\mathcal{N}'.$$

This is a coherent \mathcal{D}_X -module, and we claim that it is holonomic. Indeed, on one hand we have

$$\text{Ch}(\mathcal{M}) \subseteq \text{Ch}(\mathcal{N}) = W_f \cup \Lambda,$$

using Theorem 2.5.14. On the other hand, by construction $\text{Supp}(\mathcal{M}) \subseteq D = (f = 0)$, since by construction \mathcal{N} and \mathcal{N}' coincide away from D . Hence in fact

$$\text{Ch}(\mathcal{M}) \subseteq (W_f \cap \pi_X^{-1}(D)) \cup \Lambda.$$

It remains to note that by definition W_f dominates X , and therefore the right hand side has dimension n .

We can therefore apply Lemma 2.5.12, since by Corollary 2.5.15 we know that $\mathcal{N}/t\mathcal{N}$ is holonomic. We conclude that there exists an integer $N \geq 0$ such that

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s+1) \cdots b_{\mathcal{N}}(s+N).$$

By Lemma 2.5.16 we also know that $b_{\mathcal{N}}(s) \mid b_{f'}(s)$. It suffices then to show that $b_f(s) \mid b_{\mathcal{N}'}(s)$. But this is clear, since \mathcal{N}_f is a quotient of \mathcal{N}' , so $b_{\mathcal{N}'}(s) \cdot \mathcal{N}_f \subseteq t\mathcal{N}_f$. \square

2.6. Log resolutions, log canonical thresholds, multiplier ideals

In this section we review a few basic concepts from birational geometry. Most of the material below is covered in great detail in [La, Ch.9]. For general singularities of pairs, a great introduction is [KM].

We start by recalling a few notions and facts related to resolution of singularities. We always consider X to be a smooth complex variety of dimension n .

DEFINITION 2.6.1. Let D be an effective \mathbf{Q} -divisor on X . A *log resolution* of the pair (X, D) is a proper birational morphism $\mu: Y \rightarrow X$, where Y is a smooth variety and the divisor $\mu^*D + \text{Exc}(\mu)$ has simple normal crossing (SNC) support.

Log resolutions are known to exist in characteristic 0 by Hironaka's theorem. Moreover, one can always obtain such a resolution as a sequence of blow-ups along smooth centers contained in the locus where D does not have SNC support. Hence we will often assume that μ is an isomorphism away from this locus, so in particular away from $U = X \setminus \text{Supp}(D)$.

REMARK 2.6.2. For concrete calculations, the existence of such a resolution can be rephrased locally as saying that for any regular function f on X , there exists a proper birational map as above, and around each point $y \in Y$ a system of algebraic coordinates y_1, \dots, y_n such that in this neighborhood

$$\mu^*f = h \cdot y_1^{a_1} \cdot \dots \cdot y_n^{a_n},$$

with $k_i \geq 0$ and h an invertible function.

We will write

$$(2.6.1) \quad f^*D = \sum_{i=1}^m a_i \cdot E_i,$$

where $a_i \geq 0$ are rational numbers, and E_i are either components of the proper transform \tilde{D} or exceptional divisors. Another standard point to note is that while we cannot talk about canonically defined divisors K_X and K_Y , there is a canonically defined *relative* canonical divisor $K_{Y/X}$, namely the zero locus of the Jacobian

$$\text{Jac}(\mu) = \det(\partial\mu_i/\partial y_j)_{i,j}$$

of the map μ . This supported on the exceptional locus of μ . We will also write

$$(2.6.2) \quad K_{Y/X} := \text{div}(\text{Jac}(\mu)) = \sum_{i=1}^m b_i \cdot E_i.$$

with $b_i \geq 0$. When μ is an isomorphism away from $U = X \setminus \text{Supp}(D)$, all exceptional divisors appear nontrivially in both sums.

EXAMPLE 2.6.3. Let's review a few well-known examples; all the statements are left as exercises.

(1) If $\mu: Y = \text{Bl}_W X \rightarrow X$ is the blow-up of X along a smooth subvariety of codimension c , and F is the exceptional divisor over W , then

$$K_{Y/X} = (c - 1)F.$$

In particular, if $W = \{x\}$ is a point, then $K_{Y/X} = (n - 1)F$.

(2) If $D = (x^2 + y^3 = 0) \subset \mathbf{C}^2$ is a cusp, then a log resolution $\mu: Y \rightarrow X = \mathbf{C}^2$ of (X, D) can be obtained as the composition of three successive blow-ups at points; see the picture

in [La, Example 9.1.13]. If we denote by F_1 , F_2 and F_3 the exceptional divisors arising from the three blow-ups (in this order), then an easy calculation gives

$$(2.6.3) \quad \mu^*D = \tilde{D} + 2F_1 + 3F_2 + 6F_3 \quad \text{and} \quad K_{Y/X} = F_1 + 2F_2 + 4F_3.$$

(3) Let $D = (f = 0)$ and assume for simplicity that $f(0) = 0$. Write $f = f_m + f_{m+1} + \cdots$, with f_i homogeneous of degree i for all i , and $f_m \neq 0$, so that $\text{mult}_0 D = m$. Recall that the tangent cone of D at 0 is

$$TC_x D = (f_m = 0) \subset \mathbf{A}^n,$$

while the projectivized tangent cone is $\mathbf{P}(TC_0 D) \subset \mathbf{P}^{n-1}$. We say that D has an *ordinary singularity* at 0 if $\mathbf{P}(TC_0 D)$ is smooth.

The main example is when D is the cone in \mathbf{A}^n over a smooth hypersurface in \mathbf{P}^{n-1} . For instance one can take $f = x_1^m + \cdots + x_n^m$, the Fermat hypersurface of degree m .

If $x \in D$ is an ordinary singularity of multiplicity m , then it is an isolated singularity, and a log resolution is given by $\mu: Y = \text{Bl}_x X \rightarrow X$. We then have

$$\mu^*D = \tilde{D} + mF \quad \text{and} \quad K_{Y/X} = (n-1)F.$$

DEFINITION 2.6.4. (1) Let $\mu: Y \rightarrow X$ be a log resolution of (X, D) , and fix a prime divisor E on Y . The *discrepancy* of E (with respect to D) is

$$a(E) = a(E; X, D) := \text{ord}_E(K_{Y/X} - \mu^*D) \in \mathbf{Q}.$$

Consequently we have

$$K_Y - \mu^*(K_X + D) = \sum_E a(E) \cdot E,$$

where the sum is taken over all prime divisors E in Y , or equivalently

$$K_Y + \tilde{D} - \mu^*(K_X + D) = \sum_{E \text{ exceptional}} a(E) \cdot E.$$

With the notation introduced above, we of course have $a(E_i) = b_i - a_i$.

(2) The pair (X, D) is called *log-canonical* if $a(E) \geq -1$ for all exceptional divisors E on any log resolution. It is called *klt* (*Kawamata log terminal*) if $a(E) > -1$ for all prime divisors E on any log resolution.

(3) The *log canonical threshold* of D is

$$\text{lct}(D) := \inf\{c \in \mathbf{Q} \mid (X, cD) \text{ is not log canonical}\}.$$

In particular, if D is an integral divisor, then (X, D) is log canonical if and only if $\text{lct}(D) = 1$. There is also a more refined local invariant, defined for any $x \in X$ to be

$$\text{lct}_x(D) := \inf\{c \in \mathbf{Q} \mid (X, cD) \text{ is not log canonical around } x\}.$$

EXERCISE 2.6.5. We have $\text{lct}(D) = \min_{x \in X} \text{lct}_x(D)$.

LEMMA 2.6.6. *We have $\text{lct}_x(D) \in \mathbf{Q}$, and the infimum in the definition is in fact a minimum that can be computed on any resolution of singularities.*

PROOF. We fix a log resolution $\mu: Y \rightarrow X$, and use the notation introduced in (2.6.1) and (2.6.2). We then have, for $c \in \mathbf{Q}$,

$$K_Y - \mu^*(K_X + cD) = \sum_{i=1}^m (b_i - ca_i)E_i,$$

and so with respect to the pair (X, cD) we get, for every exceptional divisor E_i , that

$$a(E_i) \geq -1 \iff b_i - ca_i \geq -1 \iff \frac{b_i + 1}{a_i} \geq c.$$

Hence we obtain that

$$(2.6.4) \quad \text{lct}_x(D) = \min_{1 \leq i \leq m} \frac{b_i + 1}{a_i},$$

where the minimum is taken over all E_i such that $x \in \mu(E_i)$, at least after showing that this quantity is independent of the choice of log resolution. This is left as an exercise, but note that it will also follow from the analytic interpretation given below. \square

EXERCISE 2.6.7. Show that the function $x \mapsto \text{lct}_x(D)$ is lower semicontinuous on X .

REMARK 2.6.8. When D is a reduced integral divisor, the proof above shows that

$$\text{lct}(D) = \min\{1, \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}\}.$$

EXAMPLE 2.6.9. Going back to Example 2.6.3, we see that:

- (1) The log canonical threshold of the cusp $(x^2 + y^3 = 0)$ is $5/6$.
- (2) If D has an ordinary singularity of multiplicity m at x , then $\text{lct}_x(D) = \min\{1, \frac{n}{m}\}$.

We now come to a concept of great importance in modern birational geometry.

DEFINITION 2.6.10. The *multiplier ideal* of the \mathbf{Q} -divisor D on X is defined as

$$\mathcal{J}(D) := \mu_* \mathcal{O}_Y(K_{Y/X} - [\mu^* D]) \subseteq \mathcal{O}_X,$$

where $\mu: Y \rightarrow X$ is any log resolution of (X, D) . The fact that it is an ideal sheaf follows since clearly $\mathcal{J}(D) \subseteq \mu_* \omega_{Y/X} \simeq \mathcal{O}_X$.

It is well known, and originally due to Esnault-Viehweg, that this definition is independent of the choice of resolution of singularities. By dominating any two log resolutions by a third, this reduces to the following statement, which can be found in [La, Lemma 9.2.19].

LEMMA 2.6.11. *Let D be an effective \mathbf{Q} -divisor on X with SNC support, and $\mu: Y \rightarrow X$ a log resolution of (X, D) . Then*

$$\mu_* \mathcal{O}_Y(K_{Y/X} - [\mu^* D]) \simeq \mathcal{O}_X(-[D]).$$

The behavior of multiplier ideals under birational maps is not hard to describe:

EXERCISE 2.6.12 (Birational transformation rule). Show that if $\mu: Y \rightarrow X$ is a proper birational map of smooth varieties, and D an effective \mathbf{Q} -divisor on X , then

$$\mathcal{J}(D) \otimes_{\mathcal{O}_X} \omega_X \simeq \mu_* \mathcal{O}_Y (\mathcal{J}(\mu^* D) \otimes_{\mathcal{O}_Y} \omega_Y).$$

(Hint: take a log resolution of the pair $(Y, \mu^* D + \text{Exc}(\mu))$, and use the projection formula.)

EXERCISE 2.6.13. The notions of singularities of pairs defined above can be interpreted in terms of multiplier ideals as follows:

$$(X, D) \text{ is klt} \iff \mathcal{J}(D) = \mathcal{O}_X$$

and

$$(X, D) \text{ is log canonical} \iff \mathcal{J}((1 - \varepsilon)D) = \mathcal{O}_X, \quad \text{for all } 0 < \varepsilon < 1.$$

In fact we have

$$\text{lct}(D) = \inf\{c \in \mathbf{Q} \mid \mathcal{J}(cD) \neq \mathcal{O}_X\}$$

and more precisely

$$\text{lct}_x(D) = \inf\{c \in \mathbf{Q} \mid \mathcal{J}(cD)_x \subseteq \mathfrak{m}_x\}.$$

EXAMPLE 2.6.14. (1) Using the definition and the projection formula, if D is an integral effective divisor we have $\mathcal{J}(D) = \mathcal{O}_X(-D)$. For the same reason, if D is an arbitrary effective \mathbf{Q} -divisor and E is an integral divisor, then

$$\mathcal{J}(D + E) = \mathcal{J}(D) \otimes \mathcal{O}_X(-E).$$

(2) If D has SNC support, then $\mathcal{J}(D) = \mathcal{O}_X(-[D])$.

(3) If $D = (x^2 + y^3 = 0)$ is a cusp in \mathbf{C}^2 , using the resolution and calculations in Example 2.6.3(2) we have

$$\begin{aligned} \mathcal{J}(cD) &= \mu_* \mathcal{O}_Y (K_{Y/X} - [c\mu^* D]) = \\ &= \mu_* \mathcal{O}_Y ((1 - [2c])E_1 + (2 - [3c])E_2 + (4 - [6c])E_3 - [c]\tilde{D}). \end{aligned}$$

We can focus on the case $0 < c < 1$ (see (1)), and so we obtain $\mathcal{J}(cD) = \mathcal{O}_X$ as long as all the coefficients in the parenthesis are nonnegative, i.e.

$$\mathcal{J}(cD) = \mathcal{O}_X \iff 0 < c < 5/6$$

and

$$\mathcal{J}(cD) = \mu_* \mathcal{O}_Y(-E_3) = \mathfrak{m}_0 \iff 5/6 \leq c < 1.$$

(4) A more general example that will be discussed later in the analytic setting is that of a general $D = (x^a + y^b = 0) \subset \mathbf{C}^2$, with $a, b \geq 2$. We will see in Example 2.6.25 that

$$\mathcal{J}(cD) = \mathcal{O}_X \iff c < \frac{1}{a} + \frac{1}{b},$$

so in particular $\text{lct}(D) = 1/a + 1/b$.

The log canonical threshold is the first in a sequence of rational numbers describing the “jumps” of the multiplier ideals associated to multiples of a fixed divisor.

PROPOSITION 2.6.15. *If D is an effective \mathbf{Q} -divisor and $x \in X$ is a point, then there exists an increasing sequence of rational numbers*

$$0 = c_0 < c_1 < c_2 < \cdots$$

such that:

- $\mathcal{J}(cD)_x = \mathcal{J}(c_i D)_x$ for $c \in [c_i, c_{i+1})$.
- $\mathcal{J}(c_{i+1}D)_x \neq \mathcal{J}(c_i D)_x$ for all i .

(Here by convention $\mathcal{J}(0 \cdot D) = \mathcal{O}_X$.) In particular $c_1 = \text{lct}_x(D)$.

PROOF. Just as in the proof of Lemma 2.6.6, we can write

$$\mathcal{J}(cD) = \mu_* \mathcal{O}_Y \left(\sum_{i=1}^m (b_i - [ca_i]) E_i \right),$$

and clearly the coefficients $b_i - [ca_i]$ are constant on intervals as indicated. Moreover, the endpoints of these intervals belong to the set

$$\left\{ \frac{b_i + m}{a_i} \mid \text{some } i \text{ and some } m \geq 1 \right\} \subset \mathbf{Q}.$$

□

DEFINITION 2.6.16. The numbers c_i in Proposition 2.6.15 are called the *jumping numbers* (or *jumping coefficients*) of D at x .

EXAMPLE 2.6.17. (1) If D is an integral divisor, we have noted earlier that $\mathcal{J}((c+1)D) = \mathcal{J}(cD) \otimes \mathcal{O}_X(-D)$, and therefore c is a jumping number for D if and only if $c+1$ is one. So all the jumping numbers are determined by those in the interval $[0, 1]$, which form a finite set according to the formula at the end of the proof of Proposition 2.6.15.

(2) We will see later that if $f = x_1^{d_1} + \cdots + x_n^{d_n}$, then the jumping numbers of f are all the rational numbers of the form

$$\frac{e_1 + 1}{d_1} + \cdots + \frac{e_n + 1}{d_n}, \quad \text{for all } e_1, \dots, e_n \in \mathbf{N}.$$

We record the following for later use:

LEMMA 2.6.18. *The jumping numbers of D at x satisfy the inequalities*

$$c_{i+1} \leq c_1 + c_i.$$

PROOF. This is a consequence of the Subadditivity Theorem for multiplier ideals, for which I refer to [La, Theorem 9.5.20]. It says that for any \mathbf{Q} -divisors D_1 and D_2 on X we have

$$\mathcal{J}(D_1 + D_2) \subseteq \mathcal{J}(D_1) \cdot \mathcal{J}(D_2).$$

In our case, we then have

$$\mathcal{J}((c_i + c_1)D)_x \subseteq \mathcal{J}(c_i D)_x \cdot \mathcal{J}(c_1 D)_x \subsetneq \mathcal{J}(c_i D)_x,$$

and therefore by definition $c_{i+1} \leq c_1 + c_i$. □

Analytic interpretation. An excellent source for this material is [De, §5.B].

Let $D = \sum_{i=1}^k a_i D_i$ be an effective divisor on X , with D_i prime divisors and $a_i \in \mathbf{Q}$. We fix an open set U on which we have $D_i = (f_i = 0)$ for a regular function f_i . We next view X as a complex manifold.

DEFINITION 2.6.19. The *analytic multiplier ideal* of D (on U) is

$$\mathcal{J}_{\text{an}}(D) := \{g \in \mathcal{O}_X(U) \mid \frac{|g|^2}{\prod_{i=1}^k |f_i|^{2a_i}} \text{ is locally integrable}^5\}.$$

In particular, if D is an integral divisor given by $f \in \mathcal{O}_X(U)$, then for every $c \in \mathbf{Q}_{>0}$ we have

$$\mathcal{J}_{\text{an}}(cD) := \{g \in \mathcal{O}_X(U) \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally integrable}\}.$$

It is not hard to see that the local definitions glue to give a global sheaf of ideals on X^{an} .

REMARK 2.6.20 (Plurisubharmonic functions). The multiplier ideal of D defined above is a special example of a more general analytic notion. To this end, note that

$$\varphi_D := \sum_{i=1}^k a_i \log |f_i|$$

is a *plurisubharmonic* function⁶ (since $\log |z|$ is so). Now for any plurisubharmonic function φ on X , one defines its multiplier ideal $\mathcal{J}(\varphi)$ via

$$\mathcal{J}(\varphi)(U) := \{g \in \mathcal{O}_X(U) \mid |g|^2 \cdot e^{-2\varphi} \text{ is locally integrable}\}.$$

It can be shown that this is a coherent sheaf of ideals; see [De, Proposition 5.7].

It turns out that analytic multiplier ideals satisfy the same birational transformation formula as the algebraic ones:

LEMMA 2.6.21. *Let $\mu: Y \rightarrow X$ be a proper bimeromorphic holomorphic map, and φ any plurisubharmonic function on X . Then*

$$\mathcal{J}(\varphi) \otimes_{\mathcal{O}_X} \omega_X \simeq \mu_* \mathcal{O}_Y (\mathcal{J}(\varphi \circ \mu) \otimes_{\mathcal{O}_Y} \omega_Y).$$

PROOF. Using the definition of $\mathcal{J}(\varphi)$, we can interpret the sheaf $\mathcal{J}(\varphi) \otimes_{\mathcal{O}_X} \omega_X$ as the subsheaf of ω_X consisting, for each open set U , of n -forms ω such that

$$\omega \wedge \bar{\omega} \cdot e^{-2\varphi} \quad \text{is locally integrable on } U.$$

⁵Recall that this means that its integral with respect to the Lebesgue measure is finite on every compact subset of U .

⁶This means a function $\varphi: X \rightarrow [-\infty, \infty)$ that is upper semicontinuous, locally integrable, and satisfies the mean-value inequality

$$(\varphi \circ \gamma)(0) \leq \frac{1}{\pi} \int_{\Delta} (\varphi \circ \gamma) d\mu$$

for any holomorphic map $\gamma: \Delta \rightarrow X$ from the open unit disk $\Delta \subset \mathbf{C}$. See [De, §1B] for basic properties.

It is in fact enough to restrict to forms defined on the open set $V \subseteq U$ over which μ is a biholomorphism, as they automatically extend over the complement of V by virtue of being locally L^2 . Thus the change of variables formula gives

$$\int_U \omega \wedge \bar{\omega} \cdot e^{-2\varphi} = \int_{\mu^{-1}(U)} \mu^* \omega \wedge \mu^* \bar{\omega} \cdot e^{-2(\varphi \circ \mu)},$$

and therefore

$$\omega \in \Gamma(U, \mathcal{J}(\varphi) \otimes_{\mathcal{O}_X} \omega_X) \iff \mu^* \omega \in \Gamma(\mu^{-1}(U), \mathcal{J}(\varphi \circ \mu) \otimes_{\mathcal{O}_Y} \omega_Y).$$

□

PROPOSITION 2.6.22. *For every effective \mathbf{Q} -divisor D on X we have*

$$\mathcal{J}_{\text{an}}(D) = \mathcal{J}(D)^{\text{an}}.$$

PROOF. Since by Exercise 2.6.12 and Lemma 2.6.21 the two types of multiplier ideals satisfy the same birational transformation formula, by passing to a log resolution it is enough to check the statement when D is assumed to have SNC support.

Let's assume then that $D = \sum_{i=1}^k a_i D_i$, where $a_i \in \mathbf{Q}_{>0}$ and $\sum D_i$ is SNC. We need to show that

$$\mathcal{J}_{\text{an}}(D) = \mathcal{O}_X(-[D])^{\text{an}}.$$

Choosing local coordinates x_i such that $D_i = (x_i = 0)$, this can be reinterpreted as saying that for a holomorphic function g on such a neighborhood we have

$$\frac{|g|^2}{\prod_{i=1}^k |x_i|^{2a_i}} \text{ is locally integrable } \iff x_1^{[a_1]} \cdots x_n^{[a_n]} \mid g.$$

A standard reduction allows us to assume that g is a monomial in the x_i , in which case by Fubini's theorem we can separate the variables and reduce the statement to the fact that for a single variable z , on a ball B around the origin, say of radius ε , we have

$$\int_B \frac{1}{|z|^{2c}} < \infty \iff c < 1.$$

In polar coordinates $z = re^{i\theta}$ the integral on the left is $2\pi \int_0^\varepsilon r^{1-2c} dr$, and this is easily checked. □

COROLLARY 2.6.23. *If D is as in Definition 2.6.19, then*

$$\text{lct}(D) = \sup\{c > 0 \mid \frac{1}{\prod_{i=1}^k |f_i|^{2ca_i}} \text{ is locally integrable around } x\}.$$

PROOF. We have seen that

$$\text{lct}_x(D) = \sup\{c \in \mathbf{Q} \mid \mathcal{J}(cD)_x = \mathcal{O}_{X,x}\}.$$

Thanks to Proposition 2.6.22 we then have

$$\text{lct}_x(D) = \sup\{c \in \mathbf{Q} \mid 1 \in \mathcal{J}_{\text{an}}(cD)_x\},$$

which is equivalent by definition to the assertion in the Corollary. □

REMARK 2.6.24. Note that Corollary 2.6.23 (and the calculation in Proposition 2.6.22) shows that the invariant defined in terms of discrepancies in (2.6.4) is indeed independent of the choice of log resolution.

EXAMPLE 2.6.25. Let $D = (x^a + y^b = 0) \subset \mathbf{C}^2$. We check using the interpretation in Corollary 2.6.23 that

$$\text{lct}(D) = \frac{1}{a} + \frac{1}{b}.$$

According to its statement, we have

$$\text{lct}(D) = \sup\{c > 0 \mid \int_B |x^a + y^b|^{-2c} < \infty\}$$

on any ball B in the neighborhood of the origin. Consider now the unit ball $B = B(0; 1) \subset \mathbf{C}^2$, and the transformation

$$T: B \rightarrow B, \quad (x, y) \mapsto \left(\frac{x}{2^a}, \frac{y}{2^b}\right).$$

Denote $T^{(r)} = T \circ \dots \circ T$, the r -fold composition, and set

$$\Omega_r := T^{(r-1)}(B) \setminus T^{(r)}(B).$$

We clearly have

$$\bigcup_{r \geq 1} \Omega_r = B \setminus \{0\}.$$

Denoting $I_r = \int_{\Omega_r} |x^a + y^b|^{-2c}$, and using the change of variables $x \mapsto 2^{-b}x$, $y \mapsto 2^{-a}y$ (hence $dx dy \mapsto 2^{-(a+b)} dx dy$), we have

$$I_{r+1} = \int_{\Omega_r} 2^{2abc} |x^a + y^b|^{-2c} 2^{-(a+b)} = 2^{2(abc-(a+b))} \cdot I_r.$$

Hence

$$\int_{B \setminus \{0\}} |x^a + y^b|^{-2c} = \sum_{r \geq 1} I_r = I_1 \cdot (1 + u + u^2 + \dots),$$

where $u = 2^{2(abc-(a+b))}$, and so this is finite if and only if $abc < a + b$.

2.7. Log canonical threshold and jumping numbers as roots

We go back to the discussion of the archimedean zeta function. Let f be a regular function on X . It turns out that the greatest pole of the distribution $|f|^{2s}$ is the well-known invariant of the singularities of f discussed in the previous section.

PROPOSITION 2.7.1. *The greatest pole of $|f|^{2s}$ is $-\text{lct}(f)$.*

PROOF. We use the analytic interpretation of the log-canonical threshold discussed in Corollary 2.6.23, namely

$$c_0 := \text{lct}(f) = \sup\{c > 0 \mid |f|^{-2c} \text{ is locally integrable}\}.$$

By the same argument as in Proposition 2.4.2, we have that $Z_\varphi(s) = \int_{\mathbf{C}^n} |f|^{2s} \varphi$ is analytic at $s = -c$ for any $c < c_0$ and any $\varphi \in C_c^\infty(\mathbf{C}^n)$. Therefore we need to show that there exists a φ such that the continuation of $Z_\varphi(s)$ does have a pole at $s = -c_0$.

Let B be a ball in the neighborhood of a point x in the zero locus of f such that

$$\int_B |f|^{-2c_0} = \infty,$$

and let φ be a bump function in a neighborhood of x such that φ is identically 1 on B . If we assumed that the continuation of $Z_\varphi(s)$ were analytic at $s = -c_0$, and so the limit of $|Z_\varphi(-c)|$ would be finite as $c \rightarrow c_0$ from the left. On the other hand, by Fatou's Lemma we have that

$$\liminf_{c \rightarrow c_0} \int_{\mathbf{C}^n} |f|^{-2c} \varphi \geq \int_{\mathbf{C}^n} |f|^{-2c_0} \varphi \geq \int_B |f|^{-2c_0} = \infty,$$

which is a contradiction. \square

According to Theorem 2.4.3 and Proposition 2.7.1, the log canonical threshold of f should then be an integral shift of a root of $b_f(s)$. The best possible scenario does in fact happen, according to the following theorem appearing in works of Yano, Lichtin, and Kollár. We follow Kollár's approach in [K05], similar to the methods discussed above.

THEOREM 2.7.2. *Let $f \in \mathbf{C}[X_1, \dots, X_n]$ be a nontrivial polynomial (or germ of analytic function), and let α_f be the negative of the greatest root of the Bernstein-Sato polynomial $b_f(s)$. Then*

$$\alpha_f = \text{lct}(f).$$

PROOF. *Step 1.* In this step we only show that $-\text{lct}(f)$ is a root of $b(s) = b_f(s)$. We use the analytic interpretation of the log canonical threshold, namely

$$c_0 := \text{lct}(f) = \sup\{c > 0 \mid \frac{1}{|f|^{2c}} \text{ is locally } L^1\}.$$

Therefore for some point x in the zero locus of f and some small ball B around x , the function $\frac{1}{|f|^{2c}}$ with $c = c_0 - \varepsilon$ for $0 < \varepsilon \ll 1$ is integrable on B , but $\frac{1}{|f|^{2c_0}}$ is not integrable on some compact ball B' strictly contained in B .

Let us now take $s = -c$ in Lemma 2.4.4, so that

$$b(-c)^2 |f|^{-2c} = (P(-c)\bar{P}(-c)) |f|^{2(-c+1)}.$$

We can think of this as being an equality of distributions, since both sides are integrable on B . Therefore, for any smooth positive test function φ supported on B , we have

$$(2.7.1) \quad \int_B b(-c)^2 |f|^{-2c} \varphi = \int_B (P(-c)\bar{P}(-c)) |f|^{2(-c+1)} \varphi.$$

We can in fact take φ to be a bump function with support in B , identically equal to 1 on B' , in which case we obtain that the left-hand side of (2.7.1) is at least

$$b(-c)^2 \int_{B'} |f|^{-2c}.$$

Using integration by parts in a way similar to Exercise 2.4.5, we see that the right-hand side of (2.7.1) is equal to

$$\int_B |f|^{2(-c+1)} (P(-c)\bar{P}(-c)\varphi),$$

and therefore if ε is in a fixed interval $(0, \delta]$, then it is bounded above by some $M > 0$ depending only on φ (as $c - 1$ belongs to a closed interval of values for which the integral is finite and depends continuously on c). We deduce that

$$b(-c)^2 \int_{B'} |f|^{-2c} \leq M < \infty$$

for every such c . On the other hand, $\frac{1}{|f|^{2c_0}}$ is not integrable on B' , hence by Fatou's Lemma we have

$$\int_{B'} |f|^{-2c} \rightarrow \infty \quad \text{as } c \rightarrow c_0.$$

The only way this can happen is if $b(-c_0) = 0$.

Step 2. To show in addition that $-c_0$ is the greatest root of $b(s)$, we need to use in addition Lichtin's refinement of Kashiwara's result on the rationality of the roots of $b(s)$, Theorem 3.5.1 below. If $f: Y \rightarrow \mathbf{C}^n$ is log resolution of $D = (f = 0)$ with the property that it is an isomorphism away from D and \tilde{D} smooth (which can always be achieved by a few extra blow-ups), then writing

$$f^*D = \sum_{i=1}^m a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i E_i,$$

Lichtin's theorem tells us that all the roots of $b(s)$ are of the form

$$-\frac{b_i + 1 + \ell}{a_i} \quad \text{with } 1 \leq i \leq m,$$

where $\ell \geq 0$ is an integer. Since on the other hand we know by the proof of Lemma 2.6.6 that

$$c_0 = \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i},$$

and $c_0 \leq 1$, it is then clear that no root can exceed $-c_0$. \square

In [ELSV], Theorem 2.7.2 was extended along similar lines to the following statement:

THEOREM 2.7.3. *With the same hypothesis as in Theorem 2.7.2, let ξ be a jumping coefficient of f in the interval $(0, 1]$. Then $-\xi$ is a root of $b_f(s)$.*

PROOF. Let ξ' be the previous jumping coefficient (taken by convention to be equal to 0 if ξ is the first jumping coefficient, i.e. the log canonical threshold). Recall that this means that for every $c \in [\xi', \xi)$ we have

$$\mathcal{I}(\xi' \cdot f) = \mathcal{I}(c \cdot f),$$

but there exists $x \in Z(f)$ such that

$$\mathcal{I}(\xi \cdot f)_x \subsetneq \mathcal{I}(c \cdot f)_x.$$

Recall now that the analytic interpretation of multiplier ideals gives

$$\mathcal{I}(c \cdot f) = \{g \in \mathcal{O}_{\mathbf{C}^n} \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally } L^1\}.$$

Therefore for any $c \in [\xi', \xi)$, there exists a function g and a small ball B around x such that

$$\int_B \frac{|g|^2}{|f|^{2c}} < \infty \quad \text{but} \quad \int_B \frac{|g|^2}{|f|^{2\xi}} = \infty.$$

The argument then goes through exactly as in the proof of Theorem 2.7.2, after multiplying the two integrands in (2.7.1) by $|g|^2$. \square

REMARK 2.7.4. The converse of the Theorem above is not true. For instance, Saito [Sa3, Example 3.5] shows that if $f(x, y) = x^5 + y^4 + x^3y^2$ then $b_f(-s)$ has roots in $(0, 1]$ that are not jumping numbers for f ; cf. also [ELSV, Example 2.5]. Despite this, we nevertheless have the following behavior, similar to that of jumping numbers:

COROLLARY 2.7.5. *Let*

$$-1 = \alpha_m < \alpha_{m-1} < \cdots < \alpha_1 < 0$$

be the distinct roots of $b_{f,x}(s)$ in the interval $[-1, 0)$, for some $x \in X$. Then

$$\alpha_i + \alpha_1 \leq \alpha_{i+1}, \quad \text{for all } 1 \leq i < m.$$

PROOF. Set $\beta_i = -\alpha_i$, so that the inequality to be shown is

$$\beta_{i+1} \leq \beta_i + \beta_1.$$

Denote by c_j and c_{j+1} the two consecutive jumping numbers of f at x such that $c_j \leq \beta_i < c_{j+1}$. Note also that the first nontrivial jumping number is $c_1 = \text{lct}_x(f) = \beta_1$, according to Theorem 2.7.2. Now according to Lemma 2.6.18, we have

$$c_{j+1} \leq c_j + c_1 \leq \beta_i + \beta_1.$$

On the other hand, clearly $c_j < 1$, hence $c_{j+1} \leq 1$. Therefore by Theorem 2.7.3 there exists some $k > i$ such that $c_{j+1} = \beta_k \geq \beta_{i+1}$, and the result follows. \square

Note. A detailed account of the developments leading to Theorems 2.7.2 and 2.7.3 can be found in [ELSV]. These theorems are also consequences of the stronger Budur-Saito theorem [BS], comparing multiplier ideals with the V -filtration, explained in §3.6. Another generalization, giving a criterion for jumping numbers of higher Hodge ideals to be roots of the Bernstein-Sato polynomial, is proved in [MP4, Proposition 6.14].

CHAPTER 3

The V -filtration, and more on Bernstein-Sato polynomials

The aim of this chapter is to introduce and study the V -filtration on \mathcal{D} -modules along hypersurfaces, according to Kashiwara, Malgrange, and Saito. Based on this notion, we introduce in passing nearby and vanishing cycles for \mathcal{D} -modules, which will be needed in later chapters. We then look at generalized Bernstein-Sato polynomials for arbitrary elements in certain \mathcal{D} -modules, and connect both the roots of these polynomials and the V -filtration to objects in birational geometry.

3.1. V -filtration: the smooth case

Let Y be a smooth complex variety, of dimension n . We consider a hypersurface X in Y which we assume to be defined globally by a function $f \in \mathcal{O}_Y$. In this section X is always assumed to be *smooth*.

DEFINITION 3.1.1. For any $k \in \mathbf{Z}$ we define

$$V_k \mathcal{D}_Y := \{P \in \mathcal{D}_Y \mid P \cdot (f)^i \subseteq (f)^{i-k}\},$$

with the convention that $(f)^i = \mathcal{O}_Y$ for $i < 0$.

Since X is smooth, we can consider local algebraic coordinates $x_1, \dots, x_{n-1}, t = f$ on Y . The following exercise summarizes the main properties of this filtration on \mathcal{D}_Y , some of them described in these local coordinates.

EXERCISE 3.1.2. The filtration $V_\bullet \mathcal{D}_Y$ satisfies the following properties:

- (1) $V_0 \mathcal{D}_Y = \{P \in \mathcal{D}_Y \mid P = \sum_{\alpha, k, \ell} a_{\alpha, k, \ell}(x) \partial_x^\alpha t^\ell \partial_t^k, \alpha \in \mathbf{N}^{n-1}, k \geq \ell\} = \mathcal{D}_Y[t, t\partial_t]$.
- (2) For all $k \in \mathbf{N}$, we have

$$P \in V_{-k} \mathcal{D}_Y \iff P = t^k \cdot Q, \quad Q \in V_0 \mathcal{D}_Y.$$

- (3) For all $k \in \mathbf{N}$, we have

$$P \in V_k \mathcal{D}_Y \iff P = \sum_{j=0}^k Q_j \cdot \partial_t^j, \quad Q_j \in V_0 \mathcal{D}_Y \text{ for all } j.$$

- (4) $V_\bullet \mathcal{D}_Y$ is increasing and exhaustive.
- (5) For all $k, \ell \in \mathbf{Z}$ we have

$$V_k \mathcal{D}_Y \cdot V_\ell \mathcal{D}_Y \subseteq V_{k+\ell} \mathcal{D}_Y,$$

with equality if $k, \ell \leq 0$ or $k, \ell \geq 0$.

- (6) $\bigcap_{k \in \mathbf{Z}} V_k \mathcal{D}_Y = \{0\}$.

- (7) $\mathrm{gr}_0^V \mathcal{D}_Y \simeq \mathcal{D}_Y[t\partial_t]$.
(8) The class of $t\partial_t$ in $\mathrm{gr}_0^V \mathcal{D}_Y$ does not depend on the choice of local coordinates.

Fix now a coherent left \mathcal{D} -module \mathcal{M} on Y .

DEFINITION 3.1.3. A V -filtration on \mathcal{M} along X is a rational filtration $(V^\gamma = V^\gamma \mathcal{M})_{\gamma \in \mathbf{Q}}$ that is exhaustive, decreasing, discrete, and left continuous,¹ such that the following conditions are satisfied:

- (1) Each V^γ is a coherent module over $V_0 \mathcal{D}_Y = \mathcal{D}_X[t, \partial_t]$.
(2) For every $\gamma \in \mathbf{Q}$, we have an inclusion

$$t \cdot V^\gamma \subseteq V^{\gamma+1},$$

with equality if $\gamma > 0$.

- (3) For every $\gamma \in \mathbf{Q}$, we have

$$\partial_t \cdot V^\gamma \subseteq V^{\gamma-1}.$$

- (4) For every $\gamma \in \mathbf{Q}$, if we set $V^{>\gamma} = \bigcup_{\gamma' > \gamma} V^{\gamma'}$, then $\partial_t t - \gamma$ acts nilpotently on

$$\mathrm{gr}_V^\gamma := V^\gamma / V^{>\gamma}.$$

EXAMPLE 3.1.4. (1) Let $\mathcal{M} = \mathcal{O}_Y$ and $X = (t = 0)$. We can then consider the t -adic filtration, more precisely

$$V^m \mathcal{O}_Y = (t)^{m-1}, \quad \forall m \in \mathbf{Z}$$

(with the convention that $(t)^j = \mathcal{O}_Y$ for $j < 0$) and $V^\alpha \mathcal{O}_Y = V^{[\alpha]} \mathcal{O}_Y$ for $\alpha \in \mathbf{Q}$. This is easily seen to be a V -filtration. For instance, since

$$\partial_t t(t^{m-1}) = mt^{m-1},$$

we have that $\partial_t t - m$ is identically zero on $\mathrm{gr}_V^m \mathcal{O}_Y$.

(2) More generally, let E be a vector bundle with flat connection on Y . We can then consider the filtration on E where $V^m E$ is the subsheaf generated by $t^{m-1} \cdot E$ for $m \in \mathbf{Z}$, with the same convention as above, and $V^\alpha E = V^{[\alpha]} E$ for $\alpha \in \mathbf{Q}$. Again, this is easily seen to be a V -filtration. For instance, for every local section s of E we have

$$\partial_t(t^{m-1} \cdot s) = (m-1)t^{m-2} \cdot s + t^{m-1} \partial_t s \in t^{m-2} \cdot E,$$

which verifies (3), and

$$(\partial_t t - m)(t^{m-1} \cdot s) = t^m \partial_t s,$$

which shows that $\partial_t t - m$ is identically zero on $\mathrm{gr}_V^m E$ and hence verifies (4).

(3) Let \mathcal{M} be an arbitrary coherent \mathcal{D}_Y -module supported on X . In this case an explicit V -filtration on \mathcal{M} is provided by the proof of Kashiwara's theorem. Indeed, for every $j \in \mathbf{Z}$ consider the eigenspace

$$\mathcal{M}^j := \{s \in \mathcal{M} \mid (\partial_t t - j) \cdot s = 0\}$$

¹More precisely, the filtration has the property that there is a positive integer ℓ such that V^γ takes constant value in each interval $(\frac{i}{\ell}, \frac{i+1}{\ell}]$, for all $i \in \mathbf{Z}$.

for the operator $\partial_t t$. In the course of the proof of Kashiwara's theorem we showed that

$$\mathcal{M} = \bigoplus_{j \leq 0} \mathcal{M}^j.$$

We define

$$V^m \mathcal{M} := \bigoplus_{j \geq m} \mathcal{M}^j, \quad \forall m \in \mathbf{Z}$$

and $V^\alpha \mathcal{M} = V^{[\alpha]} \mathcal{M}$ for $\alpha \in \mathbf{Q}$. This is a V -filtration; all the required properties were shown in the course of the proof of Kashiwara's theorem. Note that (4) follows by definition, since $\text{gr}_V^m \mathcal{M} = \mathcal{M}^m$. Note also that $V^\alpha \mathcal{M} = 0$ for all $\alpha > 0$.

Also as part of the proof of Kashiwara's theorem we have seen that

$$\mathcal{M} \simeq \mathcal{M}^0 \otimes_{\mathbf{C}} \mathbf{C}[\partial_t] \quad \text{and} \quad \mathcal{M}^{-k} = \partial_t^k \cdot \mathcal{M}^0.$$

Hence we have the alternative interpretation

$$V^{-m} \mathcal{M} = \bigoplus_{j=0}^m \mathcal{M}^0 \otimes \partial_t^j.$$

(4) By analogy with (1), we can also describe the V -filtration on the localization $\mathcal{M} = \mathcal{O}_Y(*X)$, where $X = (t = 0)$. We now consider the t -adic filtration in the generalized sense that we also allow negative powers, namely

$$V^m \mathcal{O}_Y(*X) = (t)^{m-1}, \quad \forall m \in \mathbf{Z}$$

and again $V^\alpha \mathcal{O}_Y = V^{[\alpha]} \mathcal{O}_Y$ for $\alpha \in \mathbf{Q}$. It is again straightforward to check that this is a V -filtration along X .

PROPOSITION 3.1.5. *If it exists, a V -filtration on \mathcal{M} along X is unique.*

PROOF. Let $W^\bullet = W^\bullet \mathcal{M}$ be another filtration satisfying all the properties in Definition 3.1.3. It suffices to show

$$V^\alpha \subseteq W^\alpha, \quad \forall \alpha \in \mathbf{Q},$$

since then the process can be reversed. We first claim that for all rational numbers $\alpha \neq \beta$ we have

$$(3.1.1) \quad V^\alpha \cap W^\beta = V^{>\alpha} \cap W^\beta + V^\alpha \cap W^{>\beta}.$$

Indeed, consider the quotient module

$$U_{\alpha,\beta} := \frac{V^\alpha \cap W^\beta}{V^{>\alpha} \cap W^\beta + V^\alpha \cap W^{>\beta}}.$$

It is clear that $U_{\alpha,\beta}$ is a sub-quotient of both gr_V^α and gr_V^β , and hence both operators $\partial_t t - \alpha$ and $\partial_t t - \beta$ act nilpotently on it by property (4) in the definition of the V -filtration. It follows that so does multiplication by $\alpha - \beta$, hence $U_{\alpha,\beta} = 0$.

We next claim that for every $\alpha \in \mathbf{Q}$ we have

$$(3.1.2) \quad V^\alpha \subseteq V^{>\alpha} + W^\alpha.$$

To this end, fix $v \in V^\alpha$. Since W^\bullet is decreasing and exhaustive, there exists $\beta < \alpha$ such that $v \in W^\beta$. From (3.1.1) we deduce that we can write

$$v = v_1 + v_2, \quad v_1 \in V^{>\alpha}, \quad v_2 \in V^\alpha \cap W^{>\beta}.$$

We can therefore replace v by v_2 ; note that

$$\widehat{v} = \widehat{v}_2 \in \text{gr}_V^\alpha,$$

but for v_2 can choose a larger β . We can now repeat this process as long as $\beta < \alpha$. Since W^\bullet is discrete, with bounded denominators, after finitely many jumps we can then reach the situation where $\beta \geq \alpha$. Therefore $W^\beta \subseteq W^\alpha$, and the claim is proved.

The next claim is that in fact

$$(3.1.3) \quad V^\alpha \subseteq V^\beta + W^\alpha, \quad \forall \beta \geq \alpha.$$

Indeed, let $\beta_0 > \alpha$ such that $V^{>\alpha} = V^{\beta_0}$. Then by (3.1.2) we have

$$V^\alpha \subseteq V^{\beta_0} + W^\alpha \subseteq V^{>\beta_0} + W^{\beta_0} + W^\alpha = V^{>\beta_0} + W^\alpha.$$

We can then repeat the process, and again since V^\bullet is discrete with bounded denominators, after finitely many steps we reach (3.1.3).

Finally from (3.1.3) and property (2) in the definition of the V -filtration we deduce that for $\beta \gg 0$ we have

$$(3.1.4) \quad V^\alpha \subseteq t^q \cdot V^\beta + W^\alpha, \quad \forall q \geq 0.$$

On the other hand, V^β is finitely generated over $\mathcal{D}_X[t, t\partial_t]$, and W^\bullet is exhaustive, so there exists $\gamma \in \mathbf{Q}$ such that $V^\beta \subseteq W^\gamma$. Choosing q such that $q + \gamma \geq \alpha$, using (3.1.4) we obtain

$$V^\alpha \subseteq t^q \cdot W^\gamma + W^\alpha \subseteq W^{q+\gamma} + W^\alpha = W^\alpha.$$

□

REMARK 3.1.6. What we have in fact shown above is the more general fact that if the filtration W^\bullet satisfies properties (2) and (4) in Definition 3.1.3, without being assumed to be finitely generated, then $V^\alpha \subseteq W^\alpha$ for all $\alpha \in \mathbf{Q}$.

Before stating the existence theorem, we list a few other basic properties of the V -filtration.

EXERCISE 3.1.7. Let $u: \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of \mathcal{D}_Y -modules endowed with a V -filtration along X . Then u is compatible with the V -filtration, and in fact *strictly compatible*, in the sense that

$$u(V^\alpha \mathcal{M}) = u(\mathcal{M}) \cap V^\alpha \mathcal{N}$$

for all α . In particular, the category of \mathcal{D}_X -modules endowed with V -filtration is abelian, and its morphisms are strict.

LEMMA 3.1.8. Let $u: \mathcal{M} \rightarrow \mathcal{N}$ be a morphism of \mathcal{D}_Y -modules such that $u|_U$ is an isomorphism, where $U = Y \setminus X$. Then the induced morphisms

$$u: V^\alpha \mathcal{M} \longrightarrow V^\alpha \mathcal{N}$$

are isomorphisms for all $\alpha > 0$. In particular $V^{>0} \mathcal{M}$ depends only on the restriction of \mathcal{M} to U .

PROOF. Denoting by \mathcal{K} and \mathcal{Q} the kernel and cokernel of u , by Exercise 3.1.7 we obtain, for each α , an exact sequence

$$0 \longrightarrow V^\alpha \mathcal{K} \longrightarrow V^\alpha \mathcal{M} \longrightarrow V^\alpha \mathcal{N} \longrightarrow V^\alpha \mathcal{Q} \longrightarrow 0.$$

But since \mathcal{K} and \mathcal{Q} are supported on D , we have $V^\alpha \mathcal{K} = V^\alpha \mathcal{Q} = 0$ for all $\alpha > 0$ by Example 3.1.4(3). \square

The main existence theorem for the V -filtration is the following:

THEOREM 3.1.9 (Malgrange, Kashiwara). *Let \mathcal{M} be a regular, holonomic, coherent \mathcal{D}_Y -module, with quasi-unipotent monodromy around X . Then \mathcal{M} admits a V -filtration along X .*

REMARK 3.1.10. (1) The quasi-unipotence hypothesis is necessary in order to obtain a rationally indexed V -filtration. It is a condition that is automatically satisfied in Hodge theory, in particular for all Hodge \mathcal{D} -modules. It can be removed if one is willing to work with a \mathbf{C} -indexed filtration instead.

(2) The more precise condition that implies the existence of the V -filtration is that \mathcal{M} is a holonomic \mathcal{D} -module such that every element $u \in \mathcal{M}$ admits a Bernstein-Sato polynomial whose roots are rational numbers. We will use this approach in the next section to prove Theorem 3.1.9 for our main \mathcal{D} -modules of interest, following Malgrange.

3.2. V -filtration: the general case

Let now X be a smooth complex variety of dimension n , and $f \in \mathcal{O}_X$ an arbitrary nontrivial function. One reduces to the smooth case using the graph embedding

$$\iota: X \hookrightarrow X \times \mathbf{C}, \quad x \mapsto (x, f(x))$$

described in §2.1. Denote $Y = X \times \mathbf{C}$, and let t be the coordinate on the second factor \mathbf{C} . For a left \mathcal{D}_X -module \mathcal{M} , according to the previous section we can consider the notion of a V -filtration on $\iota_+ \mathcal{M}$ along the hypersurface ($t = 0$). We recall its definition for convenience.

DEFINITION 3.2.1. A V -filtration on $\iota_+ \mathcal{M}$ is a rational filtration $(V^\gamma = V^\gamma \iota_+ \mathcal{M})_{\gamma \in \mathbf{Q}}$ that is exhaustive, decreasing, discrete, and left continuous such that the following conditions are satisfied:

- (1) Each V^γ is a coherent module over $\mathcal{D}_X[t, \partial_t t]$.
- (2) For every $\gamma \in \mathbf{Q}$, we have an inclusion

$$t \cdot V^\gamma \subseteq V^{\gamma+1},$$

with equality if $\gamma > 0$.

- (3) For every $\gamma \in \mathbf{Q}$, we have

$$\partial_t \cdot V^\gamma \subseteq V^{\gamma-1}.$$

- (4) For every $\gamma \in \mathbf{Q}$, if we set $V^{>\gamma} = \bigcup_{\gamma' > \gamma} V^{\gamma'}$, then $\partial_t t - \gamma$ acts nilpotently on $\text{gr}_V^\gamma := V^\gamma / V^{>\gamma}$.

EXERCISE 3.2.2. Show that the definition depends only on the hypersurface $D = (f = 0)$ and not on the particular f chosen. Hence it can be extended to the global setting.

REMARK 3.2.3. If a V -filtration on $\iota_+ \mathcal{M}$ exists, then it induces a filtration on \mathcal{M} as well, by indentifying it with $\mathcal{M} \otimes 1 \subset \iota_+ \mathcal{M}$. In other words

$$V^\alpha \mathcal{M} := V^\alpha \iota_+ \mathcal{M} \cap (\mathcal{M} \otimes 1), \quad \forall \alpha \in \mathbf{Q}.$$

In the case $\mathcal{M} = \mathcal{O}_X$, this is intimately linked to important objects in birational geometry.

The main point we want to study here is the existence of the V -filtration on the \mathcal{D}_Y -module $\iota_+ \mathcal{O}_X$. To this end, it will be convenient to work in the larger \mathcal{D} -module $\iota_+ \mathcal{O}_X(*D)$, where D is the hypersurface $(f = 0)$ in X . We will use the description of these two \mathcal{D} -modules in §2.1. More generally, we will consider any \mathcal{D}_X -module \mathcal{M} on which multiplication by f is bijective, as it is the case with $\mathcal{O}_X(*D)$.

REMARK 3.2.4. If we assume that \mathcal{M} is such a \mathcal{D}_X -module and there is a V -filtration on $\iota_+ \mathcal{M}$, then

$$t \cdot V^\alpha = V^{\alpha+1} \quad \text{for all } \alpha \in \mathbf{Q},$$

where for simplicity we denote $V^\alpha = V^\alpha \iota_+ \mathcal{M}$. Indeed, the inclusion “ \subseteq ”, as well as the reverse inclusion for $\alpha > 0$, follow from general properties of the V -filtration. Moreover, the induced map

$$\text{gr}(t): \text{gr}_V^\delta \rightarrow \text{gr}_V^{\delta+1}$$

is an isomorphism if $\delta \neq 0$. (EXPLAIN; we are only using injectivity below, which is clear.)

Suppose now that $\alpha \leq 0$ and $u = tw \in V^{\alpha+1}$. Let $\delta \ll 0$ be such that $w \in V^\delta$. If $\delta \geq \alpha$, then we are done. On the other hand, if $\delta < \alpha$, then $\delta \neq 0$ since $\alpha \leq 0$; since $tw \in V^{>\delta+1}$, we conclude that $w \in V^{>\delta}$. After repeating this argument finitely many times, we obtain $w \in V^\alpha$.

REMARK 3.2.5. In light of Remarks 2.1.5 and 3.2.4, if \mathcal{M} is a \mathcal{D}_X -module on which multiplication by f is bijective, a V -filtration on $\iota_+ \mathcal{M}$ can be characterized as an exhaustive, decreasing, discrete, left continuous, rational filtration $(V^\gamma = V^\gamma \iota_+ \mathcal{M})_{\gamma \in \mathbf{Q}}$, that satisfies the following conditions:

- (1) Each V^γ is a coherent module over $\mathcal{D}_X \langle t, t^{-1}, s \rangle$.
- (2) For every $\gamma \in \mathbf{Q}$, we have $t \cdot V^\gamma = V^{\gamma+1}$.
- (3) For every $\gamma \in \mathbf{Q}$, we have $\partial_t \cdot V^\gamma \subseteq V^{\gamma-1}$.
- (4) For every $\gamma \in \mathbf{Q}$, the operator $s + \gamma$ acts nilpotently on Gr_V^γ .

We now address the main result of this section. The proof makes use of the rationality of the roots of the Bernstein-Sato polynomial $b_f(s)$, and of the $\mathcal{D}_X[s]$ module $\mathcal{N}_f = \mathcal{D}_X[s]f^s$ studied in §2.5.

THEOREM 3.2.6. *There exists a V -filtration on $\iota_+ \mathcal{O}_X$ and $\iota_+ \mathcal{O}_X(*D)$.*

PROOF. The proof will be divided into a few steps:

Step 1. In this step we construct a $\mathcal{D}_X[s]$ -submodule \mathcal{N}' of $\iota_+ \mathcal{O}_X(*D)$ with the following two properties:

- (1) $t^k \mathcal{N}_f \subseteq \mathcal{N}' \subseteq t^m \mathcal{N}_f$ for some $k, m \in \mathbf{Z}$.
- (2) $t \mathcal{N}' \subseteq \mathcal{N}'$, and the action of $-s = \partial_t t$ on $\mathcal{N}'/t \mathcal{N}'$ has a minimal polynomial whose roots belong to $\mathbf{Q} \cap (0, 1]$.

We start by noting that since the action of t on $\iota_+ \mathcal{O}_X(*D)$ is bijective, for every $k \in \mathbf{Z}$ we have an induced isomorphism of \mathcal{D}_X -modules

$$t^k: \mathcal{N}_f/t \mathcal{N}_f \rightarrow t^k \mathcal{N}_f/t^{k+1} \mathcal{N}_f.$$

There is an induced action of $-s$ on $t^k \mathcal{N}_f/t^{k+1} \mathcal{N}_f$, and note that its minimal polynomial is equal to $b_f(-s+k)$. This follows from the identities

$$b_f(s+k) \cdot t^k \cdot f^s = t^k \cdot b_f(s) \cdot f^s = t^k \cdot P(s) \cdot f \cdot f^s \in t^{k+1} \mathcal{N}_f,$$

where the first equality uses (2.1.2), and from the fact that we can also go in reverse.

Let now

$$\alpha_1 < \cdots < \alpha_r$$

be the distinct roots of the polynomial $b_f(-s)$. By Kashiwara's Theorem 2.5.1 we know that they are positive rational numbers. We also choose $k, m \in \mathbf{Z}$ such that

$$k + \alpha_1 > 1 \quad \text{and} \quad m + \alpha_r \leq 1.$$

In particular we have $k > m$. Using the observation in the previous paragraph, we deduce that the action of $-s$ on $t^m \mathcal{N}_f/t^k \mathcal{N}_f$ has a minimal polynomial whose roots are of the form

$$\lambda_{ij} := \alpha_i + j, \quad \text{for } i = 1, \dots, r \quad \text{and } j = m, \dots, k-1.$$

In what follows we will denote any of the λ_{ij} by λ .

We now write

$$t^m \mathcal{N}_f/t^k \mathcal{N}_f = \bigoplus_{\lambda} P_{\lambda},$$

where P_{λ} is the submodule on which $s + \lambda$ acts nilpotently. Note that since the action of $-s$ is \mathcal{D}_X -linear, each P_{λ} is a \mathcal{D}_X -submodule. We now consider the submodule

$$t^k \mathcal{N}_f \subseteq \mathcal{N}' \subseteq t^m \mathcal{N}_f$$

satisfying

$$\mathcal{N}'/t^k \mathcal{N}_f = \bigoplus_{\lambda > 0} P_{\lambda}.$$

We first claim that

$$(3.2.1) \quad t \cdot \mathcal{N}' \subseteq \mathcal{N}'.$$

To this end, note that if $u \in P_{\lambda}$, then $(s + \lambda)^e \cdot u = 0$ for some $e \geq 1$. Hence using (2.1.2) we have

$$t \cdot (s + \lambda)^e \cdot u = (s + \lambda + 1)^e \cdot tu = 0,$$

so the action of t on $t^m \mathcal{N}_f/t^k \mathcal{N}_f$ maps P_{λ} to $P_{\lambda+1}$, which gives the claim.

The next claim is that the operator

$$\sigma := \prod_{0 < \lambda \leq 1} (s + \lambda)$$

acts nilpotently on $\mathcal{N}'/t\mathcal{N}'$, which would finish the construction described at the beginning of this step. To check this, let $v \in \mathcal{N}'$. By definition, there exists a positive integer N , depending only on $b_f(s)$, such that if $w = \sigma^N(v)$, then

$$\widehat{w} \in \bigoplus_{\lambda > 1} P_\lambda \subseteq \mathcal{N}'/t^k \mathcal{N}_f.$$

We next show that $w \in t\mathcal{N}'$; this means that $\sigma^N(\widehat{w}) = 0$, which proves the claim. Recall that the minimal polynomial of the action of $-s$ on $t^m \mathcal{N}_f/t^{m+1} \mathcal{N}_f$ is equal to $b_f(-s+m)$, hence by definition $b_f(-s+m)$ divides a power of

$$\prod_{j=1}^r (s - (\alpha_j + m)).$$

On the other hand we have $\alpha_j + m \leq 1$ for all j , hence the projection of (the preimage of) P_λ to $t^m \mathcal{N}_f/t^{m+1} \mathcal{N}_f$ is 0 if $\lambda > 1$. Hence we have

$$w \in t^{m+1} \mathcal{N}_f.$$

The proof of the claim will then be finished by showing that there exists an integer $N_1 > 0$ such that

$$(3.2.2) \quad \prod_{\lambda > 0} (s + \lambda)^{N_1} \cdot t^{-1} w \in t^k \mathcal{N}_f.$$

Indeed, we would then have that $t^{-1} w \in \mathcal{N}'$, or equivalently $w \in t\mathcal{N}'$. Before proving this, let's also record the fact that, again using (2.1.2), we have

$$(3.2.3) \quad \prod_{\lambda > 0} (s + \lambda)^{N_1} \cdot t^{-1} w = t^{-1} \cdot \prod_{\lambda > 0} (s + \lambda + 1)^{N_1} w$$

To see (3.2.2), first we know that by definition there exists an integer $N_2 > 0$ such that

$$w' := \prod_{\lambda > 0} (s + \lambda)^{N_2} w \in t^k \mathcal{N}_f.$$

Now the minimal polynomial of the action of $-s$ on $t^k \mathcal{N}_f/t^{k+1} \mathcal{N}_f$ is $b_f(-s+k)$, hence there also exists an integer $N_3 > 0$ such that

$$\prod_{i=1}^r (s + \alpha_i + k)^{N_3} w' \in t^{k+1} \mathcal{N}_f.$$

Recall that all λ are of the form $\alpha_i + j$ where $j = m, \dots, k-1$, and so if $\lambda > 1$, then $\lambda - 1$ is also of this form. Thus all the factors in the product in (3.2.2) appear also on the right hand side in (3.2.3). Similarly, all the factors $(s + \alpha_i + k)$ appear on the right hand side in (3.2.3). Hence it simply suffices to take

$$N_1 \geq \max\{N_2, N_3\}.$$

This concludes the first step.

Step 2. In this step we define a filtration W^α on $\iota_+ \mathcal{O}_X(*D)$, with $\alpha \in \mathbf{Q}$, and show that it is a V -filtration.

Recall from the previous step that $-s$ acts on $\mathcal{N}'/t\mathcal{N}'$ with a minimal polynomial with roots in $\mathbf{Q} \cap (0, 1]$. Let's denote these roots

$$\beta_1 < \cdots < \beta_\ell.$$

We also denote by $P_i \subseteq \mathcal{N}'/t\mathcal{N}'$ the submodule on which $s + \beta_i$ acts nilpotently. For $i = 1, \dots, \ell$, we consider the submodule $t\mathcal{N}' \subseteq W^{\beta_i} \subseteq \mathcal{N}'$ such that

$$W^{\beta_i}/t\mathcal{N}' = \bigoplus_{j \geq i} P_j.$$

In particular, $W^{\beta_1} = \mathcal{N}'$. For every $q \in \mathbf{Z}$ and every i , we also define

$$W^{\beta_i+q} := t^q \cdot W^{\beta_i} \subseteq \iota_+ \mathcal{O}_X(*D).$$

Finally, for an arbitrary $\alpha \in \mathbf{Q}$, we can write $\gamma_1 < \alpha \leq \gamma_2$, where γ_1 and γ_2 are consecutive numbers in the set $\{\beta_i + q \mid i = 1, \dots, \ell; q \in \mathbf{Z}\}$, and define

$$W^\alpha := W^{\gamma_2}.$$

The claim is that this gives a V -filtration on $\iota_+ \mathcal{O}_X(*D)$. It is clear that W^α is finitely generated, decreasing, discrete and left continuous, and also that $t \cdot W^\alpha = W^{\alpha+1}$ for all α .

To check that $\partial_t \cdot W^\alpha \subseteq W^{\alpha-1}$ for all α , note first that by construction we have $\partial_t t \cdot W^{\beta_i} \subseteq W^{\beta_i}$ for all i . On the other hand, for all q we have $\partial_t t^q = t^{q-1}(\partial_t t + q - 1)$. We obtain

$$\partial_t t^q \cdot W^{\beta_i} \subseteq t^{q-1} \cdot W^{\beta_i},$$

which gives the assertion.

To check that W^\bullet is exhaustive, let $v \in \iota_+ \mathcal{O}_X(*D)$. Then by Lemma 2.5.5 there exists $r \geq 0$ such that $t^r v \in \mathcal{N}_f$, so

$$t^{k+r} v \in t^k \mathcal{N}_f \subseteq \mathcal{N}' = W^{\beta_1},$$

which implies that $v \in W^{\beta_1-q}$.

Finally, we check that $s + \alpha$ is nilpotent on gr_W^α . Take $\gamma_2 = \beta_i + q$ as in the definition of W^α , so that $\alpha \leq \gamma_2$ and $W^\alpha = W^{\gamma_2}$. If $\alpha < \gamma_2$, then $\text{gr}_W^\alpha = 0$ and there is nothing to prove. Hence we can assume $\alpha = \beta_i + q$, and so we are looking at the action of $s + \beta_i + q$ on $\text{gr}_W^{\beta_i+q} = t^q \cdot \text{gr}_W^{\beta_i}$. But

$$(s + \beta_i + q)t^q = t^q(s + \beta_i),$$

and we know that $s + \beta_i$ is nilpotent on (the lift of) P_i , which is precisely $\text{gr}_W^{\beta_i}$. This concludes the proof.

Step 3. In this final step we define a filtration V^α on $\iota_+ \mathcal{O}_X$, with $\alpha \in \mathbf{Q}$, and show that it is a V -filtration. This is in fact straightforward: taking into account Exercise 2.1.1, we simply define it as

$$V^\alpha \iota_+ \mathcal{O}_X := W^\alpha \iota_+ \mathcal{O}_X \cap \iota_+ \mathcal{O}_X, \quad \forall \alpha \in \mathbf{Q}.$$

The fact that V^\bullet is a V -filtration follows immediately from the properties of W^\bullet studied in the previous step.

We record further basic properties of this V -filtration. For the first, we need the following:

LEMMA 3.2.7. *In the language of Step 1, we have $\mathcal{N}' \subseteq \iota_+ \mathcal{O}_X$.*

PROOF. Using Exercise 2.1.1, it is enough to show that for every $u \in \mathcal{N}'$ and $m \in \mathbf{N}$ such that $t^m u \in \iota_+ \mathcal{O}_X$, we have $u \in \iota_+ \mathcal{O}_X$. By induction on m , it is also enough to assume $m = 1$. By the definition of \mathcal{N}' , there exist $\lambda_1, \dots, \lambda_\ell > 0$ (perhaps non-distinct) such that

$$\prod_{i=1}^{\ell} (s + \lambda_i) \cdot u \in t^k \mathcal{N}_f \subseteq \iota_+ \mathcal{O}_X.$$

Note also that $s \cdot \iota_+ \mathcal{O}_X \subseteq \iota_+ \mathcal{O}_X$, and applying this inductively to the formula above, we obtain

$$\prod_{i=1}^{\ell} (s + \lambda_i) \cdot u = w + (-1)^\ell \prod_{i=1}^{\ell} \lambda_i \cdot u \in \iota_+ \mathcal{O}_X,$$

with $w \in \iota_+ \mathcal{O}_X$. But all $\lambda_i > 0$, so $u \in \iota_+ \mathcal{O}_X$. \square

Having this lemma at our disposal we observe that

$$(3.2.4) \quad V^\alpha = W^\alpha, \quad \forall \alpha > 0.$$

Indeed, this amounts to checking that $W^\alpha \subset \iota_+ \mathcal{O}_X$ for all $\alpha > 0$. But by definition, for such α we have $W^\alpha = t^q \cdot W^{\beta_i}$ for some i and some $q \geq 0$. Now by construction $W^{\beta_i} \subset \mathcal{N}'$, and so using (3.2.1) and Lemma 3.2.7 we have

$$W^\alpha \subseteq \mathcal{N}' \subseteq \iota_+ \mathcal{O}_X.$$

Another thing to note is that

$$(3.2.5) \quad \bigcap_{\alpha \in \mathbf{Q}} V^\alpha = 0.$$

To see this, note that if v is in this intersection, then since $t^q \cdot V^\alpha = V^{\alpha+q}$ for $\alpha > 0$ and $q \geq 0$, we have

$$v \in \bigcap_{q \in \mathbf{N}} t^q \cdot \iota_+ \mathcal{O}_X.$$

But if $w = \sum_{j=0}^p g_j \partial_t^j \delta \in \iota_+ \mathcal{O}_X$, then

$$t^q \cdot w = \sum_{j=0}^p h_j \partial_t^j \delta$$

with $h_p = f^q \cdot g_p$, and from this it follows easily that $v = 0$. \square

The proof of Theorem 3.2.6 also leads to the following statement regarding the nontrivial jumps in the V -filtration.

COROLLARY 3.2.8. *Let $\alpha > 0$ be a rational number such that the V -filtration on $\iota_+ \mathcal{O}_X$ jumps at α , i.e. $V^{\alpha+\varepsilon} \subsetneq V^\alpha$ for all $\varepsilon > 0$. Then there exists a root λ of $b_f(s)$ and $k \in \mathbf{Z}$ such that $\alpha = \lambda + k$.*

PROOF. We have seen that each eigenvalue of the action of $-s$ on $\mathcal{N}'/t\mathcal{N}'$ is one of the β_i , which is in turn congruent to some λ modulo \mathbf{Z} . On the other hand, by construction V^α jumps at rational numbers of the form $\beta_i + \mathbf{Z}$ for all i . \square

3.3. Nearby and vanishing cycles

The standard reference for the material in this section is [Sa1], especially Sections 3.1 and 5.1.

First take. Let first \mathcal{M} be a \mathcal{D}_X module endowed with a V -filtration along a smooth hypersurface defined by a function t . Note that by definition, for each $\alpha \in \mathbf{Q}$ we have induced operators

$$t: \mathrm{gr}_V^\alpha \mathcal{M} \longrightarrow \mathrm{gr}_V^{\alpha+1} \mathcal{M}$$

and

$$\partial_t: \mathrm{gr}_V^\alpha \mathcal{M} \longrightarrow \mathrm{gr}_V^{\alpha-1} \mathcal{M}.$$

The defining properties of the V -filtration imply the following:

EXERCISE 3.3.1. The operator t is an isomorphism for all $\alpha \neq 0$, while the operator ∂_t is an isomorphism for all $\alpha \neq 1$.

This leaves us with two interesting homomorphisms, which are crucial in what follows, namely

$$(3.3.1) \quad (\mathrm{var} :=) t: \mathrm{gr}_V^0 \mathcal{M} \longrightarrow \mathrm{gr}_V^1 \mathcal{M}$$

and

$$(3.3.2) \quad (\mathrm{can} :=) \partial_t: \mathrm{gr}_V^1 \mathcal{M} \longrightarrow \mathrm{gr}_V^0 \mathcal{M}.$$

The names of these morphisms are motivated by analogous maps in the theory of perverse sheaves. This theory (and results of Kashiwara and Malgrange on the Riemann-Hilbert correspondence) also motivates the following:

DEFINITION 3.3.2. The *unipotent nearby cycles* \mathcal{D} -module of \mathcal{M} along t is defined as

$$\psi_{t,1} \mathcal{M} := \mathrm{gr}_V^1 \mathcal{M}.$$

The *vanishing cycles* \mathcal{D} -module of \mathcal{M} along t is

$$\varphi_{t,1} \mathcal{M} := \mathrm{gr}_V^0 \mathcal{M}.$$

Note that these objects are supported on the hypersurface $D = (t = 0)$, and in fact by the definition of the V -filtration they are \mathcal{D}_D -modules.

In the general case we consider a left \mathcal{D}_X -module \mathcal{M} on the smooth variety X , and a nontrivial function $f \in \mathcal{O}_X(X)$. We denote as always by

$$\iota: X \hookrightarrow Y = X \times \mathbf{C}, \quad x \mapsto (x, f(x))$$

the closed embedding given by the graph of f , and by t be the coordinate on the second factor \mathbf{C} , so that $(t = 0)$ is the smooth hypersurface $X \times \{0\}$ in Y . We assume that there exists a rational V -filtration on $\iota_+ \mathcal{M}$ along t , for instance as in Theorem 3.1.9 and Remark 3.1.10. We define the unipotent nearby cycles and the vanishing cycles as

$$\psi_{f,1} \iota_+ \mathcal{M} := \mathrm{gr}_V^1 \iota_+ \mathcal{M} \quad \text{and} \quad \varphi_{f,1} \iota_+ \mathcal{M} := \mathrm{gr}_V^0 \iota_+ \mathcal{M}.$$

Relationship with strict support decomposition. One of the uses of the nearby and vanishing cycle is to provide a criterion for when the \mathcal{D} -module \mathcal{M} endowed with a V -filtration along a hypersurface has no nontrivial sub-objects or quotient objects supported on that hypersurface.

PROPOSITION 3.3.3. *Assume that D is a smooth hypersurface defined by a function t , and let $\mathcal{M}' \subseteq \mathcal{M}$ be the smallest submodule such that $\mathcal{M}'|_U = \mathcal{M}|_U$, where $U = X \setminus D$. Denote by $i: D \hookrightarrow X$ the inclusion map. Then:*

- (1) $\mathcal{M}' = \mathcal{D}_X \cdot (V^{>0} \mathcal{M}) = \mathcal{D}_X \cdot (V^\alpha \mathcal{M})$ for any $\alpha > 0$.
- (2) $\mathcal{M}/\mathcal{M}' = i_+ \mathrm{Coker}(\partial_t: \mathrm{gr}_V^1 \mathcal{M} \rightarrow \mathrm{gr}_V^0 \mathcal{M})$.
- (3) $\mathcal{H}_D^0 \mathcal{M} = i_+ \mathrm{Ker}(t: \mathrm{gr}_V^0 \mathcal{M} \rightarrow \mathrm{gr}_V^1 \mathcal{M})$, where $\mathcal{H}_D^0 \mathcal{M}$ is the sub-object of \mathcal{M} generated by sections whose support is contained in D . (The notation is meant to suggest local cohomology.)

PROOF. (1) Since \mathcal{M}' and \mathcal{M} coincide on U , we have

$$V^\alpha \mathcal{M}' = V^\alpha \mathcal{M} \quad \text{for all } \alpha > 0$$

by Lemma 3.1.8. Thus

$$\mathcal{D}_X \cdot (V^\alpha \mathcal{M}) \subseteq \mathcal{M}' \quad \text{for all } \alpha > 0.$$

Note also that

$$(\mathcal{D}_X \cdot (V^\alpha \mathcal{M}))|_U = \mathcal{M}|_U \quad \text{for all } \alpha > 0,$$

since the V -filtration is trivial outside of D . By the definition of \mathcal{M}' , this gives the opposite inclusion

$$\mathcal{M}' \subseteq \mathcal{D}_X \cdot (V^\alpha \mathcal{M}).$$

(2) Denote $\mathcal{Q} = \mathcal{M}/\mathcal{M}'$. According to Lemma 3.1.7, we have

$$\mathrm{gr}_V^0 \mathcal{Q} = \mathrm{Coker}(\mathrm{gr}_V^0 \mathcal{M}' \rightarrow \mathrm{gr}_V^0 \mathcal{M}).$$

The claim is that we also have

$$(3.3.3) \quad \mathrm{gr}_V^0 \mathcal{Q} = \mathrm{Coker}(\mathrm{gr}_V^1 \mathcal{M} \xrightarrow{\partial_t} \mathrm{gr}_V^0 \mathcal{M}).$$

Let's grant this for the moment. Now obviously \mathcal{Q} has support contained in D , and therefore by Example 3.1.4(3) we have $\mathcal{Q} = \mathcal{Q}^0 \otimes_{\mathbf{C}} \mathbf{C}[\partial_t] = i_+ \mathcal{Q}^0$ and $V^{>0} \mathcal{Q} = 0$. We therefore obtain $\mathrm{gr}_V^0 \mathcal{Q} = V^0 \mathcal{Q} = \mathcal{Q}^0$, and so

$$\mathcal{Q} = i_+ \mathrm{gr}_V^0 \mathcal{Q}.$$

We are left with proving (3.3.3). To this end, note that $\mathrm{gr}_V^0 \mathcal{M}' \subseteq \mathrm{gr}_V^0 \mathcal{M}$, and so it suffices to show that

$$\mathrm{gr}_V^0 \mathcal{M}' = \mathrm{Im} \partial_t = \frac{\partial_t \cdot V^1 \mathcal{M} + V^{>0} \mathcal{M}}{V^{>0} \mathcal{M}}.$$

Now

$$\mathrm{gr}_V^0 \mathcal{M}' = \frac{V^0 \mathcal{M}'}{V^{>0} \mathcal{M}'} = \frac{\mathcal{M}' \cap V^0 \mathcal{M}}{V^{>0} \mathcal{M}},$$

where we used Exercise 3.1.7 and Lemma 3.1.8. We conclude that it is enough to show the identity

$$\mathcal{M}' \cap V^0 \mathcal{M} = \partial_t \cdot V^1 \mathcal{M} + V^{>0} \mathcal{M}.$$

This follows in turn from the inclusion

$$(3.3.4) \quad \mathcal{M}' \cap V^0 \mathcal{M} \subseteq \partial_t \cdot V^1 \mathcal{M} + V^1 \mathcal{M}.$$

Indeed, the right hand side is contained in $\partial_t \cdot V^1 \mathcal{M} + V^{>0} \mathcal{M}$, which in turn is contained in $\mathcal{M}' \cap V^0 \mathcal{M}$ by part (1).

To establish (3.3.4), we need the following two claims:

- (i) $\mathcal{M}' = \sum_{i \geq 0} \partial_t^i \cdot V^1 \mathcal{M}$.
- (ii) $\partial_t^i: V^0 \mathcal{M}/V^1 \mathcal{M} \xrightarrow{\cong} V^{-i} \mathcal{M}/V^{-i+1} \mathcal{M}$.

Claim (i) is clear, since we've seen in (1) that

$$\mathcal{M}' = \mathcal{D}_X \cdot (V^1 \mathcal{M}),$$

and the action of functions and other ∂_{x_i} leaves all $V^\alpha \mathcal{M}$ fixed. Claim (ii) follows from Exercise 3.3.1.

Let's finally deduce (3.3.4) from the two claims. Using (i), we see that it is enough to show that for all $i \geq 2$ we have

$$\partial_t^i \cdot V^1 \mathcal{M} \cap V^0 \mathcal{M} \subseteq \partial_t^{i-1} \cdot V^1 \mathcal{M} + V^0 \mathcal{M},$$

since then, continuing inductively, we eventually get that the left hand side is contained in $\partial_t \cdot V^1 \mathcal{M}$ (which is contained in $V^0 \mathcal{M}$). But now

$$\partial_t^i \cdot V^1 \mathcal{M} \cap V^0 \mathcal{M} \subseteq \partial_t^{i-1} \cdot V^0 \mathcal{M} \cap V^0 \mathcal{M} \subseteq V^{-i+1} \mathcal{M},$$

If we pick an element $x \in \partial_t^i \cdot V^1 \mathcal{M} \cap V^0 \mathcal{M}$, via these inclusions we can write $x = \partial_t^{i-1} \cdot y$ for some $y \in V^0 \mathcal{M}$. If $y \in V^1 \mathcal{M}$, we are done. Otherwise we have $0 \neq y \in V^0 \mathcal{M}/V^1 \mathcal{M}$, and so by (ii) we obtain that $\partial_t \cdot x = \partial_t^i \cdot y \notin V^{-i+1} \mathcal{M}$. But $\partial_t \cdot x \in V^{-1} \mathcal{M}$, which gives a contradiction since $i \geq 2$.

(3) It is straightforward to check that

$$\mathcal{H}_D^0 \mathcal{M} = i_+ \mathrm{Ker}(t: \mathcal{M} \longrightarrow \mathcal{M}),$$

and we will produce an isomorphism between this kernel, denoted by K , and

$$K' := \mathrm{Ker}(t: \mathrm{gr}_V^0 \mathcal{M} \longrightarrow \mathrm{gr}_V^1 \mathcal{M}).$$

We first show that

$$K \subseteq V^0 \mathcal{M}.$$

Indeed, consider $u \in K$, so that $tu = 0$, and say $u \in V^\alpha \mathcal{M}$. If $\alpha \geq 0$, then we are done, so let's assume $\alpha < 0$. We then have

$$(\partial_t t - \alpha)u = -\alpha u \in V^\alpha \mathcal{M},$$

and we take its image in $\mathrm{gr}_V^\alpha \mathcal{M}$, where the action of $\partial_t t - \alpha$ is nilpotent. It follows that there exists $p > 0$ such that $\alpha^p u = 0$, i.e. $u = 0$ in $\mathrm{gr}_V^\alpha \mathcal{M}$, and so $u \in V^\beta \mathcal{M}$ for some $\beta > \alpha$. Repeating this process, since the V -filtration is discrete, with bounded denominators, we eventually obtain that $u \in V^0 \mathcal{M}$.

With this in mind, note that we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^{>0} \mathcal{M} & \longrightarrow & V^0 \mathcal{M} & \longrightarrow & \mathrm{gr}_V^0 \mathcal{M} \longrightarrow 0 \\ & & \downarrow t & & \downarrow t & & \downarrow t \\ 0 & \longrightarrow & V^{>1} \mathcal{M} & \longrightarrow & V^1 \mathcal{M} & \longrightarrow & \mathrm{gr}_V^1 \mathcal{M} \longrightarrow 0, \end{array}$$

Recalling that t acts bijectively on $V^{>0} \mathcal{M}$, the Snake Lemma then implies $K \simeq K'$. \square

COROLLARY 3.3.4. *Let f be a nonconstant function on X , and \mathcal{M} a \mathcal{D}_X -module endowed with a V -filtration along f . Then*

- (1) \mathcal{M} has no nonzero sub-object supported on $D = f^{-1}(0) \iff \mathrm{var}: \varphi_{f,1} \iota_+ \mathcal{M} \longrightarrow \psi_{f,1} \iota_+ \mathcal{M}$ is injective.
- (2) \mathcal{M} no nonzero quotient object supported on $D = f^{-1}(0) \iff \mathrm{can}: \psi_{f,1} \iota_+ \mathcal{M} \longrightarrow \varphi_{f,1} \iota_+ \mathcal{M}$ is surjective.

PROOF. This follows from Proposition 3.3.3 applied to $\iota_+ \mathcal{M}$ along t , noting that the sub-objects or quotient objects of \mathcal{M} supported on D correspond via Kashiwara's equivalence to those of $\iota_+ \mathcal{M}$ supported on $t^{-1}(0)$. \square

We have in fact the following more precise statement:

PROPOSITION 3.3.5. *Under the same hypotheses, the following are equivalent:*

- (1) $\varphi_{f,1} \iota_+ \mathcal{M} = \mathrm{Ker}(\mathrm{var}: \varphi_{f,1} \iota_+ \mathcal{M} \longrightarrow \psi_{f,1} \iota_+ \mathcal{M}) \oplus \mathrm{Im}(\mathrm{can}: \psi_{f,1} \iota_+ \mathcal{M} \longrightarrow \varphi_{f,1} \iota_+ \mathcal{M})$.
- (2) There is a decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$, with $\mathrm{Supp} \mathcal{M}' \subseteq D = (f = 0)$, and \mathcal{M}'' with no nontrivial sub-objects or quotient objects supported on D .

PROOF. Assume first that there is a decomposition as in (2). We then have

$$\varphi_{f,1} \iota_+ \mathcal{M} = \varphi_{f,1} \iota_+ \mathcal{M}' \oplus \varphi_{f,1} \iota_+ \mathcal{M}''.$$

Moreover, in the case of \mathcal{M}'' , by Corollary 3.3.4, we have that var is injective and can is surjective. On the other hand, by the discussion in the proof of Proposition 3.3.3, in the case of \mathcal{M}' the action of var and can is trivial, as it is supported on D . This implies that

$$\mathrm{Ker}(\mathrm{var}) = \varphi_{f,1} \iota_+ \mathcal{M}' \quad \text{and} \quad \mathrm{Im}(\mathrm{can}) = \varphi_{f,1} \iota_+ \mathcal{M}'',$$

and so gives (1).

We now assume (1). We define

$$\widetilde{\mathcal{M}}' := \mathcal{D}_{X \times \mathbf{C}} \cdot V^{>0} \iota_+ \mathcal{M}$$

and

$$\widetilde{\mathcal{M}}'' := \text{Ker}(\text{var}: \varphi_{f,1} \iota_+ \mathcal{M} \longrightarrow \psi_{f,1} \iota_+ \mathcal{M}),$$

so that as in Proposition 3.3.3, $\iota_+ \mathcal{M} / \widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{M}}''$ are the largest quotient object, respectively sub-object, of $\iota_+ \mathcal{M}$ with support in $t^{-1}(0)$ (so in fact in $t^{-1}(0) \cap \iota(X) = D$). Property (1) implies that we have a direct sum decomposition

$$\iota_+ \mathcal{M} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''$$

and we obtain (2) by Kashiwara's equivalence. \square

DEFINITION 3.3.6. Let $Z \subset X$ be an irreducible closed subset. A \mathcal{D}_X -module \mathcal{M} has *strict support* Z if \mathcal{M} is supported on Z and has no nontrivial quotients or sub-objects supported on a proper subset of Z .²

In the following two important corollaries of the results above, \mathcal{M} is a \mathcal{D}_X -module admitting a rational V -filtration along any hypersurface, for instance a regular holonomic \mathcal{D}_X -module with quasi-unipotent monodromy; see Theorem 3.1.9 and Remark 3.1.10.

COROLLARY 3.3.7. \mathcal{M} has strict support X if and only if $\text{var}: \varphi_{f,1} \iota_+ \mathcal{M} \longrightarrow \psi_{f,1} \iota_+ \mathcal{M}$ is injective and $\text{can}: \psi_{f,1} \iota_+ \mathcal{M} \longrightarrow \varphi_{f,1} \iota_+ \mathcal{M}$ is surjective for all nonconstant $f \in \mathcal{O}_X(X)$.

COROLLARY 3.3.8. \mathcal{M} has a strict support decomposition $\mathcal{M} = \bigoplus_{Z \subset X} \mathcal{M}_Z$ (i.e. a direct sum over a finite collection of irreducible closed subsets Z such that each \mathcal{M}_Z is a \mathcal{D}_X -module with strict support Z) if and only if the decomposition

$$\varphi_{f,1} \iota_+ \mathcal{M} = \text{Ker}(\text{var}) \oplus \text{Im}(\text{can})$$

in Proposition 3.3.5 holds for all $f \in \mathcal{O}_X$.

REMARK 3.3.9. Note that a direct sum decomposition as the Corollary above is necessarily *unique*, since there can be no nontrivial morphisms between \mathcal{D} -modules with different strict support.

3.4. Sabbah's description of the V -filtration

Keeping the notation of the previous sections, here we extend the notion of Bernstein-Sato polynomial to arbitrary elements of $\iota_+ \mathcal{O}_X$, and use it in order to give an alternative description of the V -filtration.

PROPOSITION 3.4.1. Let $w \in \iota_+ \mathcal{O}_X$. Then there exists a non-zero polynomial $b(s) \in \mathbf{C}[s]$ and $P \in \mathcal{D}_X[s]$ such that

$$b(s) \cdot w = P \cdot tw.$$

²These should be thought of as the analogues of the simple objects (intersection complexes) in the theory of perverse sheaves.

PROOF. Let us consider, by analogy with the $\mathcal{D}_X[s]$ -module $\mathcal{N}_f = \mathcal{D}_X[s]f^s$ which played a key role earlier, the submodule

$$\mathcal{N}_w := \mathcal{D}_X[s] \cdot w \subseteq \iota_+ \mathcal{O}_X.$$

We need to show that there exists $b(s) \in \mathbf{C}[s]$ such that

$$b(-\partial_t t) \cdot w \in t\mathcal{N}_w = \mathcal{D}_X[\partial_t t] \cdot tw.$$

Fix now $\alpha \in \mathbf{Q}$ such that $w \in V^\alpha \iota_+ \mathcal{O}_X$, and consider any $\beta \in \mathbf{Q}$.

Claim. There exists a polynomial $b(s) \in \mathbf{C}[s]$, all of whose roots are rational and $\leq -\alpha$, such that

$$b(s) \cdot w \in V^\beta \iota_+ \mathcal{O}_X.$$

To see this, let $\alpha' \in \mathbf{Q}$ such that $V^{>\alpha} = V^{\alpha'}$. Hence we have

$$(s + \alpha)^{N'} \cdot w \in V^{\alpha'} \quad \text{for some } N' \geq 1.$$

If $\beta \leq \alpha'$, then $V^{\alpha'} \subseteq V^\beta$ and we are done. If $\beta > \alpha'$, then we consider $\alpha'' > \alpha'$ and $N'' \geq 1$ such that

$$(s + \alpha)^{N''} (s + \alpha)^{N'} \cdot w \in V^{\alpha''} = V^{>\alpha'}.$$

We can continue in this fashion, and since V^\bullet is discrete with bounded denominators, after a finite number of steps we reach an index which is greater than β .

Continuing with the proof of the Propostion, due to the Claim above it suffices to show that there exists $\beta \in \mathbf{Q}$ such that

$$V^\beta \cap \mathcal{N}_w \subseteq t\mathcal{N}_w.$$

To this end, first note that there exists $\beta_0 > 0$ such that $V^{\beta_0} \subseteq \mathcal{N}_f$, and so for every $q \in \mathbf{N}$ we have

$$t^q \cdot V^{\beta_0} = V^{\beta_0+q} \subseteq t^q \cdot \mathcal{N}_f.$$

It therefore suffices in turn to show that there exists $q \in \mathbf{N}$ such that

$$(3.4.1) \quad t^q \cdot \mathcal{N}_f \cap \mathcal{N}_w \subseteq t\mathcal{N}_w.$$

Note also that by definition $\mathcal{N}_{tw} = t\mathcal{N}_w$, hence we can replace w by $t^p w$ with $p \gg 0$. Thus by Lemma 2.5.5, finally we can assume in (3.4.1) that $w \in \mathcal{N}_f$. In particular $\mathcal{N}_w \subseteq \mathcal{N}_f$.

We now proceed to proving (3.4.1) under this assumption. We consider the sheaf of rings

$$\mathcal{R} := \mathcal{D}_X[s] = \mathcal{D}_X[\partial_t t] \subseteq \mathcal{D}_Y.$$

Exercise 2.5.6 implies that for all $p \geq 0$ we have

$$t^p \cdot \mathcal{R} = \mathcal{R} \cdot t^p = \bigoplus_{i-j=p} \mathcal{D}_X \cdot t^i \partial_t^j.$$

Using this and (2.1.2), we deduce that

$$\mathcal{S} := \bigoplus_{i \geq j} \mathcal{D}_X \cdot t^i \partial_t^j$$

is a subring of \mathcal{D}_Y , with a decreasing filtration

$$F_p \mathcal{S} := \bigoplus_{i-j \geq p} \mathcal{D}_X \cdot t^i \partial_t^j$$

for $p \geq 0$. Denoting $t \partial_t$ by u and t by v , they satisfy the relation $uv = v(u+1)$, and we obtain

$$\mathrm{gr}_F^\bullet \mathcal{S} \simeq \mathcal{D}_X[u, v] / (uv - v(u+1)).$$

This ring is Noetherian; see Exercise 3.4.2 below.

Consider now on \mathcal{N}_f the decreasing filtration

$$G_p \mathcal{N}_f := t^p \cdot \mathcal{N}_f, \quad p \geq 0.$$

It is easily checked that G_\bullet is compatible with the filtration F_\bullet on \mathcal{S} , and that the total associated graded $\mathrm{gr}_\bullet^G \mathcal{N}_f$ is generated over $\mathrm{gr}_\bullet^F \mathcal{S}$ by the class of δ . Since $\mathcal{N}_w \subseteq \mathcal{N}_f$, we can consider on \mathcal{N}_w the filtration

$$G_p \mathcal{N}_w := G_p \mathcal{N}_f \cap \mathcal{N}_w, \quad p \geq 0.$$

As $\mathrm{gr}_\bullet^G \mathcal{N}_f$ is finitely generated over the (locally) Noetherian $\mathrm{gr}_\bullet^F \mathcal{S}$, we deduce that $\mathrm{gr}_\bullet^G \mathcal{N}_w$ is also finitely generated. It follows that for $q \gg 0$ we have (3.4.1), which concludes the proof. \square

EXERCISE 3.4.2. Show that $\mathcal{D}_X[u, v] / (uv - v(u+1))$ is a sheaf of Noetherian rings. (Hint: define on it locally an analogue of the Bernstein filtration, and check that its associated graded is a polynomial ring in $2n$ variables.)

DEFINITION 3.4.3. The set of all polynomials satisfying the conclusion of Proposition 3.4.1 forms an ideal in $\mathbf{C}[s]$. The unique monic generator $b_w(s)$ of this ideal is called the *Bernstein-Sato polynomial of w* . (Note that the usual Bernstein-Sato polynomial of f is the special case $b_f(s) = b_\delta(s)$ of this construction.)

REMARK 3.4.4. The proof of Proposition 3.4.1 shows that for every $w \in \iota_+ \mathcal{O}_X$, the roots of $b_w(s)$ are rational numbers. Moreover, if $w \in V^\alpha$ then they are all $\leq -\alpha$.³

The result above leads to a useful alternative description of the V -filtration due to Sabbah [Sab]:

THEOREM 3.4.5. For every $\alpha \in \mathbf{Q}$ we have

$$V^\alpha \iota_+ \mathcal{O}_X = \{w \in \iota_+ \mathcal{O}_X \mid \text{all the roots of } b_w(s) \text{ are } \leq -\alpha\}.$$

PROOF. The inclusion from left to right follows from the Remark above. In order to prove the opposite inclusion, it suffices to show that if $w \in V^\alpha \setminus V^{>\alpha}$, then $b_w(-\alpha) = 0$.

Recall that $s + \alpha$ is nilpotent on gr_V^α , and therefore for every $\beta \neq \alpha$ we have that $s + \beta$ is invertible on gr_V^α . On the other hand, we know that

$$b_w(s) \cdot w \in \mathcal{D}_X[\partial_t] \cdot tw \in V^{>\alpha}.$$

Since $\widehat{w} \neq 0$ in gr_V^α , this would be impossible if $-\alpha$ were not among the roots of $b_w(s)$. \square

³Note however that this does not recover Kashiwara's theorem on the rationality of the roots of $b_f(s)$, since this theorem was used in the proof of the existence of the V -filtration.

An important consequence regards the greatest root of the Bernstein-Sato polynomial of f . Consider in fact

$$\alpha_f := -(\text{greatest root of } b_f(s)).$$

Recall also that we can define a decreasing V -filtration on \mathcal{O}_X , by considering it embedded in $\iota_+ \mathcal{O}_X$ as $\mathcal{O}_X \otimes 1$, and taking

$$V^\alpha \mathcal{O}_X = V^\alpha \iota_+ \mathcal{O}_X \cap \mathcal{O}_X.$$

For every $\alpha \in \mathbf{Q}$, $V^\alpha \mathcal{O}_X$ is a coherent ideal sheaf in \mathcal{O}_X .

COROLLARY 3.4.6. *We have the following equivalences:*

$$V^\alpha \mathcal{O}_X = \mathcal{O}_X \iff \delta \in V^\alpha \iota_+ \mathcal{O}_X \iff \text{all the roots of } b_f(s) \text{ are } \leq -\alpha.$$

In particular,

$$\alpha_f = \max \{ \beta \in \mathbf{Q} \mid V^\beta \mathcal{O}_X = \mathcal{O}_X \}.$$

We will see later that α_f coincides with the *log canonical threshold* of f . We also have another interesting consequence about the jumps of $V^\bullet \mathcal{O}_X$ in the interval $(0, 1)$.

COROLLARY 3.4.7. *If $\alpha \in (0, 1) \cap \mathbf{Q}$ corresponds to a jump in $V^\bullet \mathcal{O}_X$, i.e. $V^{\alpha+\varepsilon} \mathcal{O}_X \subsetneq V^\alpha \mathcal{O}_X$ for all $\varepsilon > 0$, then $-\alpha$ is a root of $b_f(s)$.*

PROOF. We continue to denote $V^\alpha = V^\alpha \iota_+ \mathcal{O}_X$. Let's first note that for every $\beta \in (0, 1) \cap \mathbf{Q}$ we have $f\delta \in V^\beta$. Indeed, we know that $\delta \in V^{\alpha_f}$, so $f\delta = t\delta \in V^{\alpha_f+1}$. But since $\alpha_f > 0$, we have $V^{\alpha_f+1} \subseteq V^\beta$.

We know that

$$b_f(s)\delta \in \mathcal{D}_X[s]f\delta \subseteq V^{>\alpha},$$

where the inclusion follows from the discussion above. Hence we also have

$$b_f(s)h\delta \in V^{>\alpha}.$$

On the other hand, the condition on α is equivalent to the existence of an $h \in \mathcal{O}_X$ such that $h\delta \in V^\alpha \setminus V^{>\alpha}$. By the definition of the V -filtration, for $N \gg 0$ we also have

$$(s + \alpha)^N h\delta \in V^{>\alpha}.$$

If the two polynomials $b_f(s)$ and $(s + \alpha)^N$ were coprime, we would infer that $h\delta \in V^{>\alpha}$, which is a contradiction. Thus we deduce that $b_f(-\alpha) = 0$. \square

It is worth noting that the obvious possible improvements of Corollary 3.4.7 do not hold, as shown by the examples below.

EXAMPLE 3.4.8. (1) The converse of the statement in Corollary 3.4.7 does not necessarily hold, even for isolated singularities. For instance, Saito shows in [Sa3, Example 3.5] that if $f = x^5 + y^4 + x^3y^2 \in \mathbf{C}[x, y]$, then $b_f(s)$ has roots in $(-1, 0)$ that do not correspond to jumps of the filtration $V^\bullet \mathcal{O}_X$.

(2) In Corollary 3.4.7 one cannot replace the set of jumping exponents of the V -filtration on \mathcal{O}_X by the analogous (but often larger) set for $\iota_+ \mathcal{O}_X$. For example, let $f = x^2 + y^3 \in \mathbf{C}[x, y]$ be a cusp. Then the V -filtration on $\iota_+ \mathcal{O}_X$ is known to jump at $\alpha = 1/6$; see for

instance the combinatorial calculation of the microlocal V -filtration for isolated quasi-homogeneous singularities in [Sa6, (4.1.2)], combined with the fact that for $\alpha \leq 1$ it coincides with $V^\bullet \iota_+ \mathcal{O}_X$. On the other hand, the only negative of a root of $b_f(s)$ that is less than 1 is $5/6$; see Example 2.2.11(2).

Going back to the statement of Corollary 3.4.6, we will later see that the condition

$$\partial_t^p \delta \in V^\alpha \iota_+ \mathcal{O}_X$$

for $p \geq 1$ is also significant; it is equivalent to the triviality of the Hodge ideal $I_p(D)$, where D is the \mathbf{Q} -divisor $D = \alpha \cdot \text{div}(f)$ (when $\text{div}(f)$ is reduced).

EXAMPLE 3.4.9. In view of the paragraph above, it is interesting to have some understanding of $b_{\partial_t^p \delta}(s)$ for $p \geq 1$, at least in terms of the standard $b_f(s) = b_\delta(s)$. Recall that for all nontrivial f , $(s+1)$ divides $b_f(s)$, so we can also consider the *reduced* Bernstein-Sato polynomial

$$\tilde{b}_f(s) := \frac{b_f(s)}{s+1}.$$

We claim that we have the divisibility relation

$$b_{\partial_t^p \delta}(s) \mid (s+1) \tilde{b}_f(s-p).$$

We begin by noting that for every polynomial $Q(s)$, we have

$$(3.4.2) \quad \partial_t \cdot Q(\partial_t t) = Q(\partial_t t + 1) \cdot \partial_t \quad \text{and} \quad t \cdot Q(\partial_t t) = Q(\partial_t t - 1) \cdot t.$$

Indeed, it is enough to check this when $Q(s) = s^q$ is a monomial, and in this case both equalities can be easily verified by induction on q ; the second relation is simply (2.1.2).

By the definition of $b_f(s)$, there exists $P \in \mathcal{D}_X[s]$ such that

$$b_f(-\partial_t t) \delta = P(-\partial_t t) t \delta.$$

Using (3.4.2), we obtain

$$P(-\partial_t t) t \delta = t P(-\partial_t t - 1) \delta \quad \text{and}$$

$$b_f(-\partial_t t) = (1 - \partial_t t) \tilde{b}_f(-\partial_t t) = -\tilde{b}_f(-\partial_t t) t \partial_t = -t \cdot \tilde{b}_f(-\partial_t t - 1) \partial_t.$$

Since the action of t on $\iota_+ \mathcal{O}_X$ is injective, we deduce that

$$(3.4.3) \quad \tilde{b}_f(-\partial_t t - 1) \partial_t \delta = R(-\partial_t t) \delta, \quad \text{where} \quad R(s) = -P(s-1).$$

Again using (3.4.2), we also obtain

$$\begin{aligned} \tilde{b}_f(-\partial_t t - p) \partial_t^p \delta &= \partial_t^{p-1} \cdot \tilde{b}_f(-\partial_t t - 1) \partial_t \delta = \partial_t^{p-1} \cdot R(-\partial_t t) \delta \\ &= R(-\partial_t t - p + 1) \partial_t^{p-1} \delta, \end{aligned}$$

hence

$$\begin{aligned} (1 - \partial_t t) \tilde{b}_f(-\partial_t t - p) \partial_t^p \delta &= R(-\partial_t t - p + 1) \cdot (1 - \partial_t t) \partial_t^{p-1} \delta \\ &= -R(-\partial_t t - p + 1) \cdot t \partial_t^p \delta \in \mathcal{D}_X[-\partial_t t] \cdot t \partial_t^p \delta. \end{aligned}$$

By the definition of $b_{\partial_t^p \delta}(s)$, we thus conclude that

$$b_{\partial_t^p \delta}(s) \mid (s+1) \tilde{b}_f(s-p).$$

It is also the case that

$$\tilde{b}_f(s-p)|b_{\partial_t^p \delta}(s),$$

but this requires further background; see [MP4, Proposition 6.12].

3.5. Lichtin's theorem and generalizations

In this section we discuss a refinement of Kashiwara's theorem on the rationality of the roots of the Bernstein-Sato polynomial. This is due to Lichtin [Li], and relates the roots of Bernstein-Sato polynomials with invariants appearing on resolutions of singularities. We will in fact prove a more general recent theorem of Dirks-Mustață [DM], extending Lichtin's theorem to the Bernstein-Sato polynomials of certain elements in the \mathcal{D} -module $\iota_+ \mathcal{O}_X$ as in Proposition 3.4.1.

We start by stating Lichtin's theorem. We fix a nonzero regular function f on X , and set $D = (f = 0)$. We fix a log log resolution $\mu: Y \rightarrow X$, with the property that it is an isomorphism away from $\text{Supp}(D)$, and that the proper transform \tilde{D} is smooth. We write $K_{Y/X}$ and μ^*D as in (2.6.1) and (2.6.2), and we have that all $a_i \neq 0$. The following is [Li, Theorem 5].⁴

THEOREM 3.5.1. *With the notation above, all the roots of the Bernstein-Sato polynomial $b_f(s)$ are of the form*

$$-\frac{b_i + 1 + \ell}{a_i}$$

for some $1 \leq i \leq m$ and some integer $\ell \geq 0$.

The proof is a refinement of Kashiwara's arguments described in the previous section. A recent result of Dirks-Mustață uses arguments similar to Lichtin's (and Kashiwara's) to extend this further to other Bernstein-Sato polynomials related to f . The following is [DM, Theorem 1.2].

THEOREM 3.5.2. *Let $g \in \mathcal{O}_X(X)$, and let $u = g\partial_t^p f^s \in \iota_+ \mathcal{O}_X$. With the same notation as in Theorem 3.5.1, we set $k_i = \text{ord}_{E_i}(g)$. Then the following hold:*

(1) *The greatest root of b_u is at most $\max\{-1, p - \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}\}$.*

(2) *If $p = 0$, then the greatest root of b_u is at most $-\min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}$.*

(3) *If $g = 1$, then every root of b_u is either a negative integer or of the form*

$$p - \frac{b_i + 1 + \ell}{a_i} \quad \text{for some integers } 1 \leq i \leq m \text{ and } \ell \geq 0.$$

If we assume in addition that $D = (f = 0)$ is reduced and the proper transform \tilde{D} is smooth, then we may consider only those i such that E_i is exceptional.

⁴Note that the proof of Kashiwara's result, Theorem 2.5.1, only shows that the roots of $b_f(s)$ are all of the form $-\frac{1+\ell}{a_i}$ for some $1 \leq i \leq m$ and some integer $\ell \geq 0$.

Lichtin's theorem above is then the special case $g = 1$ and $p = 0$ of this result. (Note that in this case all negative integers are also of the second type described in (3).) The extra ingredient introduced by Lichtin compared to the proof of Kashiwara's theorem is to pass from left to right \mathcal{D} -modules, and work with a slightly modified \mathcal{D} -module on the resolution in order to absorb the relative canonical divisor $K_{Y/X}$ in the calculations.

We now introduce the necessary ingredients for the proof. We use the notation introduced in §2.5 and §3.4.

Generalities of Bernstein-Sato polynomials. A useful point for later is the following:

LEMMA 3.5.3. *Let u be a section of $\iota_+ \mathcal{O}_X$, and $h \in \mathcal{O}_X$. Then the greatest root of $b_{hu}(s)$ is at most equal to the greatest root of $b_u(s)$.*

PROOF. Let $-\alpha$ be the greatest root of u . By Sabbah's description of the V -filtration, Theorem 3.4.5, we then have $u \in V^\gamma \iota_+ \mathcal{O}_X$. But V^γ is an \mathcal{O}_X -module, and therefore $hu \in V^\gamma \iota_+ \mathcal{O}_X$ as well. Applying Sabbah's result again, we obtain that the greatest root of $b_{hu}(s)$ is at most $-\alpha$. \square

The following lemma is a generalization of Exercise 2.2.8:

LEMMA 3.5.4. *Let p and q be invertible functions on X , and let $u = g\partial_t^p f^s$ for some regular functions f and g . If $v = (qg)\partial_t^p (pf)^s$, then $b_u(s) = b_v(s)$.*

PROOF. We consider the sheaf

$$\iota_+ \mathcal{O}_X(*D) \simeq \mathcal{O}_X[s, \frac{1}{f}]f^s$$

as an \mathcal{O}_X -module; it already has the standard $\mathcal{D}_X\langle t, \partial_t \rangle$ -action, but we endow it with a new one, denoted \star and given by:

- $D \star w = Dw + swD(p)p^{-1}$, for all $D \in \text{Der}_{\mathbf{C}}(\mathcal{O}_X)$.
- $t \star w = (pt)w$.
- $\partial_t \star w = (p^{-1}\partial_t)w$.

We denote this $\mathcal{D}_X\langle t, \partial_t \rangle$ -module by $\iota_+ \mathcal{O}_X(*D)^\star$. Note that the new action of $s = -\partial_t t$ coincides with the old one.

A simple calculation shows that the map

$$\nu: \iota_+ \mathcal{O}_X(*D) \rightarrow \iota_+ \mathcal{O}_X(*D)^\star, \quad P(s)(pf)^s \mapsto P(s)f^s$$

is an isomorphism of $\mathcal{D}_X\langle t, \partial_t \rangle$ -modules, mapping v to $qp^{-m}g\partial_t^p f^s$. Recalling that $b_v(s)$ is the monic polynomial of minimal degree such that $b_v(s)v \in \mathcal{D}_X\langle t, s \rangle tv$, it follows that it is also the monic polynomial of minimal degree satisfying

$$b_v(s)qp^{-m}g\partial_t^p f^s \in \mathcal{D}_X\langle t, s \rangle t \star qp^{-m}g\partial_t^p f^s.$$

On the other hand, for every section w of $\iota_+ \mathcal{O}_X(*D)$ and every invertible function $\varphi \in \mathcal{O}_X$ we have

$$\mathcal{D}_X\langle t, s \rangle t \star w = \mathcal{D}_X\langle t, s \rangle tw \quad \text{and} \quad \mathcal{D}_X\langle t, s \rangle \varphi w = \mathcal{D}_X\langle t, s \rangle w.$$

Hence we deduce that $b_v(s) = b_u(s)$. \square

REMARK 3.5.5. If $g \in \mathcal{O}_X$ is such that g/f is not a regular function, then

$$(s+1) \mid b_{gf^s}(s).$$

Indeed, we have that

$$b_{gf^s}(s)gf^s = P(s)gf^{s+1} \quad \text{with } P \in \mathcal{D}_X[s],$$

and taking $s = -1$ we obtain that $b_{gf^s}(-1) \cdot (g/f)$ is a regular function, which cannot happen unless $b_{gf^s}(-1) = 0$.

LEMMA 3.5.6. *In local algebraic coordinates x_1, \dots, x_n , let $u = g\partial_t^p f^s$ be the section of $\iota_+ \mathcal{O}_X$ given by $f = x_1^{c_1} \cdots x_n^{c_n}$ and $g = x_1^{d_1} \cdots x_n^{d_n}$, for some non-negative integers c_i, d_i and p . Then the following hold:*

- (1) $b_u(s)$ divides $(s+1) \prod_{i=1}^n \prod_{j=1}^{c_i} \left(s - p + \frac{d_i+j}{c_i} \right)$. (Here the second product is taken to be 1 if $c_i = 0$.)
- (2) If $p = 0$, then $b_u(s)$ divides $\prod_{i=1}^n \prod_{j=1}^{c_i} \left(s + \frac{d_i+j}{c_i} \right)$.
- (3) If $c_1 = 1$ and $d_1 = 0$, then $b_u(s)$ divides $(s+1) \prod_{i=2}^n \prod_{j=1}^{c_i} \left(s - p + \frac{d_i+j}{c_i} \right)$.

PROOF. We use the notation

$$gf^s = x^{cs+d} := \prod_{i=1}^n x_i^{c_i s + d_i} \quad \text{and} \quad tgf^s = x^{c(s+1)+d} := \prod_{i=1}^n x_i^{c_i(s+1)+d_i}.$$

Note that we can rewrite $u = \partial_t^p x^{cs+d}$.

Consider now the polynomial

$$c(s) := \prod_{i=1}^n \prod_{j=1}^{c_i} (c_i(s-p) + d_i + j).$$

Using the identity $\partial_t P(s) = P(s-1)\partial_t$ for all $P \in \mathbf{C}[s]$, we derive

$$(3.5.1) \quad \partial_{x_1}^{c_1} \cdots \partial_{x_n}^{c_n} \partial_t^p x^{c(s+1)+d} = \partial_t^p \prod_{i=1}^n \prod_{j=1}^{c_i} (c_i s + d_i + j) x^{cs+d} = c(s) \partial_t^p x^{cs+d} = c(s)u.$$

This immediately implies (2), since for $p = 0$ it gives

$$c(s)u \in \mathcal{D}_X[s] \cdot tu.$$

By repeatedly applying the formula $\partial_t t = t\partial_t + 1$, and noting that $s = -\partial_t t$, we obtain

$$\partial_t^p t = -(s-p+1)\partial_t^{p-1},$$

and therefore we have

$$(s+1)\partial_t^p t = (s-p+1)t\partial_t^p.$$

We obtain

$$(s+1)\partial_t^p x^{c(s+1)+d} = (s-p+1)tu.$$

Hence for arbitrary $p \geq 1$, going back to (3.5.1), this gives

$$(s+1)c(s)u = (s-p+1)\partial_{x_1}^{c_1} \cdots \partial_{x_n}^{c_n} tu,$$

which implies (1).

For (3), note first that under the assumption $c_1 = 1$ and $d_1 = 0$, g/f cannot be a regular function. By Remark 3.5.5 we then know that $(s+1)$ divides b_{gfs} , so we can talk about the reduced version \tilde{b}_{gfs} . Moreover, exactly as in Example 3.4.9, we have that

$$b_u(s) \mid (s+1)\tilde{b}_{gfs}(s-p).$$

It therefore suffices to show that

$$\tilde{b}_{gfs}(s) \mid \prod_{i=2}^n \prod_{j=1}^{c_i} \left(s + \frac{d_i + j}{c_i} \right).$$

But by the assumption on c_1 and d_1 , this is the same as what we saw in (2). \square

Left to right correspondence. Recall that there is an equivalence of categories between left and right \mathcal{D}_X -modules, taking a left \mathcal{D}_X -module \mathcal{M} to the right \mathcal{D}_X -module $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$. In local coordinates x_1, \dots, x_n , this is given by an involution

$$\mathcal{D}_X \rightarrow \mathcal{D}_X, \quad P \mapsto P^*,$$

where P^* is the adjoint of P , determined uniquely by the rules: $(PQ)^* = Q^*P^*$, $f^* = f$ for $f \in \mathcal{O}_X$, and $\partial_{x_i}^* = -\partial_{x_i}$. For a section u of \mathcal{M} , we define the section

$$u^* = dx \otimes u$$

of $\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$, where $dx := dx_1 \wedge \cdots \wedge dx_n$. We then have

$$(Pu)^* = u^*P^* \quad \text{for all } P \in \mathcal{D}_X.$$

We now extend this to $\mathcal{D}_X\langle s, t \rangle$ -modules. The same rule

$$\mathcal{M} \mapsto \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

takes a left $\mathcal{D}_X\langle s, t \rangle$ -module to a right one as follows. The involution of \mathcal{D}_X we just described extends to one of $\mathcal{D}_X\langle t, \partial_t \rangle$ by mapping $t \mapsto t$ and $\partial_t \mapsto -\partial_t$, hence mapping

$$s = -\partial_t t \mapsto t \partial_t = -\partial_t t - 1 = -s - 1.$$

We again have $(Pu)^* = u^*P^*$ for all sections u of \mathcal{M} and P of $\mathcal{D}_X\langle s, t \rangle$.

Just as with $\iota_+ \mathcal{O}_X(*D)$, we can also consider Bernstein-Sato polynomials for sections of the right $\mathcal{D}_X\langle s, t \rangle$ -module $\omega_X \otimes_{\mathcal{O}_X} \iota_+ \mathcal{O}_X(*D)$. More precisely, for a section u of $\iota_+ \mathcal{O}_X(*D)$, a Bernstein-Sato relation $b_u(s)u = P(tu)$ becomes, after passing to adjoints,

$$u^* b_u(-s-1) = (u^* t) P^*,$$

hence we have

$$b_{u^*}(s) = b_u(-s-1).$$

Main construction, and proof of the theorem. Fix $g \in \mathcal{O}_X(X)$ and $p \geq 0$, and consider the $\mathcal{D}_X\langle t, s \rangle$ -module

$$\mathcal{N}_{f,p}(g) := \mathcal{D}_X\langle t, s \rangle \cdot g \partial_t^p f^s \subseteq \iota_+ \mathcal{O}_X.$$

where $\iota: X \hookrightarrow Y = X \times \mathbf{C}$ is the graph embedding induced by f . Note that when $g = 1$ and $p = 0$ we have

$$\mathcal{N}_{f,0}(1) = \mathcal{N}_f,$$

the \mathcal{D} -module used in the proof of Kashiwara's theorem. Just as in that case, we have:

EXERCISE 3.5.7. The action of t preserves $\mathcal{N}_{f,p}(g)$, so that $t\mathcal{N}_{f,p}(g)$ is a $\mathcal{D}_X\langle t, s \rangle$ -submodule of $\mathcal{N}_{f,p}(g)$. Furthermore, if $u = g\partial_t^p f^s$, then the Bernstein-Sato polynomial $b_u(s)$ is the minimal polynomial of the action of s on the quotient $\mathcal{N}_{f,p}(g)/t\mathcal{N}_{f,p}(g)$.

As in the proof of Theorem 2.5.1, by restricting to an open neighborhood of the zero locus $Z(f)$ we can make the harmless extra assumption that

$$(3.5.2) \quad Z(\text{Jac}(f)) \subseteq Z(f),$$

where $\text{Jac}(f)$ is the Jacobian ideal of f . Recall that we denote by W_f the closure of the subset

$$\{(x, sdf(x)) \mid f(x) \neq 0, s \in \mathbf{C}\} \subseteq T^*X.$$

This is an irreducible subvariety of T^*X , of dimension $n+1$, which dominates X . A result that Kashiwara proves at the same time as Theorem 2.5.14 is the following:

THEOREM 3.5.8 ([Ka1, Theorem 5.3]). *The \mathcal{D}_X -module \mathcal{N}_f is coherent, and $\text{Ch}(\mathcal{N}_f) = W_f$. In particular \mathcal{N}_f is subholonomic.*

Based on this, one can show the following:

PROPOSITION 3.5.9. *If f defines a divisor with SNC support and satisfies (3.5.2), denoting $\mathcal{N}_{f,p} = \mathcal{N}_{f,p}(1)$, for every $p \geq 0$ we have:*

- (1) *As a \mathcal{D}_X -module, $\mathcal{N}_{f,p}$ is generated by $\partial_t^j f^s$, with $0 \leq j \leq p$.*
- (2) $\text{Ch}(\mathcal{N}_{f,p}) = W_f$.

PROOF. Recall that for every $j \geq 0$ we have

$$t\partial_t^j f^s = f\partial_t^j f^s - j\partial_t^{j-1} f^s,$$

from which descending induction on j shows that $\partial_t^j f^s \in \mathcal{N}_{f,p}$ for $0 \leq j \leq p$. The formula also shows that

$$\mathcal{N}'_{f,p} := \sum_{j=0}^p \mathcal{D}_X \cdot \partial_t^j f^s$$

is a $\mathcal{D}_X[t]$ -submodule of $\mathcal{N}_{f,p}$. Thus by the definition of $\mathcal{N}_{f,p}$, to deduce (1) it suffices to show that

$$s\partial_t^j f^s \in \mathcal{N}'_{f,p}, \quad \text{for all } 0 \leq j \leq p.$$

Recall now that $\partial_t s = (s-1)\partial_t$, and so

$$s\partial_t^j f^s = \partial_t^j (s+j) f^s.$$

It suffices thus to show that $s f^s \in \mathcal{D}_X f^s$, which is left as Exercise 3.5.10 below.

We show (2) by induction on p ; the base case $p = 0$ is Theorem 3.5.8. Assuming that $p \geq 1$, part (1) implies that $\mathcal{N}_{f,p-1} \subset \mathcal{N}_{p,f}$, and the quotient $\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}$ is generated over \mathcal{D}_X by the class of $\partial_t^p f^s$. In particular, there is a surjective \mathcal{D}_X -module homomorphism

$$\mathcal{N}_{f,0} = \mathcal{D}_X f^s \rightarrow \mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}, \quad Pf^s \mapsto P\widehat{\partial_t^p f^s},$$

which implies that $\text{Ch}(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}) \subseteq W_f$. The fact that $\text{Ch}(\mathcal{N}_{f,p}) = W_f$ follows then from the chain of inclusions

$$W_f = \text{Ch}(\mathcal{N}_{f,p-1}) \subseteq \text{Ch}(\mathcal{N}_{f,p}) \subseteq \text{Ch}(\mathcal{N}_{f,p-1}) \cup \text{Ch}(\mathcal{N}_{f,p}/\mathcal{N}_{f,p-1}) \subseteq W_f.$$

□

EXERCISE 3.5.10. Show that if f defines a divisor with SNC support and satisfies (3.5.2), then $sf^s \in \mathcal{D}_X f^s$.

PROOF OF THEOREM 3.5.2. Step 1. Using Proposition 2.2.6 and Remark 2.2.7, we see that the statement of the theorem is local on X . We may therefore assume that X is affine, with algebraic coordinates x_1, \dots, x_n . We may also assume that f is not an invertible function on X , hence after passing to an open neighborhood of the zero locus of f , that condition (3.5.2) is satisfied.

For each u as in the statement, we consider the section

$$u^* = dx \otimes u$$

of the right $\mathcal{D}_X\langle s, t \rangle$ -module $\mathcal{N}_u := \omega_X \otimes_{\mathcal{O}_X} \mathcal{N}_{f,p}(g)$. The right \mathcal{D} -module version of Exercise 3.5.7 tells us that $b_{u^*}(s)$ is equal to $b_{\mathcal{N}_u}(s)$, the minimal polynomial of the action of s on $\mathcal{N}_u/\mathcal{N}_u t$. By the discussion above we also know that $b_{u^*}(s) = b_u(-s-1)$.

Recalling that $\mu: Y \rightarrow X$ is the fixed log resolution, we denote

$$f' := f \circ \mu \quad \text{and} \quad g' = g \circ \mu.$$

By analogy with the construction on X , on Y we consider the $\mathcal{D}_Y\langle s, t \rangle$ -module

$$\mathcal{N}_{u'} := \omega_Y \otimes_{\mathcal{O}_Y} \mathcal{N}_{f',p}(g'),$$

where $u' = g' \partial_t^p f'^s \in \iota_+ \mathcal{O}_Y$ and $\mathcal{N}_{f',p}(g') = \mathcal{D}_Y\langle s, t \rangle \cdot u'$. We also consider its submodule

$$\mathcal{N}_v := v \cdot \mathcal{D}_Y\langle s, t \rangle, \quad \text{with} \quad v := \mu^* dx \otimes u'.^5$$

We fix an open set $V \subseteq Y$ on which we have algebraic coordinates y_1, \dots, y_n in which

$$f' = h_1 y_1^{a_1} \cdots y_n^{a_n}, \quad \mu^* dx = h_2 y_1^{b_1} \cdots y_n^{b_n} dy, \quad \text{and} \quad g' = h y_1^{k_1} \cdots y_n^{k_n},$$

according to our usual notation (the only a_i which are non-zero correspond to $1 \leq i \leq m$), where h_1 and h_2 are invertible functions on V , and h is a regular function on V . Note that $b_v(s)$ (in the sense of right \mathcal{D} -modules) is the same as $b_{\mathcal{N}_v}(s)$, the minimal polynomial of the action of s on $\mathcal{N}_v/\mathcal{N}_v t$. According to the right \mathcal{D} -module version of Lemma 3.5.4, we then have that

$$b_{\mathcal{N}_v}(s) = b_{u'^*}(s) = b_{u'}(-s-1),$$

⁵Note that this is where things diverge from (the analogue of) Kashiwara's argument, which would proceed using $\mathcal{N}_{u'}$, while here this is replaced by \mathcal{N}_v .

where $w' = hw$, with

$$w := y_1^{b_1+k_1} \cdots y_n^{b_n+k_n} \partial_t^p (y_1^{a_1} \cdots y_n^{a_n})^s \in \iota_+ \mathcal{O}_Y \left[\frac{1}{f'} \right].$$

Using Lemma 3.5.3, we see that the greatest root of $b_{w'}(s)$ is at most equal to the greatest root of $b_w(s)$. On the other hand, w is an element to which we can apply Lemma 3.5.6. Using it, and recalling that we have $a_i \neq 0$ only for $1 \leq i \leq m$, we obtain the following:

- The greatest root of $b_w(s)$ is at most $\max\{-1, p - \min_{1 \leq i \leq m} \frac{b_i+1+k_i}{a_i}\}$.
- If $p = 0$, then the greatest root of $b_w(s)$ is at most $-\min_{1 \leq i \leq m} \frac{b_i+1+k_i}{a_i}$.
- If $g = 1$ (so that $h = 1$ and we don't need to use Lemma 3.5.3) every root of $b_w(s)$ is either equal to -1 or to some $p - \frac{b_i+\ell}{a_i}$, with $1 \leq i \leq m$ and $1 \leq \ell \leq a_i$. If in addition D is reduced and its proper transform \tilde{D} is smooth, then we may assume that the divisor given by $(y_i = 0)$ on Y is exceptional. Indeed, note that in this case at most one y_i satisfies $b_i = 0$, i.e. it is not exceptional, and $a_i > 0$; in this case in fact $a_i = 1$.

Covering Y by open sets V on which we have such local coordinates, the global polynomial $b_{\mathcal{N}_v}(s)$ is the least common multiple of the respective polynomials on each V , described above. Hence to conclude the proof, it suffices to show that each root of $b_{u^*}(s)$ is of the form $\alpha + \ell$, where α is a root of $b_{\mathcal{N}_v}(s)$ and ℓ is a non-negative integer.

Step 2. According to the paragraph above, it suffices to show that there exists an integer $N \geq 0$ such that

$$(3.5.3) \quad b_{u^*}(s) \mid b_{\mathcal{N}_v}(s)b_{\mathcal{N}_v}(s-1) \cdots b_{\mathcal{N}_v}(s-N).$$

This is now very similar to the proof of Kashiwara's theorem, only in the setting of right \mathcal{D} -modules, and with \mathcal{N}_v playing the role of $\mathcal{N}_{f'}$ there. We define

$$\mathcal{N} := \mathcal{H}^0 \mu_+ \mathcal{N}_v = R^0 \mu_* (\mathcal{N}_v \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}).$$

Exactly as in Lemma 2.5.16, we have the divisibility

$$(3.5.4) \quad b_{\mathcal{N}}(s) \mid b_{\mathcal{N}_v}(s).$$

We now perform Kashiwara's main construction (see §2.5) in this setting, by constructing a distinguished section $r \in \Gamma(X, \mathcal{N})$. Note first that on Y we have a morphism of \mathcal{D}_Y -modules

$$\mathcal{D}_Y \rightarrow \mathcal{N}_v, \quad 1 \mapsto v.$$

Taking the derived tensor product with $\mathcal{D}_{Y \rightarrow X}$, we obtain a homomorphism

$$\mathcal{D}_{Y \rightarrow X} \rightarrow \mathcal{N}_v \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}.$$

On the other hand, the section $1 \in \mathcal{D}_X$ induces an \mathcal{O}_Y -module homomorphism $\mathcal{O}_Y \rightarrow \mathcal{D}_{Y \rightarrow X}$, since $\mathcal{D}_{Y \rightarrow X} \simeq \mu^* \mathcal{D}_X$ as an \mathcal{O}_Y -module. By composition we obtain a section

$$\mathcal{O}_Y \rightarrow \mathcal{N}_v \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X},$$

and applying $R^0\mu_*$ we finally obtain a global section $r \in \Gamma(X, \mathcal{N})$. It is immediate from the construction that $r = u^*$ on U , the complement of the zero locus of f . We define

$$\mathcal{N}' := r \cdot \mathcal{D}_X\langle s, t \rangle \subseteq \mathcal{N},$$

a right $\mathcal{D}_X\langle s, t \rangle$ -submodule of \mathcal{N} . We can also define a $\mathcal{D}_X\langle s, t \rangle$ -module homomorphism

$$\mathcal{N}' \rightarrow \mathcal{N}_u, \quad r \mapsto u^*,$$

and precisely as in the proof of Lemma 2.5.7, this is well defined, and obviously surjective. In conclusion, we have a digram of $\mathcal{D}_X\langle s, t \rangle$ -modules

$$\begin{array}{ccc} \mathcal{N}' & \xrightarrow{i} & \mathcal{N} \\ \downarrow g & & \\ \mathcal{N}_u & & \end{array}$$

where i is the inclusion map, and g is surjective.

As in the proof of Theorem 2.5.1, we see that the $\mathcal{D}_X\langle s, t \rangle$ -module $\mathcal{M} := \mathcal{N}/\mathcal{N}'$ is holonomic as a \mathcal{D}_X -module. Indeed, as in the paragraph after (2.5.1), it suffices to have the analogue of Theorem 2.5.14, namely

$$\text{Ch}(\mathcal{M}) = W_f \cup \Lambda,$$

where Λ is a Lagrangian subvariety of T^*X . As $\mathcal{N}_v \subseteq \mathcal{N}_{f',p}(g')$, this in turn follows by using Proposition 3.5.9(2) and the obvious fact that $\mathcal{N}_{f',p}(g') \subseteq \mathcal{N}_{f',p}$ (together with a standard results on the characteristic variety of a direct image, that I have not explained yet; this is also used for proving Theorem 2.5.14, and it will be added eventually).

The right \mathcal{D} -module analogue of Lemma 2.5.12 then implies that

$$b_{\mathcal{N}'}(s) \mid b_{\mathcal{N}}(s)b_{\mathcal{N}}(s-1) \cdots b_{\mathcal{N}}(s-N).$$

for some integer $N \geq 0$. (Indeed the signs change, since now we are using the identity $b(s)t = tb(s-1)$ multiplying from the right.) In view of (3.5.4), it suffices then to show that

$$b_{u^*}(s) \mid b_{\mathcal{N}'}(s).$$

But this follows immediately from the surjection $\mathcal{N}' \rightarrow \mathcal{N}_u$, since $b_{u^*}(s) = b_{\mathcal{N}_u}(s)$. \square

3.6. Multiplier ideals vs. V-filtration on \mathcal{O}_X

We fix a non-invertible function f on X , and denote $D = (f = 0)$. We have seen in §2.6 and §3.2 that certain aspects of the behavior of the multiplier ideals $\mathcal{J}(\alpha D)$ and the filtration $V^\alpha \mathcal{O}_X$ induced on \mathcal{O}_X by the V-filtration $V^\bullet \iota_+ \mathcal{O}_X$ are very similar. For instance the threshold where they both become trivial is $\text{let}(f)$, and more generally they both change at the jumping coefficients of the pair (X, D) in the interval $[0, 1]$ (as by Theorem 2.7.3 these are roots of $b_f(s)$), though with different semicontinuity behavior.

Budur and Saito [BS, Theorem 0.1] have noted that this is not an accident, enhancing these numerical properties to the following statement, proved using the theory of mixed Hodge modules:

THEOREM 3.6.1. *For every $\alpha \in \mathbf{Q}_{>0}$, we have*

$$V^\alpha \mathcal{O}_X = \mathcal{J}((\alpha - \varepsilon)D) \quad \text{for } 0 < \varepsilon \ll 1.$$

In this section, following [DM] we give a different proof of this result, which does not make use of the Hodge filtration, but only of more elementary statements discussed up to now.

Recall that in analytic terms we have, for every $c > 0$, that

$$\mathcal{J}(cD)^{\text{an}} = \{g \in \mathcal{O}_X \mid \frac{|g|^2}{|f|^{2c}} \text{ is locally integrable}\}.$$

For a function $g \in \mathcal{O}_X(X)$, by analogy with the usual definition we denote

$$\text{lct}_g(f) := \sup\{c > 0 \mid g \in \mathcal{J}(cD)\}.$$

Fixing a log resolution $\mu: Y \rightarrow X$ which is an isomorphism away from the support of D , recall that we write

$$\mu^*D = \sum_{i=1}^m a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i E_i.$$

For each i , we also denote $k_i = \text{ord}_{E_i}(g)$. By analogy with (2.6.4), we have:

EXERCISE 3.6.2. The threshold $\text{lct}_g(f)$ is computed on Y by the formula

$$\text{lct}_g(f) = \min_{1 \leq i \leq m} \frac{b_i + 1 + k_i}{a_i}.$$

On the other hand, as in §3.4 we denote

$$u = gf^s \in \iota_+ \mathcal{O}_X,$$

and according to Sabbah's description of the V -filtration we have

$$g \in V^\alpha \mathcal{O}_X \iff u \in V^\alpha \iota_+ \mathcal{O}_X \iff c \leq -\alpha \text{ for all } c \text{ such that } b_u(c) = 0.$$

Therefore Theorem 3.6.1 is equivalent to the following analogue of Theorem 2.7.2:

THEOREM 3.6.3. *The greatest root of $b_u(s)$ is $-\text{lct}_g(f)$.*

PROOF. Just as with Theorem 2.7.2, first we show that $-\text{lct}_g(f)$ is a root of $b_u(s)$. Let us first recall that by definition we have

$$b_u(s) \cdot gf^s = P(s)t(gf^s), \quad P(s) \in \mathcal{D}_X[s].$$

Now the action of t on this element is $t(gf^s) = gf^{s+1}$ (recall that we are identifying f^s with δ , and see Exercise B.3 in the notes on the V -filtration), where as always $f^{s+1} := f \cdot f^s$. Hence this can be rewritten as

$$b_u(s) \cdot gf^s = P(s)gf^{s+1}.$$

We now proceed precisely as in the proof of Theorem 2.7.2; we repeat the argument for convenience. We denote $c_0 = \text{lct}_g(f)$. Therefore for some point x in the zero locus of f and some small ball B around x , the function $\frac{|g|^2}{|f|^{2c}}$ with $c = c_0 - \varepsilon$ for $0 < \varepsilon \ll 1$ is

integrable on B , but $\frac{|g|^2}{|f|^{2c_0}}$ is not integrable on some compact ball B' strictly contained in B .

Arguing as in Lemma 2.4.4, and taking $s = -c$, we obtain the identity

$$b_u(-c)^2 |g|^2 |f|^{-2c} = (P(-c)\bar{P}(-c)) |g|^2 |f|^{2(-c+1)}.$$

Both sides are integrable on B , and so for any smooth positive test function φ supported on B we have

$$(3.6.1) \quad \int_B b_u(-c)^2 |g|^2 |f|^{-2c} \varphi = \int_B (P(-c)\bar{P}(-c)) |g|^2 |f|^{2(-c+1)} \varphi.$$

We can in fact take φ to be a bump function with support in B , identically equal to 1 on B' , in which case we obtain that the left-hand side of (3.6.1) is at least

$$b_u(-c)^2 \int_{B'} |g|^2 |f|^{-2c}.$$

Using integration by parts in a way similar to Exercise 2.4.5, we see that the right-hand side of (3.6.1) is equal to

$$\int_B |g|^2 |f|^{2(-c+1)} (P(-c)\bar{P}(-c)\varphi),$$

and therefore if ε is in a fixed interval $(0, \delta]$, then it is bounded above by some $M > 0$ depending only on φ (as $c - 1$ belongs to a closed interval of values for which the integral is finite and depends continuously on c). We deduce that

$$b_u(-c)^2 \int_{B'} |g|^2 |f|^{-2c} \leq M < \infty$$

for every such c . On the other hand, $\frac{|g|^2}{|f|^{2c_0}}$ is not integrable on B' , hence by Fatou's Lemma we have

$$\int_{B'} |g|^2 |f|^{-2c} \rightarrow \infty \quad \text{as } c \rightarrow c_0.$$

The only way this can happen is if $b_u(-c_0) = 0$.

Having established that $-\text{lct}_g(f)$ is a root of $b_u(s)$, the full statement now follows from Theorem 3.5.2(2), which thanks to Exercise 3.6.2 can be rephrased as saying that the greatest root of $b_u(s)$ is at most equal to $-\text{lct}_g(f)$. \square

It makes sense to wonder whether $-\text{lct}_g(f)$ is also related to the poles of an archimedean zeta function as in §2.4, and this is indeed the case. Concretely, this time we can consider the distribution $|g|^2 |f|^{2s}$, which on any $\varphi \in C_c^\infty(\mathbf{C}^n)$ is defined by

$$(|g|^2 |f|^{2s}, \varphi) := \int_{\mathbf{C}^n} |g|^2 |f(x)|^{2s} \varphi(x) = Z_\varphi^g(s).$$

Arguments completely analogous to those in Proposition 2.4.2, Theorem 2.4.3, and Proposition 2.7.1 show the following:

THEOREM 3.6.4. *With the notation above, for every smooth complex-valued function with compact support $\varphi \in C_c^\infty(\mathbf{C}^n)$, Z_φ^g admits an analytic continuation to \mathbf{C} as a meromorphic function whose poles are of the form $\alpha - m$, where α is a root of the Bernstein-Sato polynomial $b_u(s)$ and $m \in \mathbf{N}$. Moreover, the greatest pole of the distribution $|g|^2|f|^{2s}$ (meaning the maximum over the poles of Z_φ^g for all φ) is equal to $-\text{lct}_g(f)$, hence to the greatest root of $b_u(s)$.*

3.7. Minimal exponent

Let X be a smooth variety of dimension n , and $f \in \mathcal{O}_X(X)$ a non-invertible function. We have seen that -1 is always a root of $b_f(s)$. We can therefore consider the polynomial

$$\tilde{b}_f(s) = \frac{b_f(s)}{s+1},$$

called the *reduced Bernstein-Sato polynomial* of f .

DEFINITION 3.7.1. The negative $\tilde{\alpha}_f$ of the greatest root of the reduced of $\tilde{b}_f(s)$ is called the *minimal exponent* of f .⁶

According to Theorem 2.7.2, the log canonical threshold $\text{lct}(f)$ is equal to α_f , the greatest root of $b_f(s)$. Therefore if $\tilde{\alpha}_f \leq 1$, then it coincides with $\text{lct}(f)$; more precisely

$$\alpha_f = \min\{1, \tilde{\alpha}_f\}.$$

Thus $\tilde{\alpha}_f$ is a refinement of the log canonical threshold, and it provides a new interesting invariant precisely when the pair (X, D) is log canonical.

REMARK 3.7.2 (Local version). Recall that we also have a local version $b_{f,x}$ of the Bernstein-Sato polynomial, around a point $x \in X$; see Definition 2.2.5. If f is not invertible around x , then $(s+1) \mid b_{f,x}(s)$, and we define $\tilde{\alpha}_{f,x}$ to be the negative of the greatest root of $\tilde{b}_{f,x}(s) = b_{f,x}(s)/(s+1)$.

REMARK 3.7.3 (Global version). We can also define a global version of the minimal exponent. For each non-trivial effective divisor D on X , there is an associated Bernstein-Sato polynomial $b_D(s)$ such that $(s+1) \mid b_D(s)$; see Remark 2.2.9. We have

$$b_D(s) = \text{lcm}_{x \in D} b_{D,x}(s),$$

where $b_{D,x}(s) := b_{f,x}$ for any locally defining equation f for D in a neighborhood of x . (Hence we can also write $\tilde{\alpha}_{D,x} := \tilde{\alpha}_{f,x}$ for any such f .) As above, the minimal exponent $\tilde{\alpha}_D$ is the negative of the greatest root of $\tilde{b}_D(s) = b_D(s)/(s+1)$. The description above implies that

$$\tilde{\alpha}_D = \min_{x \in D} \tilde{\alpha}_{D,x}.$$

EXAMPLE 3.7.4 (Quasi-homogeneous isolated singularities). Let $f \in \mathbf{C}[X_1, \dots, X_n]$ be a quasi-homogeneous polynomial, with weights w_1, \dots, w_n (see §2.3), having an isolated singularity. Since in the notation of that section $\rho(1) = 0$, Theorem 2.3.4 implies that

$$\tilde{\alpha}_f = |w| := w_1 + \dots + w_n.$$

⁶This is also called the *microlocal log canonical threshold* of f in [Sa9].

In particular, for a diagonal hypersurface $f = X_1^{a_1} + \cdots + X_n^{a_n}$ we have the celebrated

$$\tilde{\alpha}_f = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Just like α_f , the minimal exponent it is known to be related to standard types of singularities due the following results of Saito:

THEOREM 3.7.5. *Assume that D is reduced around a point x . Then*

- (1) [Sa3, Theorem 0.4] *D has rational singularities at x if and only if $\tilde{\alpha}_{D,x} > 1$.*
- (2) [Sa6, Theorem 0.5] *D has Du Bois singularities at x if and only if $\tilde{\alpha}_{D,x} \geq 1$.*

Note that this implies that rational hypersurface singularities are du Bois; this is in fact known to be true for arbitrary varieties, by work of Kovács [Kov3] and Saito [Sa5]. For a first look at du Bois singularities, see for instance [KS].

The proof of Theorem 3.7.5 requires the Hodge filtration, and therefore will be given later. For (2) we may alternatively not worry right now about what du Bois means; clearly $\tilde{\alpha}_D \geq 1$ is equivalent to the pair (X, D) having log canonical singularities, while on the other hand using birational geometry arguments it is shown in [KS, Corollary 6.6] that

PROPOSITION 3.7.6. *The pair (X, D) has log canonical singularities if and only if the divisor D has du Bois singularities.*

Let now D be a reduced effective divisor. One of the main questions about the minimal exponent of D is whether we can express it explicitly in terms of discrepancies on a log resolution, like in the case of the log canonical threshold as in Remark 2.6.8. We use again the notation introduced in §3.5, and denote

$$\gamma := \min_{E_i \text{ exceptional}} \frac{b_i + 1}{a_i}.$$

We have noted in Remark 2.6.8 that $\alpha_D = \min\{1, \gamma\}$, while on the other hand by definition $\alpha_D = \min\{1, \tilde{\alpha}_D\}$.

It is natural then to ask whether $\tilde{\alpha}_D = \gamma$, and Lichtin [Li, Remark 2, p.303] did indeed pose this question. This would provide a very simple description, but as noted by Kollár [Ko5, Remark 10.8] in general the answer is negative, since γ usually depends on the choice of log resolution. Nevertheless, at least if we assume that the proper transform \tilde{D} is smooth,⁷ one inequality does hold:

THEOREM 3.7.7 ([MP4, Corollary D]). *We always have $\tilde{\alpha}_D \geq \gamma$.*

It is worth noting that the inequality follows easily from Lichtin's result, Theorem 3.5.1, if $\tilde{\alpha}_D$ is not an integer; however, it is not clear how to use it otherwise. The original proof of the theorem in [MP4] relies on the theory of Hodge ideals. However a more elementary proof due to Dirks-Mustață can be given using Theorem 3.5.2 discussed in this chapter, and we present this next.

⁷This can always be achieved by performing a few more blow-ups, if needed.

PROOF OF THEOREM 3.7.7. Let's assume for simplicity that D is defined globally by a function f . We write $\gamma = p + \alpha$, where p is a non-negative integer, and $\alpha \in (0, 1]$. Recall now from Example 0.33 in the notes on the V -filtration (see also [MP4, Proposition 6.12]) that

$$b_{\partial_i^p f^s}(s) \mid (s+1)\tilde{b}_f(s-p) \quad \text{and} \quad \tilde{b}_f(s-p) \mid b_{\partial_i^p f^s}(s).$$

We deduce that all roots of $\tilde{b}_f(s)$ are $\leq -\gamma$ (which is what we want) if and only if all roots of $b_{\partial_i^p f^s}(s)$ are $\leq -\alpha$.

On the other hand, by Theorem 3.5.2(3) we know that for every root β of $b_{\partial_i^p f^s}(s)$ we either have that β is a negative integer, in which case we clearly have $\beta \leq -1 \leq -\alpha$, or we have

$$\beta = p - \frac{b_i + 1 + \ell}{a_i}$$

for some exceptional divisor E_i and some non-negative integer ℓ . Now by definition

$$\frac{b_i + 1 + \ell}{a_i} \geq \gamma = p + \alpha,$$

and therefore $\beta \leq -\alpha$. □

Mustață and I expect a substantially stronger statement to hold; we have formulated the following:⁸

CONJECTURE 3.7.8. *On every log resolution of (X, D) , there exists an exceptional divisor E_i for which $\tilde{\alpha}_D = \frac{b_i+1}{a_i}$.*

The log canonical threshold is well known to satisfy a few fundamental semicontinuity and restriction properties, as well as numerical bounds in terms of the multiplicity; see for instance [Ko5, §8]. It turns out that the same can be said about the minimal exponent; however the proofs are more complicated, and go beyond what we have studied up to this point (they depend on the theory of Hodge ideals). I am nevertheless including some statements below for completeness.

THEOREM 3.7.9 ([MP4, Theorem E]). *Let X be a smooth n -dimensional complex variety, and D an effective divisor on X .*

- (1) *If Y is a smooth subvariety of X such that $Y \not\subseteq D$, then for every $x \in D \cap Y$, we have*

$$\tilde{\alpha}_{D|_Y, x} \leq \tilde{\alpha}_{D, x}.$$

- (2) *Consider a smooth morphism $\pi: X \rightarrow T$, together with a section $s: T \rightarrow X$ such that $s(T) \subseteq D$. If D does not contain any fiber of π , so that for every $t \in T$ the divisor $D_t = D|_{\pi^{-1}(t)}$ is defined, then the function*

$$T \ni t \rightarrow \tilde{\alpha}_{D_t, s(t)}$$

is lower semicontinuous.

⁸This conjecture is also heuristically motivated by what is called Igusa's Strong Monodromy Conjecture for the local zeta function associated to polynomials $f \in \mathbf{Z}[X_1, \dots, X_n]$.

(3) For every $x \in X$, if $m = \text{mult}_x(D) \geq 2$, then

$$\frac{n - r - 1}{m} \leq \tilde{\alpha}_{D,x} \leq \frac{n}{m},$$

where r is the dimension of the singular locus of the projectivized tangent cone $\mathbf{P}(C_x D)$ of D at x (with the convention that $r = -1$ if $\mathbf{P}(C_x D)$ is smooth).

EXAMPLE 3.7.10 (Ordinary singularities). In the case of a Fermat hypersurface

$$f = X_1^m + \cdots + X_n^m,$$

Example 3.7.4 gives $\tilde{\alpha}_f = n/m$. This is however also an example of an ordinary singular point of multiplicity m (see Example 2.6.3(3)), and for these we always have

$$\tilde{\alpha}_f = \frac{n}{m}.$$

This is the highest one can go: it turns out that for any f and any singular point x of multiplicity m we have $\tilde{\alpha}_{f,x} \leq \frac{n}{m}$; see Theorem 3.7.9(3). The equality above follows from work of Saito. Alternatively, one can again use Theorem 3.7.9(3), noting that ordinary singularities are precisely the case $r = -1$. Note that the inequality $\tilde{\alpha}_f \geq \frac{n}{m}$ follows also from Theorem 3.7.7, since for an ordinary singularity $\gamma = n/m$ (just take the resolution given by blowing up the singular point).

REMARK 3.7.11. Another general fact worth mentioning is the following result due to Saito [Sa4, Theorem 0.4]: the negative of every root of \tilde{b}_f is in the interval $[\tilde{\alpha}_f, n - \tilde{\alpha}_f]$.⁹

⁹This gives another way of seeing that we always have $\tilde{\alpha}_f \leq n/2$.

CHAPTER 4

Filtered \mathcal{D} -modules and Hodge \mathcal{D} -modules

The aim of this chapter is to introduce some basic notions in the study of filtered \mathcal{D} -modules, focus on the interaction between the F -filtration and the V -filtration, and provide a first definition of Hodge modules. To get to this notion as quickly as possible, at first I will not develop things in the most systematic possible way. Later on I will rearrange the notes in a better order.

4.1. F -filtration and V -filtration

Let X be a smooth complex variety of dimension n , and let (\mathcal{M}, F) be a coherent \mathcal{D}_X -module endowed with a good filtration. We will see here that the nearby and vanishing cycles constructed in §3.3, and so implicitly the V -filtration, are useful for imposing restrictions on the filtration F . The material here follows closely [Sa1, §3.2].

Assume first that D is a smooth divisor on X , given by $t = 0$, and consider the V -filtration on \mathcal{M} induced by t . Assume that \mathcal{M} has strict support Z which is not contained in D .

For every $p \in \mathbf{Z}$ and $\alpha \in \mathbf{Q}$ we define

$$F_p V^\alpha \mathcal{M} := F_p \mathcal{M} \cap V^\alpha \mathcal{M}$$

and

$$F_p \operatorname{gr}_V^\alpha \mathcal{M} := \frac{F_p \mathcal{M} \cap V^\alpha \mathcal{M}}{F_p \mathcal{M} \cap V^{>\alpha} \mathcal{M}}.$$

Recall that in the proof of Proposition 3.3.3 we saw that

$$\mathcal{M} = \sum_{i \geq 0} \partial_t^i \cdot (V^{>0} \mathcal{M}).$$

On the other hand, we know that $V^{>0} \mathcal{M}$ is determined by $\mathcal{M}|_U$ by Lemma 3.1.8. A situation in which we can also recover $F_p V^{>0} \mathcal{M}$ from its restriction to U is provided by the following:

LEMMA 4.1.1. *Denoting by $j: U = X \setminus D \hookrightarrow X$ the inclusion map, we have the identity*

$$F_p V^{>0} \mathcal{M} = V^{>0} \mathcal{M} \cap j_* j^* F_p \mathcal{M}$$

if and only if

$$t: F_p V^\alpha \mathcal{M} \longrightarrow F_p V^{\alpha+1} \mathcal{M}$$

is surjective for all $\alpha > 0$.

PROOF. First assume the identity. Consider $y \in F_p V^{\alpha+1} \mathcal{M}$, for $\alpha > 0$. In particular we have $y \in F_p V^{>0} \mathcal{M}$, and so by hypothesis $y \in V^{\alpha+1} \mathcal{M} \cap j_* j^* F_p \mathcal{M}$. We conclude the surjectivity of t due to the fact that $t: V^\alpha \mathcal{M} \rightarrow V^{\alpha+1} \mathcal{M}$ is an isomorphism, combined with the fact that the action of t on $j_* j^* F_p \mathcal{M}$ is bijective.

Assume now that t acts surjectively on $F_p V^\alpha \mathcal{M}$ for every $\alpha > 0$. Note that the inclusion from left to right in the identity is clear. Consider now

$$x \in V^\alpha \mathcal{M} \cap j_* j^* F_p \mathcal{M} = \{x \in V^\alpha \mathcal{M} \mid x|_U \in F_p \mathcal{M}|_U\},$$

with $\alpha > 0$. Thus locally there exists $i \geq 0$ such that $t^i x \in F_p \mathcal{M}$. Since $x \in V^\alpha \mathcal{M}$, we have $t^i x \in V^{\alpha+i} \mathcal{M}$, so $t^i x \in F_p V^{\alpha+i} \mathcal{M}$. By hypothesis and the injectivity of the action of t , we then have that $x \in F_p V^\alpha \mathcal{M} \subseteq F_p V^{>0} \mathcal{M}$. \square

REMARK 4.1.2. Note that since $t: V^\alpha \mathcal{M} \rightarrow V^{\alpha+1} \mathcal{M}$ is an isomorphism for $\alpha > 0$, we have that in this range

$$t: F_p V^\alpha \mathcal{M} \longrightarrow F_p V^{\alpha+1} \mathcal{M}$$

is in any case injective, so the condition in the Lemma is that it is in fact an isomorphism.

Lemma 4.1.1 can be combined with the following result in order to obtain a criterion for recovering the full $F_p \mathcal{M}$ from its restriction to U . Note first that we always have

$$\sum_{i \geq 0} \partial_t^i \cdot (F_{p-i} V^{>0} \mathcal{M}) \subseteq F_p \mathcal{M}.$$

LEMMA 4.1.3. *Assume that $\partial_t: \text{gr}_V^1 \mathcal{M} \rightarrow \text{gr}_V^0 \mathcal{M}$ is surjective.¹ Then the following are equivalent:*

- (1) $F_p \mathcal{M} = \sum_{i \geq 0} \partial_t^i \cdot (F_{p-i} V^{>0} \mathcal{M})$.
- (2) $\partial_t: F_p \text{gr}_V^\alpha \mathcal{M} \rightarrow F_{p+1} \text{gr}_V^{\alpha-1} \mathcal{M}$ is surjective for all $\alpha \leq 1$.

PROOF. We define a new filtration $F'_\bullet \mathcal{M}$ by

$$F'_p \mathcal{M} = \sum_{i \geq 0} \partial_t^i \cdot (F_{p-i} V^{>0} \mathcal{M}).$$

We consider the following two claims:

Claim 1. For all $j \geq 0$ we have

$$F'_p V^{>-j} \mathcal{M} = \sum_{0 \leq i \leq j} \partial_t^i \cdot (F_{p-i} V^{>0} \mathcal{M}).$$

Claim 2. For all $\alpha \leq 0$ we have

$$F'_p V^\alpha \mathcal{M} = \sum_{0 \leq i \leq [-\alpha]} \partial_t^i \cdot (F_{p-i} V^{>0} \mathcal{M}) + \partial_t^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^{\alpha+[-\alpha]+1} \mathcal{M}).$$

¹By Proposition 3.3.3, this is equivalent to $\mathcal{M} = \mathcal{D}_X \cdot V^{>0} \mathcal{M}$, so to the fact that there are no proper submodules of \mathcal{M} that agree with \mathcal{M} on $U = X \setminus D$.

Let's first conclude the argument assuming these two claims. Note that the first implies

$$F'_p V^{>0} \mathcal{M} = F_p V^{>0} \mathcal{M},$$

while the second implies

$$F'_p \operatorname{gr}_V^\alpha \mathcal{M} = \partial_t^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} \operatorname{gr}_V^{\alpha+[-\alpha]+1} \mathcal{M}).$$

We clearly have $F'_p \mathcal{M} \subseteq F_p \mathcal{M}$. Descending gradually through the values of α for which the V -filtration jumps, we therefore conclude that $F'_p \mathcal{M} = F_p \mathcal{M}$ if and only if

$$F_p \operatorname{gr}_V^\alpha \mathcal{M} = \partial_t^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} \operatorname{gr}_V^{\alpha+[-\alpha]+1} \mathcal{M}),$$

which is equivalent to the condition in (2).

It remains to prove the two claims. Note first that Claim 2 reduces to Claim 1. Note that the mapping

$$\partial_t^j: \operatorname{gr}_V^{\alpha+j} \mathcal{M} \rightarrow \operatorname{gr}_V^\alpha \mathcal{M}$$

is always an isomorphism for $\alpha + j < 1$, and is surjective by assumption for $\alpha + j = 1$. We obtain that

$$\partial_t^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^{>0} \mathcal{M}) \cap V^\alpha \mathcal{M} = \partial_t^{[-\alpha]+1} \cdot (F_{p-[-\alpha]-1} V^{\alpha+[-\alpha]+1} \mathcal{M}).$$

for $\alpha \leq 0$.

Finally, let's prove Claim 1. Note that the inclusion from right to left is clear by definition. For the opposite inclusion, it suffices to show that

$$\partial_t^{j+1} \cdot (F_{p-j-1} V^{>0} \mathcal{M}) \cap V^{>-j} \mathcal{M} \subseteq \partial_t^j \cdot (F_{p-j} V^{>0} \mathcal{M}).$$

Let $u \in F_{p-j-1} V^{>0} \mathcal{M}$ such that $\partial_t^{j+1} u \in V^{>-j} \mathcal{M}$. Since

$$\partial_t^j: V^{>-1} \mathcal{M} / V^{>0} \mathcal{M} \rightarrow V^{>-j-1} \mathcal{M} / V^{>-j} \mathcal{M}$$

is injective, it follows that $\partial_t u \in V^{>0} \mathcal{M}$. This in turn implies $\partial_t^{j+1} u \in \partial_t^j \cdot (F_{p-j} V^{>0} \mathcal{M})$, which is what we want. \square

LEMMA 4.1.4. *If $\operatorname{Supp} \mathcal{M} \subseteq D$, define $\mathcal{M}^0 = \operatorname{Ker}(t: \mathcal{M} \rightarrow \mathcal{M})$; recall that by Kashiwara's theorem we have $\mathcal{M} = \iota_+ \mathcal{M}_0 \simeq \mathcal{M}^0 \otimes_{\mathbf{C}} \mathbf{C}[\partial_t]$, where $\iota: D \hookrightarrow X$. Define also $F_p \mathcal{M}^0 := F_p \mathcal{M} \cap \mathcal{M}^0$, for all $p \in \mathbf{Z}$. Then the following are equivalent:*

- (1) $F_p \mathcal{M} = \sum_{i \geq 0} F_{p-i} \mathcal{M}^0 \otimes \partial_t^i$. (In other words $(\mathcal{M}, F) \simeq \iota_+(\mathcal{M}^0, F)$; see Example 1.5.7.)
- (2) $\partial_t: F_p \operatorname{gr}_V^\alpha \mathcal{M} \rightarrow F_{p+1} \operatorname{gr}_V^{\alpha-1} \mathcal{M}$ is surjective for all $\alpha < 1$.

PROOF. As in Example 3.1.4(3), based on Kashiwara's equivalence, we know that for all $\alpha \leq 0$ we have

$$V^\alpha \mathcal{M} = \sum_{0 \leq i \leq [-\alpha]} \mathcal{M}^0 \otimes \partial_t^i,$$

and moreover $V^\alpha \mathcal{M} = 0$ for $\alpha > 0$. In particular it suffices to focus on integral α .

We define a new filtration $F'_\bullet \mathcal{M}$ by

$$F'_p \mathcal{M} = \sum_{i \geq 0} F_{p-i} \mathcal{M}^0 \otimes \partial_t^i.$$

Note that we clearly have $F'_p \mathcal{M} \subseteq F_p \mathcal{M}$, and moreover by definition

$$F'_p V^0 \mathcal{M} = F_p V^0 \mathcal{M} (= F_p \mathcal{M}^0) \quad \text{and} \quad F'_p \text{gr}_V^{-i} \mathcal{M} = \partial_t^i \cdot (F_{p-i} \text{gr}_V^0 \mathcal{M})$$

for all $p \in \mathbf{Z}$ and $i \geq 0$. This implies (proceeding inductively on i) that we have $F'_p \mathcal{M} = F_p \mathcal{M}$ if and only if

$$F_p \text{gr}_V^{-i} \mathcal{M} = \partial_t^i \cdot (F_{p-i} \text{gr}_V^0 \mathcal{M})$$

for all such p and i . This is easily seen to be equivalent to the condition in (2). \square

We now move to the case of hypersurfaces defined by arbitrary functions $f \in \mathcal{O}_X(X)$. We are led to collecting the conditions in the lemmas above into the following definition proposed by Saito [Sa1, 3.2.1]. To simplify the notation, we write

$$\mathcal{M}_f := \iota_+ \mathcal{M},$$

where ι stands as always for the graph embedding of X along f .

DEFINITION 4.1.5 (Regular and quasi-unipotent condition). We say that (\mathcal{M}, F) is *quasi-unipotent along f* if \mathcal{M}_f admits a rational V -filtration along the coordinate t on $X \times \mathbf{C}$, and the following conditions are satisfied:

- (1) $t \cdot (F_p V^\alpha \mathcal{M}_f) = F_p V^{\alpha+1} \mathcal{M}_f$ for $\alpha > 0$.
- (2) $\partial_t \cdot (F_p \text{gr}_V^\alpha \mathcal{M}_f) = F_{p+1} \text{gr}_V^{\alpha-1} \mathcal{M}_f$ for $\alpha < 1$.

Moreover, (\mathcal{M}, F) is called *regular along f* if in addition the filtration $F_\bullet \text{gr}_V^\alpha \mathcal{M}_f$ is a good filtration for $0 \leq \alpha \leq 1$.

REMARK 4.1.6. (1) Given the properties of the V -filtration discussed before, we have in fact that the actions of t and ∂_t , in the ranges in (1) and (2) respectively, will in fact be bijective.

(2) By contrast to Lemma 4.1.3, we do not include $\alpha = 1$ in (2) in order to have more flexibility for this notion. This condition will however be satisfied for Hodge modules with strict support not contained in $(t = 0)$, and consequently the lemma will apply; see also the Conclusion at the end of this section.

It is not so hard, but it is very useful in practice, to recognize when a filtered \mathcal{D} -module with support contained in D is regular and quasi-unipotent with respect to a defining equation:

PROPOSITION 4.1.7. *Let (\mathcal{M}, F) be a filtered coherent \mathcal{D} -module with support contained in $D = (f = 0)$, such that \mathcal{M}_f admits a rational V -filtration along t . Denote by $\iota: X \hookrightarrow X \times \mathbf{C}$ the graph embedding of X along f , and by $\iota_0: X = X \times \{0\} \hookrightarrow X \times \mathbf{C}$ the natural embedding. Then the following are equivalent:*

- (1) \mathcal{M} is regular and quasi-unipotent along f .

- (2) $F_k \mathcal{M} \subseteq f \cdot F_{k-1} \mathcal{M}$ for all k .
(3) There is a canonical isomorphism $(\mathcal{M}_f, F) = \iota_+(\mathcal{M}, F) \simeq \iota_{0+}(\mathcal{M}, F)$.

PROOF. Will include later. \square

Conclusion. The main point of this section can roughly be summarized as follows. Assume that (\mathcal{M}, F) is a filtered \mathcal{D}_X -module with strict support Z , admitting a rational V -filtration along a hypersurface $D = (f = 0)$ such that $f|_Z$ is not constant. If it is regular and quasi-unipotent along D , then most of the conditions in Lemmas 4.1.1 and 4.1.3 are satisfied. If we assume that the last needed condition holds as well, namely that

$$\partial_t: F_p \operatorname{gr}_V^1 \mathcal{M}_f \longrightarrow F_{p+1} \operatorname{gr}_V^0 \mathcal{M}_f$$

is surjective, then we conclude that

$$(4.1.1) \quad F_p \mathcal{M}_f = \sum_{i=0}^{\infty} \partial_t^i \cdot (V^{>0} \mathcal{M}_f \cap j_* j^* F_{p-i} \mathcal{M}_f),$$

so in particular (\mathcal{M}, F) is uniquely determined by its restriction to $Z \setminus (Z \cap D)$.

REMARK 4.1.8. (1) A similar argument to that of the proof of Lemma 4.1.3 shows that if we assume by contrast that $t: (\operatorname{gr}_V^0 \mathcal{M}_f, F) \rightarrow (\operatorname{gr}_V^1 \mathcal{M}_f, F)$ is injective and strict, then we have the alternative formula (see [Sa1, 3.2.3]):

$$(4.1.2) \quad F_p \mathcal{M}_f = \sum_{i=0}^{\infty} \partial_t^i \cdot (V^0 \mathcal{M}_f \cap j_* j^* F_{p-i} \mathcal{M}_f).$$

This happens for instance when $\mathcal{M} = \mathcal{O}_X(*D)$, in which case

$$t: (\operatorname{gr}_V^0 \mathcal{M}_f, F) \rightarrow (\operatorname{gr}_V^1 \mathcal{M}_f, F)$$

is a filtered isomorphism. (This follows thanks to the fact that the F filtration on $\mathcal{O}_X(*D)$ is constructed as a filtered direct image via an open embedding; explanation later.)

(2) When D is a smooth hypersurface, the results in (1) hold of course when working directly with \mathcal{M} and the V -filtration along D , without needing to pass to the graph embedding.

4.2. \mathcal{D} -modules with \mathbf{Q} -structure

The following definition underlies the notion of a Hodge module, which will be introduced in the next section. I will therefore start with the following:

Note. The definition that follows refers to two notions that we have not focused on in this course, namely that of a *regular holonomic \mathcal{D} -module*, and that of a *perverse sheaf*. If you are unfamiliar with them, the ideal thing to do would be to acquire some familiarity, by reading for example parts of Chapters 6-8 in [HTT]. In the meanwhile however, for temporarily following the theory, such familiarity is not really crucial. Specifically, on one hand we will not deal concretely with the theory of perverse sheaves almost at all, except for quoting a cohomology vanishing result at some point; on the other hand, in

these notes regularity is only used to ensure the existence of a V -filtration (due to the Kashiwara-Malgrange theorem), and as an input for the Riemann-Hilbert correspondence.

DEFINITION 4.2.1. (1) A *filtered \mathcal{D}_X -module with \mathbf{Q} -structure* is a triple $M = (\mathcal{M}, F, P)$ consisting of:

- A perverse sheaf P of \mathbf{Q} -vector spaces on X .
- A regular holonomic \mathcal{D}_X -module \mathcal{M} such that $\mathrm{DR}(\mathcal{M}) \simeq P \otimes_{\mathbf{Q}} \mathbf{C}$.
- A good filtration $F_{\bullet} \mathcal{M}$ by coherent \mathcal{O}_X -modules.

(2) Given such a triple, for any $k \in \mathbf{Z}$, its k -th Tate twist is defined as

$$M(k) := (\mathcal{M}, F_{\bullet-k}, P(k)),$$

where $P(k) = P \otimes_{\mathbf{Q}} \mathbf{Q}(k)$, with $\mathbf{Q}(k) = (2\pi i)^k \cdot \mathbf{Q} \subset \mathbf{C}$.

(3) Given a filtered \mathcal{D}_X -module with \mathbf{Q} -structure $M = (\mathcal{M}, F, P)$ and a function $f \in \mathcal{O}_X$, and denoting $\lambda = e^{-2\pi i \alpha}$ for $\alpha \in \mathbf{Q}$,² we define

- $\psi_{f,1} M := (\mathrm{gr}_V^1 \mathcal{M}_f, F_{\bullet-1} \mathrm{gr}_V^1 \mathcal{M}_f, {}^p\psi_{f,1} P)$.
- $\varphi_{f,1} M := (\mathrm{gr}_V^0 \mathcal{M}_f, F_{\bullet} \mathrm{gr}_V^0 \mathcal{M}_f, {}^p\varphi_{f,1} P)$.

to be the *unipotent nearby cycles* and the *unipotent vanishing cycles* of M along f , respectively. Here the last term in the parenthesis denotes the corresponding topological construction on the perverse sheaf P . Note that all of these objects have support contained in $f^{-1}(0)$. One can also define similarly the *total* nearby and vanishing cycles as

- $\psi_f M := \bigoplus_{0 < \alpha \leq 1} (\mathrm{gr}_V^\alpha \mathcal{M}_f, F_{\bullet-1} \mathrm{gr}_V^\alpha \mathcal{M}_f, {}^p\psi_{f,\lambda} P)$
- $\varphi_f M := \bigoplus_{0 \leq \alpha < 1} (\mathrm{gr}_V^\alpha \mathcal{M}_f, F_{\bullet} \mathrm{gr}_V^\alpha \mathcal{M}_f, {}^p\varphi_{f,\lambda} P)$.

Note that the summands for $0 < \alpha < 1$ coincide (and it can be checked that they do not carry a rational structure individually, which is the reason they are grouped together); in any case, our focus will be the unipotent versions above.

It is not hard to see that the criterion regarding decomposition by strict support in terms of nearby and vanishing cycles, Corollary 3.3.8, has an enhancement to the setting of filtered \mathcal{D} -modules with \mathbf{Q} -structure:

PROPOSITION 4.2.2. *Let $M = (\mathcal{M}, F, P)$ be a filtered \mathcal{D}_X -module with \mathbf{Q} -structure, and suppose that (\mathcal{M}, F) is regular and quasi-unipotent along $(f = 0)$ for all locally defined functions $f \in \mathcal{O}_X(U)$ on all open sets $U \subset X$. Then there exists a decomposition*

$$M \simeq \bigoplus_{Z \subset X} M_Z,$$

with each $M_Z = (\mathcal{M}_Z, F, P_Z)$ a filtered \mathcal{D}_X -module with \mathbf{Q} -structure and strict support Z , if and only if

$$\varphi_{f,1} M = \mathrm{Ker}(\mathrm{var}: \varphi_{f,1} M \longrightarrow \psi_{f,1} M(-1)) \oplus \mathrm{Im}(\mathrm{can}: \psi_{f,1} M \longrightarrow \varphi_{f,1} M),$$

²Note that this is the convention for left \mathcal{D} -modules; for right \mathcal{D} -modules it would be $\lambda = e^{-2\pi i \alpha}$.

where the filtration on $\text{Im}(\text{can})$ is induced by that on $\psi_{f,1}\mathcal{M}$.

Monodromy weight filtration. Before defining pure Hodge modules, we need a brief discussion of the monodromy weight filtration.

A general linear algebra result (see e.g. [Schmid, Lemma 6.4]) says that any nilpotent endomorphism $N: M \rightarrow M$ in an abelian category whose objects have finite length (e.g. finite dimensional vector spaces, holonomic \mathcal{D} -modules, perverse sheaves), induces a filtration called the *monodromy weight filtration*.³ This is a finite increasing filtration $W_\bullet = W_\bullet M$, going from $W_0 = 0$ to W_{2k} if $N^{k+1} = 0$; it is uniquely determined by the properties

- (1) $N(W_\ell) \subseteq W_{\ell-2}$
- (2) the induced morphism $N^k: \text{gr}_k^W \rightarrow \text{gr}_{-k}^W$ is an isomorphism

for every ℓ and k . Specifically, the filtration is defined by the formula

$$W_\ell := \sum_i \text{Im}(N^i) \cap \text{Ker}(N^{\ell+i+1}).$$

Moreover, if we denote $P_\ell = \text{Ker}(N^{\ell-k+1}) \subseteq \text{gr}_\ell^W M$ for $\ell \geq k$, and $P_\ell = 0$ for $\ell < k$, then we have a Lefschetz type decomposition $\text{gr}_\ell^W M = \bigoplus_i N^i(P_{\ell+2i})$.

Assume now in addition that our object M carries a finite increasing filtration $F_\bullet M$, and $N: M \rightarrow M$ is a filtered endomorphism, besides being nilpotent. Then there exists again a unique filtration $W_\bullet = W_\bullet M$ satisfying the alternative conditions

- (1) $N(W_\ell) \subseteq W_{\ell-2}$
- (2) the induced morphism $N^k: \text{gr}_{i+k}^W \text{gr}_i^F M \rightarrow \text{gr}_{i-k}^W \text{gr}_i^F M$ is an isomorphism

for every ℓ, i and k .

We now apply this to our situation. For completeness, I will first review a number of facts regarding nearby and vanishing cycles on perverse sheaves.⁴ Let P be a perverse sheaf on X , and $f: X \rightarrow \mathbf{C}$ a holomorphic function. We have associated perverse nearby and vanishing cycles ${}^p\psi_f P := \psi_f P[-1]$ and ${}^p\varphi_f P := \varphi_f P[-1]$, together with natural morphisms

$$\text{can}: {}^p\psi_f P \rightarrow {}^p\varphi_f P \quad \text{and} \quad \text{var}: {}^p\varphi_f P \rightarrow {}^p\psi_f P(-1),$$

and an action

$$T: {}^p\psi_f P \rightarrow {}^p\psi_f P$$

of the monodromy operator.

³This is due to the fact that in applications it usually arises from a monodromy operator, as we will see next.

⁴It's ok to skip if you are not familiar with this story. For more details and further references you can consult for instance D. Massey's notes arXiv:math/9908107; see also [Sch, §8].

Perverse sheaves form an abelian category; therefore there is a generalized eigenspace decomposition

$${}^p\psi_f P = \bigoplus_{\lambda \in \mathbf{C}^*} {}^p\psi_{f,\lambda} P$$

under the action of T , and similarly for ${}^p\varphi_f P$. Concretely, we have

$${}^p\psi_{f,\lambda} P = \text{Ker}(T - \lambda \cdot \text{Id})^m, \quad \text{for } m \gg 0.$$

The *unipotent* nearby cycles are ${}^p\psi_{f,1} P = \text{Ker}(\text{Id} - T)^m$, with $m \gg 0$.

Consider now the nilpotent operator

$$N = \frac{1}{2\pi i} \cdot \log T_u,$$

where T_u is the unipotent part of the monodromy T . (Recall from linear algebra that there is always a decomposition $T = T_u \cdot T_s = T_s \cdot T_u$, where T_u is unipotent and T_s is semisimple.) It turns out that we have

$$N = \text{var} \circ \text{can}: {}^p\psi_{f,1} P \rightarrow {}^p\psi_{f,1} P(-1),$$

which is a nilpotent operator (up to Tate twist), and therefore gives rise to a monodromy weight filtration $W_\bullet {}^p\psi_{f,1} P$ as above.

Going back to our main object of study, filtered \mathcal{D} -modules (\mathcal{M}, F) with V -filtration along f , an important result is that the two notions of nearby and vanishing cycles we have seen are related by the Riemann-Hilbert correspondence. (This is of course the reason for adopting this terminology in the world of \mathcal{D} -modules.) The following is a result of Kashiwara and Malgrange, enhanced by Saito to the case of the rational V -filtration.

THEOREM 4.2.3 ([[Sa1](#), Proposition 3.4.12]). *If \mathcal{M} is holonomic, and is regular and quasi-unipotent along f , then we have*

$$\text{DR}(\text{gr}_V^\alpha \mathcal{M}_f) \simeq \begin{cases} {}^p\varphi_{f,\lambda} \text{DR}(\mathcal{M}) & \text{for } 0 \leq \alpha < 1 \\ {}^p\psi_{f,\lambda} \text{DR}(\mathcal{M}) & \text{for } 0 < \alpha \leq 1 \end{cases}$$

where $\lambda = e^{-2\pi i \alpha}$. Moreover, via this identification the operators can and var at the level of perverse sheaves are identified with ∂_t and t at the level of \mathcal{D} -modules.

It is therefore natural that for any $\alpha \in \mathbf{Q}$ we focus on the nilpotent operator

$$N = \partial_t t - \alpha: \text{gr}_V^\alpha \mathcal{M}_f \rightarrow \text{gr}_V^\alpha \mathcal{M}_f.$$

Another calculation due to Saito [[Sa1](#)] says that (\mathcal{M}, F) is regular and quasi-unipotent along f if and only if the following properties are satisfied for every $\alpha \in \mathbf{Q}$:

- The induced filtration $F_\bullet \text{gr}_V^\alpha \mathcal{M}_f$ is finite.
- There exists a monodromy weight filtration $W_\bullet \text{gr}_V^\alpha \mathcal{M}_f$ induced by the operator N , as above.
- For every integer i , the $\text{gr}^F \mathcal{D}_X$ -module $\text{gr}^F \text{gr}_i^W \text{gr}_V^\alpha \mathcal{M}_f$ is coherent.

In particular, on $\psi_{f,1}\mathcal{M} = \mathrm{gr}_V^1 \mathcal{M}_f$, the nilpotent operator $N = \mathrm{var} \circ$ can be precisely the operator $t\partial_t = \partial_t t - 1$ that we have been looking at all along. This respects the filtration F , and if $M = (\mathcal{M}, F, P)$ is a filtered \mathcal{D}_X -module with \mathbf{Q} -structure therefore we have a nilpotent endomorphism

$$N: \psi_{f,1}M \longrightarrow \psi_{f,1}M(-1),$$

in this category. If moreover (\mathcal{M}, F) is regular and quasi-unipotent along f , we have a monodromy weight filtration W on nearby cycles, and we can wonder again about the properties of the filtered \mathcal{D}_X -modules with \mathbf{Q} -structure given by the associated graded objects $\mathrm{gr}_k^W \psi_{f,1}M$. This will be crucial in the definition of pure Hodge modules, coming next.

4.3. Pure Hodge modules

We can now define the category $\mathrm{HM}(X, w)$ of pure Hodge modules of weight w on a smooth complex variety X . The definition is inductive on the dimension of the support of the Hodge module, so it makes sense to successively define the categories $\mathrm{HM}_{\leq d}(X, w)$ of pure Hodge modules on X of weight w and support of dimension at most d .

In any case, the objects of $\mathrm{HM}(X, w)$ are filtered \mathcal{D}_X -modules with \mathbf{Q} -structure $M = (\mathcal{M}, F, P)$ which are subject to certain conditions; namely:

- (1) (\mathcal{M}, F) is regular and quasi-unipotent along every locally defined $f \in \mathcal{O}_X(U)$, with $U \subseteq X$ open.
- (2) M admits a strict support decomposition in the category of filtered \mathcal{D}_X -modules with \mathbf{Q} -structure. (Thus the formula in Theorem 4.2.2 holds.)

Pausing for a second from imposing conditions, note that (2) implies that it is enough to define pure Hodge modules with strict support. Therefore it is enough to define for each irreducible closed subset $Z \subseteq X$ the category $\mathrm{HM}_Z(X, w)$ of pure Hodge modules on X of weight w and strict support Z .

Moreover, because of (1) and the discussion in the previous section, for every $f \in \mathcal{O}_X(U)$ the nearby and vanishing cycles $\psi_{f,1}M$, and $\varphi_{f,1}M$ do make sense as filtered \mathcal{D}_X -modules with \mathbf{Q} -structure, this time with support contained in $f = 0$. They will be used for the inductive definition.

Continuing with the definition, for the base case we impose the following:

- (3) If $Z = \{x\}$, where $x \in X$ is a point, embedded via $i: \{x\} \hookrightarrow X$, then

$$\mathrm{H}_{\{x\}}(X, w) = \{i_*H \mid H \text{ is a } \mathbf{Q}\text{-Hodge structure of weight } w\}.$$

- (4) For arbitrary Z of dimension d , we say that M belongs to $\mathrm{HM}_{Z, \leq d}(X, w)$ if the following is satisfied: for every function $f \in \mathcal{O}_X(U)$ which does not vanish identically on $Z \cap U$, we

require for all k that

$$\mathrm{gr}_k^W \psi_{f,1} M \in \mathrm{HM}_{\leq d-1}(X, w-1+k),^5$$

where W is the monodromy weight filtration as in the previous section. Note that Theorem 4.2.2, which can be applied as discussed above, implies then that $\mathrm{gr}_k^W \varphi_{f,1} M \in \mathrm{HM}_{\leq d-1}(X, w+k)$ as well.

This completes inductively the definition of $\mathrm{HM}(X, w)$ as $\bigcup_{d \geq 0} \mathrm{HM}_{\leq d}(X, w)$, where the morphisms are morphisms of filtered \mathcal{D}_X -modules with \mathbf{Q} -structure. One can show that these are by default strictly compatible with the filtration F , so that in particular $\mathrm{HM}(X, w)$ is an abelian category.

REMARK 4.3.1. Strictly speaking the definition above is a definition for analytic pure Hodge modules, that can be given on any complex manifold. To have a category of algebraic Hodge modules on a smooth complex variety, one can use a compactification and take advantage of GAGA-type theorems in order to check the properties above with respect to regular functions. We blur this distinction for the moment, for simplicity. Furthermore, in the algebraic setting one usually works with polarizable Hodge structures, and take into account in the definition of the notion of a *polarization* on a Hodge module. See for instance [Sa8, §2.2] for a discussion. See also the discussion around Theorem 4.4.8 below.

REMARK 4.3.2. It is worth noting that the use of nearby cycles in the inductive definition of Hodge modules is rather natural, due to an important result in Hodge theory due to W. Schmid, called the *SL₂-orbit theorem*. A consequence of this result is that the nearby cycles of a polarizable VHS on the punctured disk carry a mixed Hodge structure whose Hodge filtration is the limit Hodge filtration of the variation, and whose weight filtration is the monodromy weight filtration of N (up to a shift).

EXAMPLE 4.3.3 (Variations of Hodge structure). Going back to Example 1.2.3, a VHS is the most basic type of a pure Hodge module, namely what is considered to be the “smooth” case. To a \mathbf{Q} -VHS $\mathbf{V} = (\mathcal{V}, F^\bullet, \mathbf{V}_{\mathbf{Q}})$ of weight ℓ on X one associates the pure Hodge module $M = (\mathcal{M}, F, P)$ of weight $n + \ell$, where:

- $\mathcal{M} = \mathcal{V}$
- $F_p \mathcal{M} = F^{-p} \mathcal{V}$
- $P = \mathbf{V}_{\mathbf{Q}}[n]$.

Given the complicated inductive definition that depends on looking at all (locally defined) functions on X , it is not at all obvious that this indeed defines a pure Hodge module. This goes through several fundamental theorems in Hodge theory, as suggested in the Remark above, or alternatively (and somewhat hiding what lies behind it) follows from functorial properties of Hodge modules established later on.

This applies even to the most basic example of this construction, namely the *trivial Hodge module* on X , i.e. the object

$$\mathbf{Q}_X^H[n] := (\mathcal{O}_X, F, \mathbf{Q}_X[n])$$

⁵For instance, when $Z = X$ is a curve, then $\psi_f M$ is locally supported on a point, and this data corresponds to a mixed Hodge structure.

associated to the trivial variation of Hodge structure \mathbf{Q} on X , where the filtration F is the trivial filtration on \mathcal{O}_X defined in Example 1.2.2.

We will see other examples later on, but first we need to understand some basic properties of pure Hodge modules. To begin with, just as with holonomic \mathcal{D} -modules every pure Hodge with strict support is generically “smooth”.

PROPOSITION 4.3.4. *Let X be a smooth complex variety, and $Z \subseteq X$ an irreducible closed subset of dimension m . If M is a pure Hodge module of weight w on X , with strict support Z , then there exists a Zariski open subset $U \subseteq Z$ such that $M|_U$ is a \mathbf{Q} -VHS of weight $w - m$ on U , and M is uniquely determined by this VHS.*

PROOF. Let $M = (\mathcal{M}, F, P) \in \mathrm{HM}_Z(X, w)$. Then \mathcal{M} is holonomic, hence by Proposition 1.2.9 there exists an open set $V \subseteq X$ intersecting Z nontrivially, such that if $V = Z \cap U$, then $\mathcal{M}|_V$ is an integrable connection. Similarly, a well-known fact from the theory of perverse sheaves says that we can also take V such that $P|_V = L[m]$, where L is a local system on V (more precisely P is the intersection complex of L). Restricting the filtration to V as well gives us the data of a \mathbf{Q} -VHS; this needs a little argument, see [Sa1, Lemma 3.2.7].

On the other hand, this VHS uniquely determines M (meaning an element in $\mathrm{HM}_Z(X, w)$ which restricts to it on V). By possibly shrinking U , we can assume that its complement in X is a hypersurface. The assertion then follows from the property of (\mathcal{M}, F) of being regular and quasi-unipotent along this hypersurface, as explained in the Conclusion of §4.1. (Again, the perverse sheaf P is determined as the intersection complex of the local system underlying the VHS.) \square

One of the main results of Saito’s theory is that under the extra assumption that the VHS is *polarizable* (which is satisfied by all those of algebro-geometric origin), the converse of this statement is true.

THEOREM 4.3.5 ([Sa2]). *Every polarizable \mathbf{Q} -VHS of weight $w - m$ on a Zariski open set $V \subseteq Z$ extends uniquely to a (polarizable) pure Hodge module in $\mathrm{HM}_Z(X, w)$.*

REMARK 4.3.6. Include something on polarizable Hodge modules.

The other main result of Saito’s theory for pure Hodge modules is a theorem on the behavior of direct images of polarizable Hodge modules via projective morphisms, only a part of which is stated below.

THEOREM 4.3.7 (Stability and Decomposition Theorem, [Sa1, Théorème 5.3.1]). *Let $f: Y \rightarrow X$ be a projective morphism of smooth complex varieties, and $M \in \mathrm{HM}(Y, \ell)$ a polarizable pure Hodge module on Y with underlying filtered \mathcal{D}_Y -module (\mathcal{M}, F) . Then:*

- (1) *The filtered direct image $f_+(\mathcal{M}, F)$ is strict, and $\mathcal{H}^i f_+ \mathcal{M}$ underlies a polarizable Hodge module $M_i \in \mathrm{HM}(X, \ell + i)$.*

(2) *There is an isomorphism*

$$f_+(\mathcal{M}, F) \simeq \bigoplus_{i \in \mathbf{Z}} \mathcal{H}^i f_+(\mathcal{M}, F)[-i]$$

in the derived category of filtered \mathcal{D}_X -modules.

The second statement extends in particular (via the Riemann-Hilbert correspondence) the celebrated Decomposition Theorem of Beilinson-Bernstein-Deligne-Gabber.

A nice survey of the proof of these two important theorems is given in [Sch, §16-18].

The strictness of the filtration on the direct image in the theorem is one of the most crucial inputs for the applications of Hodge modules to algebraic geometry, often by means of the following consequences:

(i) (*Laumon's formula*, see e.g. [PS1, §2.2, Theorem 9] for a proof.) If $\mathcal{A}_X := \text{Sym } T_X \simeq \text{gr}_{\bullet}^F \mathcal{D}_X$, and \mathcal{A}_Y is defined similarly on Y , then

$$\text{gr}_{\bullet}^F f_+(\mathcal{M}, F) \simeq \mathbf{R}f_*(\text{gr}_{\bullet}^F \mathcal{M} \otimes_{\mathcal{A}_X} f^* \mathcal{A}_Y),$$

Note that this statement is for a right \mathcal{D} -module (\mathcal{M}, F) , where the formula is simpler.

(ii) (*Saito's formula*, see [Sa1, 2.3.7].) The associated graded of the filtered de Rham complex satisfies

$$R^i f_* \text{gr}_k^F \text{DR}(\mathcal{M}, F) \simeq \text{gr}_k^F \text{DR}(\mathcal{H}^i f_+(\mathcal{M}, F)).$$

We are sometimes particularly interested in the lowest non-zero graded piece of a filtered \mathcal{D} -module. For one such left \mathcal{D}_X -module (\mathcal{M}, F) define

$$(4.3.1) \quad p(\mathcal{M}) := \min \{p \mid F_p \mathcal{M} \neq 0\} \quad \text{and} \quad S(\mathcal{M}) := F_{p(\mathcal{M})} \mathcal{M}.$$

For the associated right \mathcal{D}_X -module we then have

$$p(\mathcal{N}) = p(\mathcal{M}) - n \quad \text{and} \quad S(\mathcal{N}) = S(\mathcal{M}) \otimes \omega_X.$$

It is useful to note that

$$(4.3.2) \quad S(\mathcal{N}) \simeq \text{gr}_{p(\mathcal{M})-n}^F \text{DR}(\mathcal{M}, F).$$

EXAMPLE 4.3.8 (Direct images). Let $f: X \rightarrow Y$ be a projective morphism of smooth complex varieties of dimensions n and m respectively, and let \mathbf{V} be a polarizable variation of \mathbf{Q} -Hodge structure of weight k on an open dense subset $U \subset X$, inducing a pure Hodge module M of weight $n+k$ with strict support X . Here it is convenient to use right \mathcal{D} -modules: if (\mathcal{N}, F) is filtered right \mathcal{D}_X -module underlying M , Theorem 4.3.7 gives a decomposition

$$f_+(\mathcal{N}, F) \simeq \bigoplus_i (\mathcal{N}_i, F)[-i]$$

in the derived category of filtered \mathcal{D}_Y -modules, and each (\mathcal{N}_i, F) underlies a pure Hodge module $M_i = \mathcal{H}^i f_* M$ on Y , of weight $n+k+i$. Using Saito's formula (ii) above, together with (4.3.2), we obtain the isomorphism

$$(4.3.3) \quad \mathbf{R}f_* S(\mathcal{N}) \simeq F_{p(\mathcal{N})} (f_+ \mathcal{N}) \simeq \bigoplus_i F_{p(\mathcal{N})} \mathcal{N}_i[-i]$$

in the bounded derived category of coherent sheaves on Y .

For example, when \mathbf{V} is the constant variation of Hodge structure \mathbf{Q}_X , by Example 4.3.3 we have $p(\mathcal{N}) = -n$ and $S(\mathcal{N}) = \omega_X$. This implies for all i that

$$p(\mathcal{N}_i) = m - n \quad \text{and} \quad F_{m-n}\mathcal{N}_i = R^i f_* \omega_X.$$

Consequently, formula (4.3.3) specializes to

$$\mathbf{R}f_* \omega_X \simeq \bigoplus_i R^i f_* \omega_X[-i],$$

which is the well-known Kollár decomposition theorem [Ko5].⁶ In fact $R^i f_* S(\mathcal{N})$ satisfy other important properties known from [Ko1] in the case of canonical bundles; for instance, answering a conjecture of Kollár, Saito proved the following:

THEOREM 4.3.9 (Saito, [Saito-Kollar]). *Let $f: X \rightarrow Y$ be a surjective projective morphism, with X smooth, and let (\mathcal{M}, F) be the filtered \mathcal{D} -module underlying a pure Hodge module with strict support X that is generically a polarized variation of Hodge structure \mathbf{V} . For each $i \geq 0$, one has*

$$R^i f_* S(\mathcal{M}) = S(Y, \mathbf{V}^i),$$

the lowest Hodge piece of the variation of Hodge structure \mathbf{V}^i on the intersection cohomology of \mathbf{V} along the fibers of f . Consequently, $R^i f_ S(\mathcal{M})$ are torsion-free, and in particular*

$$R^i f_* S(\mathcal{M}) = 0 \quad \text{for } i > \dim X - \dim Y.$$

REMARK 4.3.10. WILL EXPAND THIS.

4.4. Mixed Hodge modules

In this section we will record a few facts from the theory of mixed Hodge modules, to at least be able to place some of the objects we've been looking at, like the localization $\mathcal{O}_X(*D)$ along a hypersurface, in the proper context. The relevant reference is [Sa2], while [Sch, §20-22] provides a quick overview emphasizing useful details that I will skip here.

Again, the initial definition is an analytic one, that works on an arbitrary complex manifold X .

DEFINITION 4.4.1. The category $\text{MHW}(X)$ of *weakly mixed Hodge modules* consists of objects (M, W_\bullet) , where $M = (\mathcal{M}, F, P)$ is a filtered \mathcal{D}_X -module with \mathbf{Q} -structure, and W_\bullet is a finite increasing filtration on M by filtered \mathcal{D}_X -modules with \mathbf{Q} -structure, compatible with the isomorphism $\text{DR}(\mathcal{M}) \simeq P \otimes_{\mathbf{Q}} \mathbf{C}$, such that for all $\ell \in \mathbf{Z}$ we have

$$\text{gr}_\ell^W M \in \text{HM}(X, \ell).$$

The morphisms in this category are morphisms of filtered \mathcal{D} -modules with \mathbf{Q} -structure that are compatible with this extra data; it turns out that they are strict with respect

⁶To recover the full statement of Kollár's theorem note that we should allow Y to be singular, and we can indeed do this. One can work with Hodge modules on singular varieties using local embeddings into smooth varieties and Kashiwara's theorem; for details see [Sa2, §2.1] or [Schne12, §7].

to both the filtration F_\bullet and the filtration W_\bullet . A weak mixed Hodge module M is called *graded-polarizable* if each $\mathrm{gr}_\ell^W M$ is polarizable. These form a subcategory of $\mathrm{MHW}(X)$ denoted $\mathrm{MHW}^p(X)$.

REMARK 4.4.2. It is not hard to see that the category $\mathrm{MHW}(X)$ is abelian; see [Sa2, Proposition 5.1.14].

Mixed Hodge modules form a subcategory of $\mathrm{MHW}(X)$. The reason for imposing further restrictions is an issue that already appears in the theory of mixed Hodge structures, which can roughly be expressed as the fact that variations of Hodge structure can have bad singularities at the boundary, that do not allow for extensions in the category of regular \mathcal{D} -modules.

DEFINITION 4.4.3. Let $(M, W_\bullet) \in \mathrm{MHW}(X)$, and let $f: X \rightarrow \mathbf{C}$ be a non-constant holomorphic function on X . We say that (M, W_\bullet) is *admissible* along f if the following hold:

- (1) (M, F) is regular and quasi-unipotent along f .
- (2) The three filtrations $F_\bullet \mathcal{M}_f$, $V_\bullet \mathcal{M}_f$, and $W_\bullet \mathcal{M}_f$ are compatible, i.e. the order does not matter when we compute the associated graded.
- (3) Consider the naive limit filtrations

$$L_i(\psi_{f,1}M) = \psi_{f,1}(W_{i+1}M) \quad \text{and} \quad L_i(\varphi_{f,1}M) = \varphi_{f,1}(W_iM),$$

preserved by the nilpotent endomorphism $N = (2\pi i)^{-1} \log T_u$. Then, the relative monodromy filtrations

$$W_\bullet(\psi_{f,1}M) = W_\bullet(N, L_\bullet(\psi_{f,1}M)) \quad \text{and} \quad W_\bullet(\varphi_{f,1}M) = W_\bullet(N, L_\bullet(\varphi_{f,1}M))$$

for the action of N exist; see [Sa2, 1.1.3-4].

REMARK 4.4.4. It is not hard to see that condition (1) above holds in fact automatically, since it holds for each of the pure Hodge modules $\mathrm{gr}_\ell^W M$.

With this preparation, one can define mixed Hodge modules; the definition is again inductive on dimension.

DEFINITION 4.4.5. A weak mixed Hodge module $(M, W_\bullet) \in \mathrm{MHW}(X)$ is a *mixed Hodge module* if for every locally defined holomorphic function $f: U \rightarrow \mathbf{C}$ we have:

- (1) The pair (M, W_\bullet) is admissible along f .
- (2) Both $(\psi_{f,1}M, W_\bullet)$ and $(\varphi_{f,1}M, W_\bullet)$ are mixed Hodge modules, whenever $f^{-1}(0)$ does not contain any irreducible components of $U \cap \mathrm{Supp}(M)$.

We denote by $\mathrm{MHM}(X)$ and $\mathrm{MHM}^p(X)$ the full subcategories of $\mathrm{MHW}(X)$ and $\mathrm{MHW}^p(X)$ the full subcategories of mixed Hodge modules and graded-polarizable mixed Hodge modules, respectively. Morphisms are given by morphisms of filtered \mathcal{D}_X -modules with \mathbf{Q} -structure that are compatible with W_\bullet .

The following theorem summarizes some important properties and functors associated with mixed Hodge modules.

THEOREM 4.4.6. [Sa2] (1) *The categories $\mathrm{MHM}(X)$ and $\mathrm{MHM}^p(X)$ are abelian.*

(2) *The categories $\mathrm{MHM}(X)$ and $\mathrm{MHM}^p(X)$ are stable under applying the nearby and vanishing cycles functors.*

(3) *If $f: X \rightarrow Y$ is a projective morphism, then there exist direct image functors*

$$\mathcal{H}^i f_* M : \mathrm{MHM}^p(X) \rightarrow \mathrm{MHM}^p(Y).$$

(4) *If $f: X \rightarrow Y$ is an arbitrary morphism, then there exist inverse image functors*

$$\mathcal{H}^i f^* M, \mathcal{H}^i f^! M : \mathrm{MHM}^p(Y) \rightarrow \mathrm{MHM}^p(X).$$

There is also an important analogue of Theorem 4.3.5 :

THEOREM 4.4.7 ([Sa2, Theorem 3.27]). *Let Z be an irreducible close subset of X . Then a graded-polarizable variation of mixed Hodge structure on an open subset of Z can be extended to an object in $\mathrm{MHM}(X)$ supported on Z if and only if it is admissible relative to Z .*

Regarding the two theorems above, note that for an open embedding $j: U \hookrightarrow X$ we cannot hope to have a general functor j_* on $\mathrm{MHM}^p(U)$, as the admissibility condition does not necessarily hold.

Our main interest however is to work on a smooth algebraic variety X , and consider subcategory $\mathrm{MHM}_{\mathrm{alg}}(X) \subseteq \mathrm{MHM}^p(X^{\mathrm{an}})$ of *algebraic* mixed Hodge modules. (Note that these are automatically polarizable.) These were originally defined in [Sa2] using a compactification of X , and taking advantage of GAGA-type theorems. However in [Sa7] Saito found an alternative intrinsic definition which can be summarized as follows:

THEOREM 4.4.8. *A weakly mixed Hodge module (M, W_\bullet) belongs to the category $\mathrm{MHM}_{\mathrm{alg}}(X)$ if and only if X can be covered by Zariski open subsets U such that:*

- (1) *There exists $f \in \mathcal{O}_X(U)$ such that $U \setminus f^{-1}(0)$ is smooth and dense in U .*
- (2) *The restriction of (M, W_\bullet) to the open subset $U^{\mathrm{an}} \setminus f^{-1}(0)$ is a graded-polarizable admissible variation of mixed Hodge structure.*
- (3) *The pair (M, W_\bullet) is admissible along f .*
- (4) *The pair $(\psi_f M, W_\bullet(\psi_f M))$ belongs to $\mathrm{MHM}_{\mathrm{alg}}(f^{-1}(0))$.*

Unlike the story in the analytic setting, for an open embedding $j: U \hookrightarrow X$ of smooth algebraic varieties the functors j_* and $j!$ are defined on $\mathrm{MHM}_{\mathrm{alg}}(U)$, and this allows one to define functors f_* and $f!$ for an arbitrary morphism of algebraic varieties.

EXAMPLE 4.4.9 (Localization). TO INCLUDE.

For practical purposes, besides the existence of these functors, what we need to know concretely at this stage is the construction of the direct image via the open embedding from the complement of an SNC divisor. The full construction of direct image via an open embedding combines this with resolution of singularities; see [Sa2, Theorem 3.27].

Extension of a VHS across an SNC divisor. Let D be an SNC divisor in a smooth variety X , and denote $j: U = X \setminus D \hookrightarrow X$. We consider a polarizable VHS

$$\mathbf{V} = (\mathcal{V}, F_\bullet, \mathbf{V}_{\mathbf{Q}})$$

over U , with quasi-unipotent local monodromies along the components D_i of D . In particular the eigenvalues of all residues are rational numbers. We call M the associated pure Hodge module on U .

For $\alpha \in \mathbf{Z}$, we denote by $\mathcal{V}^{\geq \alpha}$ (resp. $\mathcal{V}^{> \alpha}$) the Deligne extension with eigenvalues of residues along the D_i in $[\alpha, \alpha + 1)$ (resp. $(\alpha, \alpha + 1]$).⁷ Recall that $\mathcal{V}^{\geq \alpha}$ is filtered by

$$(4.4.1) \quad F_p \mathcal{V}^{\geq \alpha} = \mathcal{V}^{\geq \alpha} \cap j_* F_p \mathcal{V},$$

while the filtration on $\mathcal{V}^{> \alpha}$ is defined similarly. The terms in the filtration are locally free by Schmid's nilpotent orbit theorem (which can be extended to the quasi-unipotent case).

The mixed Hodge module $j_* M$ on X is then defined as follows:

$$j_* M = (\mathcal{V}(*D), F_\bullet, j_* \mathbf{V}_{\mathbf{Q}}).$$

Here $\mathcal{V}(*D)$ is Deligne's meromorphic connection extending \mathcal{V} . It has a lattice defined by $\mathcal{V}^{\geq \alpha}$ for any $\alpha \in \mathbf{Q}$, i.e. $\mathcal{V}(*D) = \mathcal{V}^{\geq \alpha} \otimes \mathcal{O}_X(*D)$, and its filtration is given by

$$F_p \mathcal{V}(*D) = \sum_i F_i \mathcal{D}_X \cdot F_{p-i} \mathcal{V}^{\geq -1}.$$

For the weight filtration see [Sa2, Proposition 2.8].

EXAMPLE 4.4.10 (Localization). When $M = \mathbf{Q}_U^H[n]$ is the trivial Hodge module, then $\mathcal{M} = \mathcal{O}_U$, whose Deligne canonical extension is \mathcal{O}_X . In this case

$$j_* \mathcal{O}_U = (\mathcal{O}_X(*D), F_\bullet, j_* \mathbf{Q}).$$

Thus $\mathcal{O}_X(*D)$ underlies a mixed Hodge module, and the Hodge filtration can be described as follows. First, it is well known that $\mathcal{O}_U^{\geq -1} = \mathcal{O}_X(D)$, and its filtration is

$$F_k \mathcal{O}_X(D) = \mathcal{O}_X(D) \cap j_* F_k \mathcal{O}_U = \mathcal{O}_X(D) \cap j_* \mathcal{O}_U = \mathcal{O}_X(D)$$

for all $k \geq 0$, and 0 otherwise. The formula above gives

$$F_p \mathcal{O}_X(*D) = F_p \mathcal{D}_X \cdot \mathcal{O}_X(D)$$

for $p \geq 0$, and 0 otherwise.

Since we have introduced this terminology, it is worth noting that the pure Hodge module extension of M with strict support X (see Theorem 4.3.5) is given by the minimal extension functor, which we looked at briefly earlier in this course. More precisely, as shown in [Sa2, §3.b], it is

$$(\mathcal{D}_X \cdot \mathcal{V}^{> -1}, F_\bullet, j_{!*} \mathbf{V}_{\mathbf{Q}}),$$

where the filtration is defined as

$$F_p(\mathcal{D}_X \cdot \mathcal{V}^{> -1}) = \sum_i F_i \mathcal{D}_X \cdot F_{p-i} \mathcal{V}^{> -1}.$$

⁷Unfortunately we haven't discussed yet this important construction in this course; see for instance [HTT, Theorem 5.2.17].

4.5. Saito's vanishing theorem

Mixed Hodge modules satisfy a vast generalization of Kodaira Vanishing, shown in [Sa2, §2.g]. Together with strictness, this is one of the most important tools towards applications to algebraic geometry.

THEOREM 4.5.1 (Kodaira-Saito Vanishing). *Let X be a complex projective variety, and L an ample line bundle on X . Consider an integer $m > 0$ such that $L^{\otimes m}$ is very ample and gives an embedding $X \subseteq \mathbf{P}^N$. Let (\mathcal{M}, F) be the filtered \mathcal{D} -module underlying a mixed Hodge module M on \mathbf{P}^N with support contained in X . Then:*

(1) $\mathrm{gr}_k^F \mathrm{DR}_{\mathbf{P}^N}(\mathcal{M})$ is an object in $\mathbf{D}^b(X)$ for each k , independent of the embedding of X in \mathbf{P}^N .⁸

(2) We have the hypercohomology vanishing

$$\mathbf{H}^i(X, \mathrm{gr}_k^F \mathrm{DR}_{\mathbf{P}^N}(\mathcal{M}) \otimes L) = 0 \text{ for all } i > 0.$$

and

$$\mathbf{H}^i(X, \mathrm{gr}_k^F \mathrm{DR}_{\mathbf{P}^N}(\mathcal{M}) \otimes L^{-1}) = 0 \text{ for all } i < 0.$$

Before giving the proof, let's see how Saito's vanishing theorem can be used to deduce many of the standard vanishing theorems.

Kodaira-Nakano vanishing. Let X be a smooth projective complex variety of dimension n . We consider the trivial Hodge module $M = \mathbf{Q}_X^H[n]$. According to Example 4.3.3, the corresponding right \mathcal{D} -module is ω_X , with filtration $F_p \omega_X = \omega_X$ if $p \geq -n$ and $F_p \omega_X = 0$ if $p < -n$, so that

$$\mathrm{gr}_{-p}^F \mathrm{DR}_X(\omega_X) = \Omega_X^p[n-p] \text{ for all } p.$$

Theorem 4.5.1 gives

$$H^q(X, \Omega_X^p \otimes L) = 0 \text{ for } p+q > n$$

and the dual statement, for any L ample, i.e. Kodaira-Nakano vanishing.

If we restrict to the Kodaira vanishing theorem, which corresponds to the lowest non-zero piece of the filtration on ω_X , then we can see it as an example of the following more easily stated special case of Theorem 4.5.1; it is useful to record this for applications.

COROLLARY 4.5.2. *If (\mathcal{M}, F) is a filtered \mathcal{D} -module underlying a mixed Hodge module M on a projective variety X , and L is an ample line bundle on X , then*

$$H^i(X, S(\mathcal{M}) \otimes L) = 0 \text{ for all } i > 0.$$

Kollár vanishing. The following theorem of Kollár is a natural generalization of Kodaira vanishing to higher direct images of canonical bundles.

⁸In fact it can be shown that each $\mathrm{gr}_k^F \mathrm{DR}_{\mathbf{P}^N}(\mathcal{M})$ is independent of the embedding of X into any smooth complex variety.

THEOREM 4.5.3 ([[Koi](#), Theorem 2.1(iii)]). *Let $f : X \rightarrow Y$ be a morphism between complex projective varieties, with X smooth, and let L be an ample line bundle on Y . Then*

$$H^i(X, R^j f_* \omega_X \otimes L) = 0 \quad \text{for all } i > 0 \text{ and all } j.$$

To deduce the statement from [Theorem 4.5.1](#), we consider the push-forward $M = f_* \mathbf{Q}_X^H[n]$ of the trivial Hodge module on X , with $n = \dim X$. According to [Example 4.3.8](#), for the underlying \mathcal{D} -modules we have

$$f_+(\omega_X, F) \simeq \bigoplus_i (\mathcal{N}_i, F)[-i]$$

in the derived category of filtered \mathcal{D}_Y -modules (so compatible with inclusions into smooth varieties), and for each i we have $S(\mathcal{N}_i) = R^i f_* \omega_X$. [Theorem 4.5.3](#) then follows from [Corollary 4.5.2](#). More generally, the same argument shows the following vanishing theorem due to Saito: the statement of [Theorem 4.5.3](#) holds for $R^i f_* S(\mathcal{M})$, where \mathcal{M} corresponds to the unique pure Hodge module with strict support X extending a polarized variation of Hodge structure on an open set $U \subseteq X$.

Nadel vanishing. Let X be a smooth projective variety, and D an effective Cartier divisor on X . Recall that $\mathcal{O}_X(*D)$ is equipped with a natural Hodge filtration F , as the left \mathcal{D} -module associated to the mixed Hodge module $j_* \mathbf{Q}_U^H[n]$, where $j : U = X \setminus D \hookrightarrow X$ is the inclusion; see [Example 4.4.9](#) and the whole of the next chapter. Looking at the first step in this filtration, one can recognize multiplier ideals from the formula

$$(4.5.1) \quad F_0 \mathcal{O}_X(*D) \simeq \mathcal{O}_X(D) \otimes \mathcal{J}((1 - \varepsilon)D),$$

where $0 < \varepsilon \ll 1$ is a rational number, and in general $\mathcal{J}(B)$ stands for the multiplier ideal of a \mathbf{Q} -divisor B ; see [Proposition 6.1.18](#) below.

This allows us to deduce the following special case of the Nadel vanishing theorem (see e.g. [[La](#), Theorem 9.4.8]).

THEOREM 4.5.4. *With the notation above, if L is a line bundle on X such that $L - D$ is ample, then*

$$H^i(X, \omega_X \otimes L \otimes \mathcal{J}((1 - \varepsilon)D)) = 0 \quad \text{for all } i > 0.$$

Indeed note that since $F_0 \mathcal{O}_X(*D)$ is the lowest nonzero term in the Hodge filtration, for the associated right \mathcal{D} -module $\mathcal{M} = \omega_X(*D)$ we have

$$S(\mathcal{M}) = \omega_X \otimes \mathcal{O}_X(D) \otimes \mathcal{J}((1 - \varepsilon)D),$$

to which we can apply [Corollary 4.5.2](#).

REMARK 4.5.5. The general form of Nadel vanishing is as follows: if B is an effective \mathbf{Q} -divisor on X and L is a line bundle such that $L - B$ is ample, then

$$H^i(X, \omega_X \otimes L \otimes \mathcal{J}(B)) = 0 \quad \text{for all } i > 0.$$

This can also be seen as a consequence of Saito Vanishing, but requires working with some twisted versions of the localization \mathcal{D} -module $\mathcal{O}_X(*D)$; see [[MP3](#)].

Proof of Saito's vanishing theorem. TO BE ADDED.

Vanishing on abelian varieties. On abelian varieties, following [PS1, §2.3] Saito's vanishing theorem can be upgraded to the associated graded pieces of the \mathcal{D} -module itself. We denote by (\mathcal{M}, F) be the filtered \mathcal{D} -module underlying a mixed Hodge module on an abelian variety A of dimension g .

PROPOSITION 4.5.6. *Let (\mathcal{M}, F) be the filtered \mathcal{D} -module underlying a mixed Hodge module on an abelian variety A , and let L be an ample line bundle. Then for each $k \in \mathbf{Z}$, we have*

$$H^i(A, \mathrm{gr}_k^F \mathcal{M} \otimes L) = 0 \quad \text{for all } i > 0.$$

PROOF. Recall that for each $k \in \mathbf{Z}$ we have

$$\mathrm{gr}_{k-g}^F \mathrm{DR}(\mathcal{M}) = \left[\mathrm{gr}_{k-g}^F \mathcal{M} \rightarrow \Omega_A^1 \otimes \mathrm{gr}_{k-g+1}^F \mathcal{M} \rightarrow \cdots \rightarrow \Omega_A^g \otimes \mathrm{gr}_k^F \mathcal{M} \right],$$

supported in degrees $-g, \dots, 0$. According to Theorem 4.5.1, this complex has the property that, for $i > 0$,

$$\mathbf{H}^i(A, \mathrm{gr}_{k-g}^F \mathrm{DR}(\mathcal{M}) \otimes L) = 0.$$

Using the fact that $\Omega_A^p \simeq \oplus \mathcal{O}_A$ for each p , one can deduce the asserted vanishing theorem for the individual sheaves $\mathrm{gr}_k^F \mathcal{M}$ by induction on k . Indeed, we have $\mathrm{gr}_k^F \mathcal{M} = 0$ for $k \ll 0$, while for each k there is a distinguished triangle

$$E_k \rightarrow \mathrm{gr}_k^F \mathcal{M} \rightarrow \mathrm{gr}_{k-g}^F \mathrm{DR}(\mathcal{M}) \rightarrow E_k[1],$$

with E_k an object supported in degrees $-g+1, \dots, 0$, and hence satisfying $\mathbf{H}^i(A, E_k \otimes L) = 0$ because each of its individual entries does so by induction. \square

This statement also leads to a generic vanishing theorem for the terms $\mathrm{gr}_k^F \mathcal{M}$; see [PS1]. For the moment though, let's record some interesting consequences noted by C. Schnell and C. Wei [Wei], useful for applications as we will see later.

LEMMA 4.5.7. *Let \mathcal{G} be the coherent sheaf on $T^*A = A \times V$ corresponding to the associated graded object $\mathcal{G}_\bullet = \mathrm{gr}_\bullet^F \mathcal{M}$, where $V = H^0(A, \Omega_A^1)$. Denote by p_1 and p_2 the two projections of $A \times V$ onto A and V respectively. Then for every ample line bundle L on A , the object*

$$\mathbf{R}p_{2*}(\mathcal{G} \otimes p_1^*L)$$

is in fact a sheaf (supported in degree 0).

PROOF. Since the projection p_1 is an affine morphism, we have

$$H^i(A \times V, \mathcal{G} \otimes p_1^*L) \simeq H^i(A, p_{1*}\mathcal{G} \otimes L) = 0 \quad \text{for } i > 0,$$

where the vanishing on the right follows from Proposition 4.5.6. (Note that by definition we have $\mathcal{G}_\bullet = p_{1*}\mathcal{G}$.)

On the other hand, since V is affine, we have

$$H^0(V, R^i p_{2*}(\mathcal{G} \otimes p_1^*L)) \simeq H^i(A \times V, \mathcal{G} \otimes p_1^*L),$$

and so this is 0 as well for $i > 0$, or equivalently $R^i p_{2*}(\mathcal{G} \otimes p_1^*L) = 0$. \square

COROLLARY 4.5.8. *With the same notation as in Lemma 4.5.7, we have that*

$$p_{2*}(\mathcal{G} \otimes p_1^*L^{-1})$$

is a torsion-free coherent sheaf on V .

PROOF. Coherence is clear, since p_2 is a proper morphism. We now apply Grothendieck duality in the following form (note that the relative dualizing complex of the morphism p_2 is $\mathcal{O}_{A \times V}[g]$, as the canonical bundles of A and V are both trivial):

$$\mathbf{R}\mathcal{H}om(\mathbf{R}p_{2*}(\mathcal{G} \otimes p_1^*L^{-1}), \mathcal{O}_V) \simeq \mathbf{R}p_{2*}\mathbf{R}\mathcal{H}om(\mathcal{G} \otimes p_1^*L^{-1}, \mathcal{O}_{A \times V}[g]).$$

Let's denote for simplicity

$$\mathcal{G}' := \mathbf{R}\mathcal{H}om(\mathcal{G}, \mathcal{O}_{A \times V}[g]),$$

so that the isomorphism above becomes

$$\mathbf{R}\mathcal{H}om(\mathbf{R}p_{2*}(\mathcal{G} \otimes p_1^*L^{-1}), \mathcal{O}_V) \simeq \mathbf{R}p_{2*}(\mathcal{G}' \otimes p_1^*L).$$

Now by general duality theory for mixed Hodge modules⁹ we know that

$$\mathcal{G}' \simeq \mathrm{gr}^F \mathcal{M}'$$

for another Hodge module (\mathcal{M}', F) (the dual of (\mathcal{M}, F)). Lemma 4.5.7 then applies to show that the right hand side of this last isomorphism is a coherent sheaf \mathcal{H} . Applying $\mathbf{R}\mathcal{H}om(\cdot, \mathcal{O}_V)$ to both sides, we then obtain

$$\mathbf{R}p_{2*}(\mathcal{G} \otimes p_1^*L^{-1}) \simeq \mathbf{R}\mathcal{H}om(\mathcal{H}, \mathcal{O}_V).$$

In particular $p_{2*}(\mathcal{G} \otimes p_1^*L^{-1}) \simeq \mathcal{H}^\vee$, and the dual of any coherent sheaf is torsion-free. \square

⁹To be discussed; see e.g. [Sch, Scn.29] for a brief overview.

CHAPTER 5

Applications to holomorphic forms and families of varieties

This chapter focuses on applications of Hodge modules to problems in complex geometry regarding the behavior of zero loci of holomorphic 1-forms on projective manifolds, or the properties of parameter spaces for families of varieties. The main input from the rest of the notes is the vanishing and positivity package for the de Rham complexes associated to pure Hodge modules.

5.1. Singular loci of morphisms

Let $f: X \rightarrow Y$ be a surjective morphism of smooth projective varieties. A standard measure of the behavior of f is its singular locus $D_f \subset Y$, i.e. the closed subset over which f fails to be smooth. Here we discuss a more refined measure of the singularities of f , namely the locus of singular cotangent vectors, and its relationship with \mathcal{D} -module theory. This circle of ideas will be very useful when analyzing the geometric problems treated in the next sections.

We use the following notation for the induced morphisms between the cotangent bundles:

$$(5.1.1) \quad \begin{array}{ccc} X & \xleftarrow{p_1} & X \times_Y T^*Y & \xrightarrow{df} & T^*X \\ \downarrow f & & \downarrow p_2 & & \\ Y & \xleftarrow{p} & T^*Y & & \end{array}$$

where df is defined by

$$df(x, (y, \xi)) = (x, f^*\xi), \quad \text{with } f(x) = y.$$

DEFINITION 5.1.1. The set of *singular cotangent vectors* of f is

$$S_f = p_2(df^{-1}(0)) \subseteq T^*Y.$$

A cotangent vector $(y, \xi) \in T^*Y$ belongs to S_f if and only if $f^*\xi$ vanishes at some point $x \in f^{-1}(y)$, or equivalently, if ξ annihilates the image of T_xX inside the tangent space T_yY . If this happens and $\xi \neq 0$, then f is not submersive at x , which means that y belongs to the singular locus $D_f \subseteq Y$. Consequently, S_f is the union of the zero-section T_Y^*Y and a closed conical subset of T^*Y whose image via p is equal to D_f , the singular locus of f .

In what follows we will use the following estimate for the behavior of the characteristic variety of a coherent \mathcal{D} -module under push-forward, due to Kashiwara; see e.g. [Ka2, Theorem 4.27].

THEOREM 5.1.2 (Kashiwara's estimate). *If $f: X \rightarrow Y$ is a proper morphism of smooth varieties, if \mathcal{M} is a coherent \mathcal{D}_X -module, then using the notation above we have*

$$\mathrm{Ch}(f_+\mathcal{M}) \subseteq p_2(df^{-1}(\mathrm{Ch}(\mathcal{M}))),$$

with equality if f is a closed immersion.

We will next use the language of *right* \mathcal{D} -modules, which is slightly more convenient for the material below. Let's also denote $d = \dim X$ and $n = \dim Y$.

EXAMPLE 5.1.3. For the trivial \mathcal{D}_X -module ω_X we have $\mathrm{Ch}(\omega_X) = T_X^*X$. Using the discussion above, it follows that for every j we have

$$\mathrm{Ch}(f_+\omega_X) = \bigcup_j \mathrm{Ch}(\mathcal{H}^j f_+\omega_X) \subseteq S_f.$$

Recall now that the complex of induced right \mathcal{D} -modules

$$0 \rightarrow \mathcal{D}_X \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \cdots \rightarrow \Omega_X^{d-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow 0,$$

placed in degrees $-d, \dots, 0$, is filtered quasi-isomorphic to ω_X with the trivial filtration; see [HTT, Lemma 1.5.27].¹ If we denote

$$\mathcal{A}_X := \mathrm{Sym} T_X \simeq \mathrm{gr}_{\bullet}^F \mathcal{D}_X,$$

passing to the associated graded in the complex above, and keeping track of the grading, provides a resolution of $\mathrm{gr}_{\bullet}^F \omega_X$ (which is nontrivial and isomorphic to ω_X only in degree $-d$), by the following complex of graded free \mathcal{A}_X -modules:

$$B_{X,\bullet}: \quad \mathcal{A}_{X,\bullet} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{A}_{X,\bullet+1} \rightarrow \cdots \rightarrow \Omega_X^{d-1} \otimes_{\mathcal{O}_X} \mathcal{A}_{X,\bullet+d-1} \rightarrow \omega_X \otimes_{\mathcal{O}_X} \mathcal{A}_{X,\bullet+d}.$$

Using the morphism f , and tensoring $B_{X,\bullet}$ over \mathcal{A}_X with \mathcal{A}_Y (via the module structure induced by the canonical morphism $T_X \rightarrow f^*T_Y$), we obtain a new complex of graded free $f^*\mathcal{A}_Y$ -modules:

$$C_{X,\bullet}: \quad f^*\mathcal{A}_{Y,\bullet} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} f^*\mathcal{A}_{Y,\bullet+1} \rightarrow \cdots \rightarrow \Omega_X^{d-1} \otimes_{\mathcal{O}_X} f^*\mathcal{A}_{Y,\bullet+d-1} \rightarrow \omega_X \otimes_{\mathcal{O}_X} f^*\mathcal{A}_{Y,\bullet+d}.$$

Therefore we have a quasi-isomorphism

$$(5.1.2) \quad C_{X,\bullet} \simeq \mathrm{gr}_{\bullet}^F \omega_X \otimes_{\mathcal{A}_X} f^*\mathcal{A}_Y.$$

Forgetting about the grading, we can think of C_X as being a complex of coherent sheaves on $X \times_Y T^*Y$, and we have the following description of its support:

LEMMA 5.1.4. *We have $\mathrm{Supp}(C_X) = df^{-1}(0)$.*

PROOF. More precisely, we have by definition that $C_X \simeq df^*B_X$, where analogously once we forget the grading we can think of B_X as a complex of sheaves on T^*X . On the other hand, by construction B_X is the Koszul complex of free modules on T^*X resolving the zero-section T_X^*X ; therefore $\mathrm{Supp}(B_X) = T_X^*X$, which implies the statement of the lemma. \square

¹We have seen the left \mathcal{D} -module version, the Spencer resolution of \mathcal{O}_X , in §1.4.

To establish a connection with the locus of singular contangent vectors S_f , at this stage it is useful to recall from §4.3 Laumon’s formula for the associated graded object of the direct image of a filtered \mathcal{D} -module underlying a mixed Hodge module; if (\mathcal{M}, F) is a right such filtered \mathcal{D} -module, then

$$\mathrm{gr}_{\bullet}^F f_+(\mathcal{M}, F) \simeq \mathbf{R}f_*(\mathrm{gr}_{\bullet}^F \mathcal{M} \otimes_{\mathcal{A}_X} f^* \mathcal{A}_Y)$$

for each i . Putting everything together for $\mathcal{M} = \omega_X$, we conclude:

COROLLARY 5.1.5. *In the derived category of graded \mathcal{A}_X -modules, we have an isomorphism*

$$\mathrm{gr}_{\bullet}^F f_+(\omega_X, F) \simeq \mathbf{R}f_* C_{X, \bullet}.$$

Moreover, from either Example 5.1.3 or Lemma 5.1.4, we deduce:

COROLLARY 5.1.6. *We have an inclusion*

$$\mathrm{Supp}(\mathbf{R}f_* C_{X, \bullet}) \subseteq S_f.$$

EXAMPLE 5.1.7. Let’s spell out a couple of special instances of the isomorphism in Corollary 5.1.5. We have $C_{X, k} = 0$ for $k < -d$, $C_{X, -d} = \omega_X$, while

$$C_{X, -d+1} = [\Omega_X^{d-1} \rightarrow \omega_X \otimes f^* T_Y] \simeq \omega_X \otimes [T_X \rightarrow f^* T_Y],$$

a complex supported in degrees -1 and 0 . Consequently, $\mathrm{gr}_k^F f_+(\omega_X, F) = 0$ for $k < -d$,

$$\mathrm{gr}_{-d}^F f_+(\omega_X, F) \simeq \mathbf{R}f_* \omega_X,$$

the lowest potentially non-zero graded piece (see also ??), while

$$\mathrm{gr}_{-d+1}^F f_+(\omega_X, F) \simeq \mathbf{R}f_*(\omega_X \otimes [T_X \rightarrow f^* T_Y]).$$

5.2. Zeros of holomorphic 1-forms

The main result, and some special cases. One of the most geometric applications of the theory to date is the following theorem on the zero of holomorphic 1-forms proved by the author and Ch. Schnell [PS2] as the solution to a conjecture by Hacon-Kovács [HK] about varieties of general type and, in the general form stated here, by Luo-Zhang [LZ].

THEOREM 5.2.1. *Let X be a smooth complex projective variety, and let $W \subseteq H^0(X, \Omega_X^1)$ be a linear subspace such that $Z(\omega)$ is empty for every nonzero one-form $\omega \in W$. Then*

$$\dim W \leq \dim X - \kappa(X).$$

The Hacon-Kovács conjecture mentioned above is the case when $\dim X = \kappa(X)$, i.e.

COROLLARY 5.2.2. *If X is a smooth complex projective variety of general type, then the zero locus of every global holomorphic one-form $\omega \in H^0(X, \Omega_X^1)$ is nonempty.*

EXAMPLE 5.2.3. If $X = C$ is a smooth projective curve of genus $g \geq 2$, then $\deg \omega_C = 2g - 2 > 0$, hence every non-trivial 1-form has precisely $2g - 2$ zeros, counted with multiplicities.

EXAMPLE 5.2.4. The result can be obtained quite quickly in dimension 2 as well. Let $X = S$ be a smooth projective surface of general type. It is not hard to see that in order to have a nowhere vanishing 1-form, S has to be minimal; indeed, otherwise it would be the blow-up of another smooth surface at a point, and any 1-form would have to vanish somewhere along the exceptional locus. Now minimal surfaces of general type satisfy the Bogomolov-Miyaoka-Yau inequality, which says that

$$3c_2(S) \geq c_1(S)^2 > 0.$$

In particular, $\chi_{\text{top}}(S) = c_2(S) > 0$. On the other hand, the general Lemma below says that if any smooth projective variety X has a nowhere vanishing global 1-form, then $\chi_{\text{top}}(X) = 0$.

LEMMA 5.2.5. *Let X be a smooth projective variety (or even compact Kähler manifold) such that there exists a nowhere vanishing $\omega \in H^0(X, \Omega_X^1)$. Then $\chi_{\text{top}}(X) = 0$.*

PROOF. The fact that ω is a nowhere vanishing section of Ω_X^1 implies that the corresponding Koszul complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\wedge \omega} \Omega_X^1 \xrightarrow{\wedge \omega} \Omega_X^2 \xrightarrow{\wedge \omega} \cdots \xrightarrow{\wedge \omega} \omega_X \longrightarrow 0$$

is exact. Therefore we have

$$0 = \sum_{i=0}^n (-1)^i \chi(\Omega_X^i) = \sum_{p,q} (-1)^{p+q} h^q(X, \Omega_X^p) = \sum_{j=0}^{2n} (-1)^j h^j(X, \mathbf{C}) = \chi_{\text{top}}(X),$$

where $n = \dim X$, and for the second and third equality we rearranged the terms involving the individual cohomologies of the bundles of holomorphic forms, and used the Hodge decomposition. \square

EXAMPLE 5.2.6. It turns out that, assuming the existence of a nowhere vanishing 1-form, much more is true: in fact $\chi(\Omega_X^i) = 0$ for all i . Even better, for general $\alpha \in \text{Pic}^0(X)$ we have

$$H^j(X, \Omega_X^i \otimes \alpha) = 0 \quad \text{for all } i, j.$$

(The first statement follows since the Euler characteristic is a deformation invariant.) This is a special case of a generic vanishing theorem due to Green-Lazarsfeld, see [GL1, Theorem 3.1]. This observation leads to an alternative way of deducing the result for a surface S : it ensures that in the presence of a nowhere vanishing 1-form we have $\chi(\mathcal{O}_S) = 0$, while on the other hand Castelnuovo's theorem (see [Be, Theorem X.4]) says that if S is of general type, then $\chi(\mathcal{O}_S) > 0$.

EXAMPLE 5.2.7. The proof in the case ω_X is ample.

Applications to families of varieties over abelian varieties. We extract some hyperbolicity-type consequence of the main result, following [PS2]. First, an immediate consequence of Theorem 5.2.1 is the following:

COROLLARY 5.2.8. *If $f: X \rightarrow A$ is a smooth morphism from a smooth complex projective variety onto an abelian variety, then*

$$\dim A \leq \dim X - \kappa(X).$$

In particular, there are no nontrivial smooth morphisms from a variety of general type to an abelian variety.

PROOF. Since f is smooth, $Z(f^*\omega)$ is empty for every nonzero $\omega \in H^1(A, \Omega_A^1)$. The inequality then follows from Theorem 5.2.1, since f^* is injective. \square

REMARK 5.2.9. The last statement in the Corollary was proved by Viehweg and Zuo [VZ1] when the base is an elliptic curve, and is implicit in Hacon and Kovács [HK] in general.

Combined with the strong subadditivity of Kodaira dimension, Corollary 5.2.8 provides an interesting application to families parametrized by abelian varieties. This should be compared to the well-known fact that there do not exist smooth non-isotrivial families of curves of genus at least 2 parametrized by elliptic curves.

COROLLARY 5.2.10. *If $f: X \rightarrow A$ is a smooth morphism onto an abelian variety, and if all fibers of f are of general type, then f is birationally isotrivial.*

PROOF. We have to show that X becomes birational to a product after a generically finite base-change; it suffices to prove that $\text{Var}(f) = 0$, in Viehweg's terminology. Since the fibers of f are of general type, we know from the main result of [Ko3] (the proof of Viehweg's $C_{n,m}^+$ conjecture when the fibers are of general type) that

$$\kappa(F) + \text{Var}(f) \leq \kappa(X).$$

On the other hand, $\kappa(X) \leq \dim F$ by Corollary 5.2.8, and so we are done. \square

REMARK 5.2.11. Earlier versions of this result go back to Migliorini [Mi] (for smooth families of minimal surfaces of general type over an elliptic curve), Kovács [Kov1] (for smooth families of minimal varieties of general type over an elliptic curve), and Viehweg-Zuo [VZ1] (for smooth families of varieties of general type over an elliptic curve). Over a base of arbitrary dimension, Kovács [Kov2] and Zhang [Zh] showed that if the fibers have ample canonical bundle, then the morphism f must actually be isotrivial.

A more general version, and the proof. We finally focus on the proof of the main result. Every holomorphic 1-form on X is the pullback of a holomorphic 1-form from the Albanese variety $\text{Alb}(X)$, so it is natural to consider more generally an arbitrary morphism from X to an abelian variety, and to ask under what conditions the pullback of a holomorphic 1-form must have a zero at some point of X . The following result in this direction implies Theorem 5.2.1.

THEOREM 5.2.12. *Let X be a smooth complex projective variety, and let $f: X \rightarrow A$ be a morphism to an abelian variety. If $H^0(X, \omega_X^{\otimes m} \otimes f^*L^{-1}) \neq 0$ for some integer $m \geq 1$ and some ample line bundle L on A , then $Z(\omega)$ is nonempty for every ω in the image of $f^*: H^0(A, \Omega_A^1) \rightarrow H^0(X, \Omega_X^1)$.*

REMARK 5.2.13. The condition in the theorem is equivalent to asking that, up to birational equivalence, the morphism f should factor through the Iitaka fibration of X . (See e.g. [Mo, Proposition 1.14].)

We first show how Theorem 5.2.12 implies the main result at the beginning of this section.

PROOF OF “THEOREM 5.2.12 \implies THEOREM 5.2.1”. The statement of Theorem 5.2.1 is vacuous when $\kappa(X) = -\infty$, and so we shall assume from now on that $\kappa(X) \geq 0$.

Let $\mu: X' \rightarrow X$ be a birational modification of X , such that $g: X' \rightarrow Z$ is a smooth model for the Iitaka fibration. The general fiber of g is then a smooth projective variety of dimension $\delta(X) = \dim X - \kappa(X)$ and Kodaira dimension zero, and therefore maps surjectively to its own Albanese variety [Kaw1, Theorem 1]. By a standard argument [Kaw1, Proof of Theorem 13], the image in $\text{Alb}(X)$ of every fiber of g must be a translate of a single abelian variety. Letting A denote the quotient of $\text{Alb}(X)$ by this abelian variety, we obtain $\dim A \geq \dim H^0(X, \Omega_X^1) - \delta(X)$, and the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \downarrow g & & \downarrow f \\ Z & \longrightarrow & A \end{array}$$

By construction, Z dominates $f(X)$, and so we conclude from Theorem 5.2.12 that all holomorphic 1-forms in the image of $f^*: H^0(A, \Omega_A^1) \rightarrow H^0(X, \Omega_X^1)$ have a nonempty zero locus. Since this subspace has codimension at most $\delta(X)$ in the space of all holomorphic 1-forms, we see that Theorem 5.2.1 holds. \square

We are left with proving the main technical result. The proof goes along the general lines of the original proof in [PS2], but with a simplification due to C. Schnell and C. Wei that appears in [Wei].²

PROOF OF THEOREM 5.2.12. Let’s first translate what we want to prove into the terminology of §5.1. Since the cotangent bundle of A is trivial, i.e. $T^*A = A \times V$, with $V = H^0(A, \Omega_A^1)$, diagram (5.1.1) becomes

$$(5.2.1) \quad \begin{array}{ccccc} X & \xleftarrow{p_1} & X \times V & \xrightarrow{df} & T^*X \\ \downarrow f & & \downarrow f \times \text{id} & & \\ A & \xleftarrow{p_1} & A \times V & \xrightarrow{p_2} & V \end{array}$$

By definition a pair $(x, \omega) \in X \times V$ belongs to $df^{-1}(0)$ if and only if $f^*\omega$ vanishes at x . It follows that the conclusion of our theorem is equivalent to the fact that the locus of singular cotangent vectors $S_f = (f \times \text{id})(df^{-1}(0)) \subseteq A \times V$ projects onto the whole of V . In other words, our final goal is to show that

$$p_2(S_f) = V.$$

To this end, we begin by performing the following reduction:

²Roughly speaking, the use of generic vanishing theory for mixed Hodge modules on abelian varieties is replaced by an application of the standard Saito vanishing theorem, in the guise of Corollary 4.5.8.

Claim: We can assume that there is an ample line bundle L and an integer $m \geq 1$ such that

$$H^0(X, (\omega_X \otimes f^*L^{-1})^{\otimes m}) \neq 0.$$

Recall that the original assumption is that

$$H^0(X, \omega_X^{\otimes m} \otimes f^*L^{-1}) \neq 0$$

for a choice of such data. Consider now the fiber product

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ \downarrow f' & & \downarrow f \\ A & \xrightarrow{\varphi_{2m}} & A \end{array}$$

where $\varphi_{2m}: A \rightarrow A$ is multiplication by $2m$. It is well known (see [?]) that

$$\varphi_{2m}^*L \simeq L'^{\otimes m}$$

for some ample line bundle L' on A . On the other hand, since ψ is étale as well, we have $\psi^*\omega_X \simeq \omega_{X'}$, and therefore we obtain the isomorphism

$$\psi^*(\omega_X^{\otimes m} \otimes f^*L^{-1}) \simeq (\omega_{X'} \otimes f'^*L'^{-1})^{\otimes m}.$$

Using the projection formula, this implies that the line bundle on the right hand side has nontrivial sections as well. Moreover, it suffices to prove the conclusion of the theorem for the morphism $f': X' \rightarrow A$, since we have the formula $Z(\psi^*\omega) = \psi^{-1}(Z(\omega))$ for any global 1-form ω on X , as ψ is étale. This concludes the proof of the Claim.

From now on we can therefore assume that there is a nontrivial section

$$0 \neq s \in H^0(X, (\omega_X \otimes f^*L^{-1})^{\otimes m}),$$

and we take m to be the smallest positive integer for which such a section exists. We now consider the cyclic cover $\pi: X_m \rightarrow X$ of degree m , branched along the divisor $D = Z(s)$; see [EV, §3], [La, §4.1.B] for details. Since m is minimal, X_m is irreducible. We also consider a resolution of singularities $\mu: Y \rightarrow X_m$, which can be taken to be an isomorphism over the complement of preimage of D . We define $\varphi = \pi \circ \mu$ and $h = f \circ \varphi$, so that we have the following commutative diagram:

$$(5.2.2) \quad \begin{array}{ccccc} & & \varphi & & \\ & & \curvearrowright & & \\ Y & \xrightarrow{\mu} & X_m & \xrightarrow{\pi} & X \\ & & & & \downarrow f \\ & & & & A \\ & & h & \curvearrowleft & \\ & & & & \end{array}$$

The reason for this construction is that the pullback $\varphi^*(\omega_X \otimes f^*L^{-1})$ now has a non-trivial section (corresponding to taking the “ m -th root” of π^*D on X_m). In other words, we have a non-trivial (injective) morphism

$$(5.2.3) \quad h^*L \hookrightarrow \varphi^*\omega_X.$$

Note that we tautologically also have morphisms

$$(5.2.4) \quad \varphi^*\Omega_X^k \rightarrow \Omega_Y^k$$

for every $k \geq 0$.

After these preliminaries, we are ready to introduce the main construction. We consider the pushforward of the trivial Hodge module $\mathbf{Q}_Y^H[d]$ via h . The underlying filtered right \mathcal{D} -module is (ω_Y, F) ; by Saito's stability and decomposition result, Theorem 4.3.7, we have

$$h_+(\omega_Y, F) \simeq \bigoplus_i (\mathcal{H}^i h_+ \omega_Y, F)[-i]$$

and each $(\mathcal{H}^i h_+ \omega_Y, F)$ underlies a pure Hodge module on A . We define

$$(\mathcal{M}, F) := (\mathcal{H}^0 h_+ \omega_Y, F),$$

with the associated graded \mathcal{A}_A -module $\mathcal{G}_\bullet = \text{gr}_\bullet^F \mathcal{M}$, corresponding to the coherent sheaf $\mathcal{G} = \text{gr}^F \mathcal{M}$ on $A \times V$.

From Example 5.1.3 we know that $\text{Supp}(\mathcal{G}) \subseteq S_h$. Note however that our goal is to prove something about S_f , and in general S_h may be much larger than S_f due to the singularities of the original section s . In order to remedy this, the plan for the rest of the proof is as follows: we will construct a graded submodule

$$\mathcal{F}_\bullet \subseteq \mathcal{G}_\bullet \otimes L^{-1}$$

with the following two properties for the corresponding sheaf \mathcal{F} on $A \times V$.

- (1) $\text{Supp}(\mathcal{F}) \subseteq S_f$.
- (2) $p_{2*} \mathcal{F} \neq 0$.

Let's first assume that this has been accomplished, and see how it finishes the proof of the Theorem. The inclusion $\mathcal{F} \subseteq \mathcal{G} \otimes p_1^* L^{-1}$ leads via pushforward to the inclusion

$$p_{2*} \mathcal{F} \subseteq p_{2*} (\mathcal{G} \otimes p_1^* L^{-1}).$$

We are aiming to show that $p_2(S_f) = V$; if this were not the case, then by (1) and (2) above $p_{2*} \mathcal{F}$ would be a nontrivial torsion sheaf on V . On the other hand, $p_{2*} (\mathcal{G} \otimes p_1^* L^{-1})$ is a torsion-free sheaf by Corollary 4.5.8, which leads to a contradiction.

We are left with constructing a graded submodule \mathcal{F}_\bullet as above. The complex $C_{Y, \bullet}:$

$$h^* \mathcal{A}_{A, \bullet} \rightarrow \Omega_X^1 \otimes h^* \mathcal{A}_{A, \bullet+1} \rightarrow \cdots \rightarrow \Omega_X^{d-1} \otimes h^* \mathcal{A}_{A, \bullet+d-1} \rightarrow \omega_X \otimes h^* \mathcal{A}_{A, \bullet+d}$$

defined as in section §5.1 (with respect to the morphism $h: Y \rightarrow A$) corresponds after forgetting the grading to a complex of sheaves C_X on $Y \times V$. There is also a similar complex C_X on $X \times V$ defined with respect to the morphism $f: X \rightarrow A$. Using the mappings in (5.2.4) we see that on $Y \times V$ there is a natural morphism of complexes

$$(\varphi \times \text{id})^* C_X \longrightarrow C_Y.$$

On the other hand, the (dual of the) nontrivial map in (5.2.3) induces by tensoring with $(\varphi \times \text{id})^* C_X$ a morphism

$$(\varphi \times \text{id})^* C_X \otimes \varphi^* p_1^* \omega_X^{-1} \longrightarrow (\varphi \times \text{id})^* C_X \otimes h^* p_1^* L^{-1},$$

which composed with the one above gives a morphism of complexes

$$\varphi^* p_1^* \omega_X^{-1} \otimes (\varphi \times \text{id})^* C_X \longrightarrow C_Y \otimes h^* p_1^* L^{-1}.$$

After applying $R^0(f \times \text{id})_*$ this finally leads to a morphism

$$\Phi: R^0(f \times \text{id})_*(p_1^* \omega_X^{-1} \otimes C_X) \longrightarrow \mathcal{G} \otimes p_1^* L^{-1},$$

where on the right hand side we used Corollary 5.1.5. We define

$$\mathcal{F} := \text{Im}(\Phi)$$

and accordingly $\mathcal{F}_\bullet = p_{1*} \mathcal{F}$.

We need to check that the required properties (1) and (2) above are satisfied. Note first that we have a surjection

$$R^0(f \times \text{id})_*(p_1^* \omega_X^{-1} \otimes C_X) \twoheadrightarrow \mathcal{F},$$

and we know that $\text{Supp}(C_X) = df^{-1}(0)$; see Lemma 5.1.4. Therefore

$$\text{Supp}(\mathcal{F}) \subseteq (f \times \text{id})(df^{-1}(0)) = S_f$$

which proves (1).

For (2) it suffices to show that

$$H^0(V, p_{2*} \mathcal{F}) \simeq H^0(A \times V, \mathcal{F}) \simeq H^0(A, p_{1*} \mathcal{F}) = \bigoplus_k H^0(A, \mathcal{F}_k)$$

is non-zero. We focus on the lowest possible non-zero graded term, which is \mathcal{F}_{-d} since $\mathcal{G}_k = 0$ for $k < -d$ by Example 5.1.7. We have seen in the same example that $C_{X,-d} = \omega_X$ and $\mathcal{G}_{-d} = h_* \omega_Y$, and therefore

$$\mathcal{F}_{-d} = \text{Im} [f_* \mathcal{O}_X \rightarrow h_* \omega_Y \otimes L^{-1}].$$

Now the composition of the morphism in the parenthesis with the inclusion $\mathcal{O}_A \rightarrow f_* \mathcal{O}_X$ is a nontrivial morphism, since it corresponds via adjunction (and tensoring with $h^* L$ to the nontrivial composition

$$h^* L \rightarrow \varphi^* \omega_X \rightarrow \omega_Y$$

arising from (5.2.3) and (5.2.4). This shows that $H^0(A, \mathcal{F}_{-d}) \neq 0$, showing (2), and thus concluding the proof of the theorem. \square

5.3. Families of varieties of general type

TO BE ADDED.

Localization and Hodge ideals

This chapter focuses on what is in some sense an extended example, namely the \mathcal{D} -module $\mathcal{O}_X(*D)$ obtained by localizing along a hypersurface D , where many of the definitions and techniques in the previous sections can be seen in action. What is currently written is based mainly on material from [MP1]; more material will be added.

6.1. Definition and basic properties

Throughout this section X is a smooth complex variety of dimension n and D is a reduced effective divisor on X . Recall that to this data we can associate the left \mathcal{D}_X -module of functions with poles along D ,

$$\mathcal{O}_X(*D) = \bigcup_{k \geq 0} \mathcal{O}_X(kD),$$

i.e. the localization of \mathcal{O}_X along D . If f is a local defining equation for D , then this is $\mathcal{O}_X[\frac{1}{f}]$, with the obvious action of differential operators. The associated right \mathcal{D}_X -module is denoted $\omega_X(*D)$.

This \mathcal{D}_X -module underlies the mixed Hodge module $j_* \mathbf{Q}_U^H[n]$, where $U = X \setminus D$ and $j : U \hookrightarrow X$ is the inclusion map. It therefore comes with an attached Hodge filtration $F_k \mathcal{O}_X(*D)$, where $k \geq 0$. Hence $(\mathcal{O}_X(*D), F)$ is a regular and quasi-unipotent filtered \mathcal{D}_X -module extending the trivial filtered \mathcal{D}_U -module (\mathcal{O}_U, F) . We will analyze this filtration by using a combination of \mathcal{D} -module theory and birational geometry techniques.

EXAMPLE 6.1.1 (Simple normal crossings case). Let's first note that the filtration $F_\bullet \mathcal{O}_X(*D)$ is well-understood when D is a simple normal crossing divisor. Indeed, we saw in Example 4.4.10, as part of the general construction of push-forward via an open embedding, that

$$F_k \mathcal{O}_X(*D) = F_k \mathcal{D}_X \cdot \mathcal{O}_X(D).$$

Note that the quotient rule for differentiation shows that $F_k \mathcal{O}_X(*D) \subseteq \mathcal{O}_X((k+1)D)$ for all $k \geq 0$, and it is not hard to see that this is an equality when D is smooth.

The fact that the filtration F has to look like this is a consequence of the property of being regular and quasi-unipotent along D . Indeed, in the case of $\mathcal{O}_X(*D)$ the formula in Remark 4.1.8 applies in order to recover F from its restriction to U using the V -filtration. Let's verify this in the case when D is smooth, where we have written down the V -filtration along D in Example 3.1.4(4), hence we know that $V^0 \mathcal{O}_X(*D) = \mathcal{O}_X(D)$.

Thus by (4.1.2) we have

$$F_k \mathcal{O}_X(*D) = \sum_{i \geq 0} \partial_t^i \cdot (\mathcal{O}_X(D) \cap j_* F_{k-i} \mathcal{O}_U) = \sum_{i=0}^k \partial_t^i \cdot \mathcal{O}_X(D) = \mathcal{O}_X((k+1)D),$$

confirming the discussion above.

For arbitrary D , note first that there is a more obvious filtration on $\mathcal{O}_X(*D)$, namely the *pole order filtration*, whose nonzero terms are taken by convention to be

$$P_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \quad \text{for } k \geq 0.$$

This filtration is in general too coarse to be of much use; it is not even a good filtration when D is not smooth. We will see however that it is very useful to measure the Hodge filtration against it, and indeed what we saw in Example 6.1.1 holds in general, as was first noted in [Sa3, Proposition 0.9]:

LEMMA 6.1.2. *For every $k \geq 0$, we have an inclusion*

$$F_k \mathcal{O}_X(*D) \subseteq P_k \mathcal{O}_X(*D).$$

PROOF. Denote by $V \subseteq X$ the open subset in which D is smooth; since D is reduced, the complement $X \setminus V$ has codimension at least 2 in X . By the discussion above, we have

$$F_k \mathcal{O}_X(*D)|_V \simeq \mathcal{O}_X((k+1)D)|_V,$$

and therefore

$$F_k \mathcal{O}_X(*D) \subseteq j_*(F_k \mathcal{O}_X(*D)|_V) \simeq j_*(\mathcal{O}_X((k+1)D)|_V) \simeq \mathcal{O}_X((k+1)D).$$

Indeed, because of the high codimension of $X \setminus V$, the last isomorphism holds since $\mathcal{O}_X((k+1)D)$ is a line bundle, while the first inclusion holds since $F_k \mathcal{O}_X(*D)$ is torsion-free (being a subsheaf of $\mathcal{O}_X(*D)$, which is clearly a torsion-free \mathcal{O}_X -module). \square

We will also see an alternative proof of this Lemma that uses resolution of singularities below.

DEFINITION 6.1.3 (**Hodge ideals**). Given the inclusion in Lemma 6.1.2, for each $k \geq 0$ we define the coherent ideal sheaf $I_k(D)$ on X by the formula

$$F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D) \otimes I_k(D).$$

We call $I_k(D)$ the k -th Hodge ideal of D .

In what follows we will analyze these ideals by making use of log resolutions and filtered push-forward for \mathcal{D} -modules underlying Hodge modules.

Simple normal crossings case. Example 6.1.1 tells us that when D is a simple normal crossing divisor, the ideals $I_k(D)$ are given by the following expression:¹

$$(6.1.1) \quad I_k(D) = (F_k \mathcal{D}_X \cdot \mathcal{O}_X(D)) \otimes \mathcal{O}_X(-(k+1)D) \quad \text{for all } k \geq 0.$$

Note that in particular $I_0(D) = \mathcal{O}_X$. We can in fact completely describe a set of generators for all of these ideals in local coordinates.

PROPOSITION 6.1.4. *Suppose that around a point $p \in X$ we have coordinates x_1, \dots, x_r such that D is defined by $(x_1 \cdots x_r = 0)$. Then, for every $k \geq 0$, the ideal $I_k(D)$ is generated around p by*

$$\{x_1^{a_1} \cdots x_r^{a_r} \mid 0 \leq a_i \leq k, \sum_i a_i = k(r-1)\}.$$

In particular, if $r = 1$ (that is, when D is smooth), we have $I_k(D) = \mathcal{O}_X$ and if $r = 2$, then $I_k(D) = (x_1, x_2)^k$.

PROOF. It is clear that $F_k \mathcal{D}_X \cdot \mathcal{O}_X(D)$ is generated as an \mathcal{O}_X -module by

$$\{x_1^{-b_1} \cdots x_r^{-b_r} \mid b_i \geq 1, \sum_i b_i = r+k\}.$$

According to (6.1.1), the expression for $I_k(D)$ now follows by multiplying these generators by $(x_1 \cdots x_r)^{k+1}$. The assertions in the special cases $r = 1$ and $r = 2$ are clear. \square

For convenience, let's record separately the case $r = 1$ in the Proposition above.

COROLLARY 6.1.5. *If D is a smooth divisor, then $I_k(D) = \mathcal{O}_X$ for all $k \geq 0$.*

REMARK 6.1.6. This is of course another way of saying what we already know: if D is smooth, the Hodge filtration coincides with the pole order filtration. It turns out that the converse of this statement holds as well, but this requires some serious work; see [MP1, Theorem A].

The general case. When D is arbitrary, we consider a log resolution $f: Y \rightarrow X$ of the pair (X, D) which is an isomorphism over $U = X \setminus D$, and let $E = (f^*D)_{\text{red}}$, an SNC divisor.

LEMMA 6.1.7. *There is a natural isomorphism*

$$f_+ \omega_Y(*E) \simeq H^0 f_+ \omega_Y(*E) \simeq \omega_X(*D).$$

PROOF. Let $V = Y \setminus E = f^{-1}(U)$, and denote by j_U and j_V the inclusions of U and V into X and Y respectively. Note that $j_{U+} \omega_U \simeq \omega_X(*D)$ and $j_{V+} \omega_V \simeq \omega_Y(*E)$. The result then follows from the fact that $j_U = f \circ j_V$. \square

¹We will see below that one can also define Hodge ideals directly using log resolutions, without appealing to the theory of Hodge modules, in which case we would need to take this as the definition.

This isomorphism continues to hold at the level of the *filtered* \mathcal{D} -modules underlying the respective Hodge modules, i.e.

$$(6.1.2) \quad f_+(\omega_Y(*E), F) \simeq (\omega_X(*D), F),$$

where the filtered push-forward on the left hand side is given by the construction described in Section 1.5. Since

$$f_+\omega_Y(*E) \simeq \mathbf{R}f_*(\omega_Y(*E) \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}),$$

the first step is to consider a convenient (filtered) representative for $\omega_Y(*E) \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}$.

It turns out that this last object is supported only in degree zero; even though not entirely necessarily, let's go one step further and provide a precise description. First, recall that since $\mathcal{D}_{Y \rightarrow X}$ is a left \mathcal{D}_Y -module, we have a canonical morphism of $(\mathcal{D}_Y, f^{-1}\mathcal{O}_X)$ bimodules

$$\varphi: \mathcal{D}_Y \longrightarrow f^*\mathcal{D}_X$$

that maps 1 to 1, and is clearly an isomorphism over $V = Y \setminus E$. Since \mathcal{D}_Y is torsion-free, we conclude that φ is injective, with cokernel supported on E . For each k , we also have induced inclusions $F_k\mathcal{D}_Y \hookrightarrow f^*F_k\mathcal{D}_X$.

PROPOSITION 6.1.8. *The canonical morphism*

$$\omega_Y(*E) \longrightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X}$$

induced by φ in the derived category of right $f^{-1}\mathcal{O}_X$ -modules is an isomorphism.

The proof we give below is inspired in part by arguments in [HTT, §5.2].

LEMMA 6.1.9. *The induced morphism*

$$\mathrm{Id} \otimes \varphi: \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^*\mathcal{D}_X$$

is an isomorphism.

PROOF. It suffices to show that the induced mappings

$$\mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} F_k\mathcal{D}_Y \longrightarrow \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} f^*F_k\mathcal{D}_X$$

are all isomorphisms for $k \geq 0$. But this follows immediately from Lemma 6.1.10 below (note that since $\mathcal{O}_Y(*E)$ is flat over \mathcal{O}_Y , these maps are injective). \square

In the proof above we used the following well-known observation; see [HTT, Lemma 5.2.7].

LEMMA 6.1.10. *If \mathcal{F} is a coherent $\mathcal{O}_X(*D)$ -module supported on D , then $\mathcal{F} = 0$.*

Let us now use the notation

$$\mathcal{D}_Y(*E) := \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y.$$

This is a sheaf of rings, and one can identify it with the subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathcal{O}_Y(*E))$ generated by \mathcal{D}_Y and $\mathcal{O}_Y(*E)$. Note that since $\mathcal{O}_Y(*E)$ is a flat \mathcal{O}_Y -module, we have $\mathcal{D}_Y(*E) \simeq \mathcal{O}_Y(*E) \otimes_{\mathcal{O}_Y}^{\mathbf{L}} \mathcal{D}_Y$. A basic fact is the following:

LEMMA 6.1.11. *The canonical morphism*

$$\mathcal{D}_Y(*E) \longrightarrow \mathcal{D}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$$

induced by φ is an isomorphism.

PROOF. Via the isomorphism $\mathcal{D}_Y(*E) \simeq \mathcal{O}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{D}_Y$, the morphism in the statement gets identified to the morphism

$$(6.1.3) \quad \mathcal{O}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{D}_Y \longrightarrow \mathcal{O}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} \mathcal{D}_Y \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X} = \mathcal{O}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_Y} f^* \mathcal{D}_X$$

induced by φ . Moreover, since $\mathcal{O}_Y(*E)$ is a flat \mathcal{O}_Y -module, the morphism (6.1.3) gets identified with the isomorphism in Lemma 6.1.9. \square

PROOF OF PROPOSITION 6.1.8. Via the right \mathcal{D} -module structure on ω_Y , we have that $\omega_Y(*E)$ has a natural right $\mathcal{D}_Y(*E)$ -module structure. The morphism in the proposition gets identified with the morphism

$$\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y(*E)} \mathcal{D}_Y(*E) \longrightarrow \omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y(*E)} \mathcal{D}_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$$

induced by φ . In turn, this is obtained by applying $\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y(*E)} -$ to the isomorphism in Lemma 6.1.11, hence it is an isomorphism. \square

Propositions 1.4.14 and 6.1.8 then have the following immediate consequence:

COROLLARY 6.1.12. *On Y there is a filtered complex of right $f^{-1}\mathcal{D}_X$ -modules*

$$\begin{aligned} 0 \rightarrow f^* \mathcal{D}_X \rightarrow \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow \cdots \\ \cdots \rightarrow \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X} \rightarrow 0 \end{aligned}$$

which is exact (though not necessarily filtered exact).

PROOF. It follows from Proposition 1.4.14 that the complex

$$0 \rightarrow f^* \mathcal{D}_X \rightarrow \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow \cdots \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow 0$$

represents the object $\omega_Y(*E) \overset{\mathbf{L}}{\otimes}_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$ in the derived category, hence Proposition 6.1.8 implies the exactness of the entire complex in the statement. \square

REMARK 6.1.13. It may be helpful to make some first comments on the filtration on the resolved object as well. On the tensor product $\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$ we consider the tensor product filtration, that is,

$$\begin{aligned} F_k(\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}) &:= \\ &= \text{Im} \left[\bigoplus_{i \geq -n} F_i \omega_Y(*E) \otimes_{\mathcal{O}_Y} f^* F_{k-i} \mathcal{D}_X \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X} \right], \end{aligned}$$

where the map in the parenthesis is the natural map between the tensor product over \mathcal{O}_Y and that over \mathcal{D}_Y .

LEMMA 6.1.14. *The definition above simplifies to*

$$F_k(\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}) = \text{Im}[\omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_{k+n} \mathcal{D}_X \rightarrow \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}].$$

PROOF. Fix $i \geq -n$ and recall that $F_i \omega_Y(*E) = \omega_Y(E) \cdot F_{i+n} \mathcal{D}_Y$. The factor $F_{i+n} \mathcal{D}_Y$ can be moved over the tensor product once we pass to the image in the tensor product over \mathcal{D}_Y , and moreover we have an inclusion

$$F_{i+n} \mathcal{D}_Y \cdot f^* F_{k-i} \mathcal{D}_X \subseteq f^* F_{k+n} \mathcal{D}_X.$$

Therefore inside $\omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$, the image of $F_i \omega_Y(*E) \otimes_{\mathcal{O}_Y} f^* F_{k-i} \mathcal{D}_X$ is contained in the image of $\omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_{k+n} \mathcal{D}_X$. \square

With these preparations in place, let's go back to understanding Hodge ideals via the isomorphism 6.1.2. We denote by A^\bullet the complex of induced $f^{-1} \mathcal{D}_X$ -modules

$$0 \rightarrow f^* \mathcal{D}_X \rightarrow \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow \cdots \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* \mathcal{D}_X \rightarrow 0$$

appearing in Corollary 6.1.12, placed in degrees $-n, \dots, 0$. Just as in Proposition 1.4.14, it is filtered by the subcomplexes of A^\bullet given by

$$\begin{aligned} C_{k-n}^\bullet = F_{k-n} A^\bullet : \quad & 0 \rightarrow f^* F_{k-n} \mathcal{D}_X \rightarrow \Omega_Y^1(\log E) \otimes_{\mathcal{O}_Y} f^* F_{k-n+1} \mathcal{D}_X \rightarrow \cdots \\ & \cdots \rightarrow \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} f^* F_{k-1} \mathcal{D}_X \rightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_k \mathcal{D}_X \rightarrow 0 \end{aligned}$$

for $k \geq 0$. (This time however these subcomplexes are not necessarily exact.)

In the language of Section 1.5, we have found a representative for

$$\omega_Y(*E) \otimes_{\mathcal{D}_Y}^{\mathbf{L}} \mathcal{D}_{Y \rightarrow X} \simeq \omega_Y(*E) \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \rightarrow X}$$

in the derived category of induced right $f^{-1} \mathcal{D}_X$ -modules, with filtration induced from the corresponding filtration on the complex of induced right \mathcal{D}_Y -modules representing $\omega_Y(*E)$. In other words, we have

$$\text{DR}_{Y/X}((\omega_Y(*E), F)) \simeq (A^\bullet, F).$$

To obtain $f_+(\omega_Y(*E), F)$, it now suffices to apply the functor

$$\mathbf{R}f_* : \mathbf{D}(\text{FM}(f^{-1} \mathcal{D}_X)) \rightarrow \mathbf{D}(\text{FM}(\mathcal{D}_X)),$$

also defined in Section 1.5. Moreover, since this comes underlies a push-forward in the category of mixed Hodge modules, we have that $f_+(\omega_Y(*E), F)$ is strict. This finally provides the following birational interpretation of Hodge ideals:

PROPOSITION 6.1.15. *With the notation above, for each $k \geq 0$, we have*

$$\omega_X((k+1)D) \otimes I_k(D) \simeq R^0 f_* C_{k-n}^\bullet.$$

PROOF. The discussion above, plus Lemma 6.1.7 combined with Corollary 1.5.5 and the subsequent discussion then, gives

$$(6.1.4) \quad F_{k-n} \omega_X(*D) = F_{k-n} \mathcal{H}^0 f_+ \omega_Y(*E) \simeq \text{Im} [R^0 f_* C_{k-n}^\bullet \rightarrow R^0 f_* A^\bullet],$$

where the map of \mathcal{O}_X -modules in the parenthesis is induced by the inclusion $C_{k-n}^\bullet \hookrightarrow A^\bullet$. On the other hand, the strictness of $f_+(\omega_Y(*E), F)$ means that the map in the parenthesis is injective; see §1.6. \square

REMARK 6.1.16. The formula in Proposition 6.1.15, more precisely (6.1.4), can be alternatively taken as a definition of the Hodge ideals $I_k(D)$ using log resolutions, without appealing to the theory of mixed Hodge modules. In this approach however, one has to prove that the definition is independent of the choice of log resolution, which can be done using a Nakano-type local vanishing theorem; see [MP1, Theorem 11.1]. We also need to have the analogue of Lemma 6.1.2, namely

$$R^0 f_* C_{k-n}^\bullet \subseteq \omega_X((k+1)D),$$

and this is seen in a similar fashion as follows: since D is reduced, we can find an open subset $V \subseteq X$ with the property that $\text{codim}(X \setminus V, X) \geq 2$, the induced morphism $f^{-1}(V) \rightarrow V$ is an isomorphism, and $D|_V$ is a smooth (possibly disconnected) divisor. Let $j: V \hookrightarrow X$ be the inclusion. By assumption, on $f^{-1}(V)$ we have $\mathcal{D}_{Y \rightarrow X} = \mathcal{D}_Y$. From Proposition 1.4.14, on V we obtain

$$R^0 f_* C_{k-n}^\bullet = \omega_X((k+1)D).$$

Now $R^0 f_* C_{k-n}^\bullet$ is torsion-free, being a subsheaf of $\omega_X(*D)$, and so the following canonical map is injective:

$$R^0 f_* C_{k-n}^\bullet \rightarrow j_*(R^0 f_* C_{k-n}^\bullet|_V) = j_*(\omega_X((k+1)D)|_V) = \omega_X((k+1)D).$$

Chain of inclusions. From the filtration property

$$F_{k-1} \mathcal{O}_X(*D) \subseteq F_k \mathcal{O}_X(*D)$$

it follows immediately that

$$I_{k-1}(D) \cdot \mathcal{O}_X(-D) \subseteq I_k(D)$$

for each $k \geq 1$. However, the Hodge ideals also satisfy another natural but more subtle sequence of inclusions.

PROPOSITION 6.1.17. *For every reduced effective divisor D on the smooth variety X , and for every $k \geq 1$, we have*

$$I_k(D) \subseteq I_{k-1}(D).$$

PROOF. Consider the canonical inclusion

$$\iota: \mathcal{O}_X \hookrightarrow \mathcal{O}_X(*D)$$

of filtered left \mathcal{D}_X -modules that underlie mixed Hodge modules. Since the category $\text{MHM}(X)$ of mixed Hodge modules on X is abelian, the cokernel \mathcal{M} of ι underlies a mixed Hodge module on X too, and it is clear that \mathcal{M} has support D . Since morphisms between Hodge \mathcal{D} -modules preserve the filtrations and are strict, for each $k \geq 0$ we have a short exact sequence

$$0 \longrightarrow F_k \mathcal{O}_X \longrightarrow F_k \mathcal{O}_X(*D) \longrightarrow F_k \mathcal{M} \longrightarrow 0.$$

Recall now that $F_k \mathcal{O}_X = \mathcal{O}_X$ for all $k \geq 0$. On the other hand, if f is a local equation of D , then by Proposition 4.1.7 we have

$$f \cdot F_k \mathcal{M} \subseteq F_{k-1} \mathcal{M}.$$

It then follows that $f \cdot F_k \mathcal{O}_X(*D) \subseteq F_{k-1} \mathcal{O}_X(*D)$ as well, which implies the assertion by the definition of Hodge ideals. \square

Computations for $k = 0, 1$. The first in the sequence of Hodge ideals can be identified with a multiplier ideal; for the general theory of multiplier ideals see [La, Ch.9] or §2.6 above.

PROPOSITION 6.1.18. *We have*

$$I_0(D) = \mathcal{J}(X, (1 - \varepsilon)D),$$

the multiplier ideal associated to the \mathbf{Q} -divisor $(1 - \varepsilon)D$ on X , for any $0 < \varepsilon \ll 1$.

PROOF. Note that $C_{-n}^\bullet = \omega_Y(E)$, hence

$$F_{-n} \omega_X(*D) = f_* \omega_Y(E) \simeq f_* \mathcal{O}_Y(K_{Y/X} + E - f^*D) \otimes \omega_X(D).$$

Therefore the statement to be proved is that

$$f_* \mathcal{O}_Y(K_{Y/X} + E - f^*D) = \mathcal{J}(X, (1 - \varepsilon)D).$$

On the other hand, the right-hand side is by definition

$$f_* \mathcal{O}_Y(K_{Y/X} - \lfloor (1 - \varepsilon)f^*D \rfloor) = f_* \mathcal{O}_Y(K_{Y/X} + (f^*D)_{\text{red}} - f^*D),$$

which implies the desired equality. \square

The following is an immediate consequence of this interpretation; see Exercise 2.6.13.

COROLLARY 6.1.19. *We have $I_0(D) = \mathcal{O}_X$ if and only if the pair (X, D) is log-canonical.*

We now move to a description of the case $k = 1$. We begin by noting the following fact about the complex C_{1-n}^\bullet .

LEMMA 6.1.20. *The morphism*

$$\Omega_Y^{n-1}(\log E) \longrightarrow \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_1 \mathcal{D}_X$$

is injective.

PROOF. Note that we have a commutative diagram

$$\begin{array}{ccc} \Omega_Y^{n-1}(\log E) & \xrightarrow{\beta} & \omega_Y(E) \otimes_{\mathcal{O}_Y} F_1 \mathcal{D}_Y \\ \downarrow \text{Id} & & \downarrow \gamma \\ \Omega_Y^{n-1}(\log E) & \xrightarrow{\alpha} & \omega_Y(E) \otimes_{\mathcal{O}_Y} f^* F_1 \mathcal{D}_X, \end{array}$$

in which γ is the canonical inclusion. Since β is injective by (the proof of) Proposition 1.4.14, it follows that α is injective, too. \square

Denoting by \mathcal{F}_1 the cokernel of the map in the Lemma above, we therefore have a short exact sequence

$$(6.1.5) \quad 0 \rightarrow \Omega_Y^{n-1}(\log E) \rightarrow \omega_Y(E) \otimes f^*F_1\mathcal{D}_X \rightarrow \mathcal{F}_1 \rightarrow 0,$$

hence in particular \mathcal{F}_1 is quasi-isomorphic to the complex C_{1-n}^\bullet , and we have

$$\omega_X(2D) \otimes I_1(D) \simeq f_*\mathcal{F}_1.$$

COROLLARY 6.1.21. *There is a four-term exact sequence*

$$0 \rightarrow f_*\Omega_Y^1(\log E) \rightarrow \omega_X(D) \otimes I_0(D) \otimes F_1\mathcal{D}_X \xrightarrow{\psi} \omega_X(2D) \otimes I_1(D) \rightarrow R^1f_*\Omega_Y^1(\log E) \rightarrow 0,$$

where the image of the map ψ corresponds to the image of the natural filtered \mathcal{D}_X -module map

$$F_{-n}\omega_X(*D) \cdot F_1\mathcal{D}_X \subseteq F_{1-n}\omega_X(*D).$$

PROOF. The exact sequence is obtained by pushing forward the short exact sequence (6.1.5). The zero on the right comes from applying the projection formula, and using the fact that

$$R^i f_*\omega_Y(E) = 0 \quad \text{for all } i > 0.$$

This follows from the Local Vanishing theorem for multiplier ideals; see also Corollary 6.1.28(2) below and the Remark thereafter. \square

The Corollary above shows the main obstruction to a having an easy calculation of $I_1(D)$ once we have good understanding of the multiplier ideal $I_0(D)$, namely the appearance of the higher direct image $R^1f_*\Omega_Y^{n-1}(\log E)$. In general this sheaf does not vanish, and is not straightforward to compute. (Similar phenomena occur for the higher Hodge ideals $I_k(D)$.) Let's analyze this a little bit better, in order to give some examples.

DEFINITION 6.1.22. We define the ideal sheaf $J_1(D) \subseteq \mathcal{O}_X$ by the formula

$$(\omega_X(D) \otimes I_0(D)) \cdot F_1\mathcal{D}_X = \omega_X(2D) \otimes J_1(D).$$

The left hand side in the formula is the image of the morphism ψ in Corollary 6.1.21, and therefore we have $J_1(D) \subseteq I_1(D)$.

The discussion above leads therefore to

LEMMA 6.1.23. *We have $J_1(D) = I_1(D)$ if and only if $R^1f_*\Omega_Y^{n-1}(\log E) = 0$.*

It turns out that everything is fine in dimension two:

PROPOSITION 6.1.24. *Assuming that X is a surface, we have*

$$R^1f_*\Omega_Y^1(\log E) = 0$$

for every reduced divisor D on X , and therefore $J_1(D) = I_1(D)$.

PROOF. We write $E = \tilde{D} + F$, where \tilde{D} is the strict transform of D and F is the reduced exceptional divisor. We consider the residue short exact sequence for forms with log poles (reference):

$$0 \longrightarrow \Omega_Y^1(\log F) \longrightarrow \Omega_Y^1(\log E) \longrightarrow \mathcal{O}_{\tilde{D}}(F|_{\tilde{D}}) \longrightarrow 0.$$

Now on one hand all fibers of $\tilde{D} \rightarrow D$ have dimension 0, and therefore

$$R^1 f_* \mathcal{O}_{\tilde{D}}(F|_{\tilde{D}}) = 0.$$

On the other hand, we also have $R^1 f_* \Omega_Y^1(\log F) = 0$, and therefore we are done; indeed, this is a special case of a general result (see [MP1, Theorem 31.1(ii)]) stating that in arbitrary dimension n if $f: Y \rightarrow X$ is a birational morphism with exceptional divisor F , then

$$R^{n-1} f_* \Omega_Y^1(\log F) = 0.$$

□

EXERCISE 6.1.25. Show directly the vanishing $R^1 f_* \Omega_Y^1(\log F) = 0$ in the proof above, in the case of a surface. (Hint: consider a resolution obtained as a composition of blow-ups of points, and use the residue exact sequence.)

The Proposition above implies that on surfaces we can always compute $I_1(D)$ once $I_0(D)$ is understood. Here are some concrete calculations:

EXAMPLE 6.1.26. • If $D = (xy = 0) \subset \mathbf{C}^2$ is a node, then $I_k(D) = (x, y)^k$ for all $k \geq 0$.

- If $D = (x^2 + y^3 = 0) \subset \mathbf{C}^2$ is a cusp, then $I_0(D) = (x, y)$ and $I_1(D) = (x^2, xy, y^3)$.
- If $D = (xy(x+y) = 0) \subset \mathbf{C}^2$ is a triple point, then $I_0(D) = (x, y)$, while $I_1(D) = (x, y)^3$.

Note how I_1 distinguishes between singularities for which I_0 is the same. This is one of the ways in which Hodge ideals become important in applications.

In higher dimension the vanishing of $R^1 f_* \Omega_Y^{n-1}(\log E) = 0$ does not necessarily hold any more, and therefore $I_1(D)$ (and the higher Hodge ideals) are usually much harder to compute.

EXAMPLE 6.1.27. Let D be the cone in $X = \mathbf{A}^3$ over a smooth plane curve of degree d , and let $f: Y \rightarrow X$ be the log resolution obtained by blowing up the origin. The claim is that if $d \geq 3 = \dim X$, then

$$R^1 f_* \Omega_Y^2(\log E) \neq 0.$$

Indeed, we have $E = \tilde{D} + F$, where F is the exceptional divisor of f , and so on Y there is a residue-type short exact sequence

$$0 \longrightarrow \Omega_Y^2(\log F) \longrightarrow \Omega_Y^2(\log E) \longrightarrow \Omega_{\tilde{D}}^1(\log F_{\tilde{D}}) \longrightarrow 0.$$

If we had

$$(6.1.6) \quad H^2(Y, \Omega_Y^2(\log F)) = 0,$$

it would follow from the long exact sequence associated to the above sequence that it is enough to show that

$$H^1(\tilde{D}, \Omega_{\tilde{D}}^1(\log F_{\tilde{D}})) \neq 0.$$

Consider however the residue short exact sequence on \tilde{D} :

$$0 \longrightarrow \Omega_{\tilde{D}}^1 \longrightarrow \Omega_{\tilde{D}}^1(\log F_{\tilde{D}}) \longrightarrow \mathcal{O}_{F_{\tilde{D}}} \longrightarrow 0.$$

Since the fibers of $f|_{\tilde{D}}$ are at most one-dimensional, we have $R^2 f_* \Omega_{\tilde{D}}^1 = 0$, and so it suffices to check that

$$R^1 f_* \mathcal{O}_{F_{\tilde{D}}} \simeq H^1(F_{\tilde{D}}, \mathcal{O}_{F_{\tilde{D}}}) \neq 0.$$

But $F_{\tilde{D}}$ is isomorphic to the original plane curve of degree $d \geq 3$, so this is clear. Therefore it is enough to prove (6.1.6).

The short exact sequence

$$0 \longrightarrow \mathcal{O}_F(-F) \longrightarrow \Omega_Y^1|_F \longrightarrow \Omega_F^1 \longrightarrow 0$$

induces for every $m \geq 0$ a short exact sequence

$$0 \rightarrow \Omega_F^1 \otimes \mathcal{O}_F(-(m+1)F) \rightarrow \Omega_Y^2|_F \otimes \mathcal{O}_F(-mF) \rightarrow \omega_F \otimes \mathcal{O}_F(-mF) \rightarrow 0.$$

Since $F \simeq \mathbf{P}^2$ and $\mathcal{O}_F(-F) \simeq \mathcal{O}_{\mathbf{P}^2}(1)$, it follows from the Euler exact sequence that

$$H^2(F, \Omega_F^1 \otimes \mathcal{O}_F(-(m+1)F)) = 0 \quad \text{for all } m \geq 0.$$

We conclude that

$$H^2(F, \Omega_Y^2|_F) \simeq \mathbf{C} \quad \text{and} \quad H^2(F, \Omega_Y^2|_F \otimes \mathcal{O}_F(-mF)) = 0 \quad \text{for } m \geq 1.$$

We now deduce from the exact sequence

$$0 \rightarrow \Omega_Y^2(-(m+1)F) \rightarrow \Omega_Y^2(-mF) \rightarrow \Omega_Y^2|_F \otimes \mathcal{O}_F(-mF) \rightarrow 0$$

that for every $m \geq 1$ the map

$$H^2(Y, \Omega_Y^2(-(m+1)F)) \rightarrow H^2(Y, \Omega_Y^2(-mF))$$

is surjective. Since $\mathcal{O}_Y(-F)$ is ample over X , we have

$$H^2(Y, \Omega_Y^2(-mF)) = 0 \quad \text{for all } m \gg 0,$$

and therefore

$$H^2(Y, \Omega_Y^2(-mF)) = 0 \quad \text{for all } m \geq 1.$$

In particular, the long exact sequence in cohomology corresponding to

$$0 \rightarrow \Omega_Y^2(-F) \rightarrow \Omega_Y^2 \rightarrow \Omega_Y^2|_F \rightarrow 0$$

gives an isomorphism

$$H^2(Y, \Omega_Y^2) \simeq H^2(Y, \Omega_Y^2|_F) \simeq \mathbf{C}.$$

Finally, consider the exact sequence

$$0 \rightarrow \Omega_Y^2 \rightarrow \Omega_Y^2(\log F) \rightarrow \Omega_F^1 \rightarrow 0,$$

which induces

$$H^1(F, \Omega_F^1) \xrightarrow{\alpha} H^2(Y, \Omega_Y^2) \rightarrow H^2(Y, \Omega_Y^2(\log F)) \rightarrow H^2(F, \Omega_F^1) = 0.$$

The composition

$$\mathbf{C} \simeq H^1(F, \Omega_F^1) \rightarrow H^2(Y, \Omega_Y^2) \rightarrow H^2(F, \Omega_F^1) \simeq \mathbf{C}$$

is given by cup-product with $c_1(\mathcal{O}_F(F))$, hence it is nonzero. Therefore α is surjective, which implies (6.1.6).

Strictness and local vanishing. We should also note that the strictness of $f_+(\omega_Y(*E), F)$ has an interesting and useful consequence beyond the injectivity of the map

$$R^0 f_* C_{k-n}^\bullet \longrightarrow R^0 f_* A^\bullet = \omega_X(*D),$$

used above.

COROLLARY 6.1.28 (Local vanishing for Hodge ideals). *For every $k \geq 0$ and $i \neq 0$ we have*

$$R^i f_* C_{k-n}^\bullet = 0.$$

PROOF. Since A^\bullet represents $\omega_Y(*E) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y \rightarrow X}$, we have $R^i f_* A^\bullet = 0$ for all $i \neq 0$ by Lemma 6.1.7. On the other hand, the strictness property (1.6.1) of the filtration on $f_+ \omega_Y(*E)$ implies that $R^i f_* C_{k-n}^\bullet$ injects in $R^i f_* A^\bullet$. \square

The case $k = 0$ of the Corollary is a special case of the well-known Local Vanishing for multiplier ideals (see [La, Theorem 9.4.1]), a consequence of the Kawamata-Viehweg vanishing theorem; in this case $C_{-n}^\bullet = \omega_Y(E)$, and the result says that $R^i f_* \omega_Y(E) = 0$ for all $i > 0$, which according to the proof of Proposition 6.1.18 can be reinterpreted as

$$R^i f_* \mathcal{O}_Y(K_{Y/X} - \lfloor (1 - \varepsilon) f^* D \rfloor) = 0 \quad \text{for } i > 0.$$

The interesting fact is that the result of the Corollary for arbitrary k leads to a natural Nakano-type extension of this local vanishing theorem:

THEOREM 6.1.29. *Let X be a smooth variety of dimension n , and D an effective divisor on X . If $f: Y \rightarrow X$ is a log resolution of (X, D) which is an isomorphism over $X \setminus D$, and $E = (f^* D)_{\text{red}}$, then*

$$R^p f_* \Omega_Y^q(\log E) = 0 \quad \text{if } p + q > n = \dim X.$$

PROOF. We proceed by descending induction on p ; the case $p > n$ is trivial. Suppose now that $p \leq n$ and $q > n - p$. We may thus apply Corollary 6.1.28 to deduce that if

$$C^\bullet = C_{-p}^\bullet [p - n]$$

(so that here $k = n - p$), then

$$(6.1.7) \quad R^j f_* C^\bullet = 0 \quad \text{for } j > n - p.$$

Note that by definition, we have

$$C^i = \Omega_Y^{p+i}(\log E) \otimes_{\mathcal{O}_Y} f^* F_i \mathcal{D}_X \quad \text{for } 0 \leq i \leq n - p.$$

Consider the spectral sequence

$$E_1^{i,j} = R^j f_* C^i \Rightarrow R^{i+j} f_* C^\bullet.$$

It follows from (6.1.7) that $E_\infty^{0,q} = 0$. Now by the projection formula we have

$$(6.1.8) \quad E_1^{i,j} = R^j f_* \Omega_Y^{p+i}(\log E) \otimes_{\mathcal{O}_X} F_i \mathcal{D}_X.$$

In particular, it follows from the inductive hypothesis that for every $r \geq 1$ we have $E_1^{r, q-r+1} = 0$, hence $E_r^{r, q-r+1} = 0$ as well. On the other hand, we clearly have $E_r^{-r, q+r-1} = 0$, since this is a first-quadrant spectral sequence. We thus conclude that

$$E_r^{0, q} = E_{r+1}^{0, q} \quad \text{for all } r \geq 1,$$

hence $E_1^{0, q} = E_\infty^{0, q} = 0$. Using (6.1.8) again, we conclude that

$$R^q f_* \Omega_Y^p(\log E) = 0.$$

□

This result was first proved by Saito [Sa6, Theorem], along similar lines. A different proof, more in line with those of standard vanishing theorems, was given in [MP1, Theorem 32.1].

6.2. Local properties

Will include proofs of the following results:

THEOREM 6.2.1. *Let D be a reduced effective divisor on a smooth variety X , and let W be an irreducible closed subset of codimension r , defined by the ideal sheaf I_W . If $m = \text{mult}_W(D)$, then for every k we have*

$$I_k(D) \subseteq I_W^{(q)}, \quad \text{where } q = \min\{m-1, (k+1)m-r\}.$$

Here $I_W^{(q)}$ is the q^{th} symbolic power of I_W , and $I_W^{(q)} = \mathcal{O}_X$ if $q \leq 0$.

EXAMPLE 6.2.2. We spell out what this criterion says when $k = 1$ and $W = \{x\}$ is a single point. If $m = \text{mult}_x(D)$, then

$$m \geq \max\left\{q+1, \frac{n+q}{2}\right\} \implies I_1(D) \subseteq \mathfrak{m}_x^q.$$

Taking $m \geq 2$ and ensuring that $q \geq 1$ in Theorem 6.2.1 gives:

COROLLARY 6.2.3. *Let D be a reduced effective divisor on the smooth variety X . If W is an irreducible closed subset of X of codimension r such that $m = \text{mult}_W(D) \geq 2$, then*

$$I_k(D) \subseteq I_W \quad \text{for all } k \geq \frac{r+1-m}{m}.$$

In particular, if $W \subseteq D_{\text{sing}}$, then

$$I_k(D) \subseteq I_W \quad \text{for all } k \geq \frac{r-1}{2}.$$

THEOREM 6.2.4. *Let D be a reduced effective divisor on a smooth variety X of dimension n , and let $x \in D$ be an ordinary singular point of multiplicity $m \geq 2$. Then*

$$I_k(D)_x = \mathcal{O}_{X,x} \iff k \leq \left\lfloor \frac{n}{m} \right\rfloor - 1.$$

REMARK 6.2.5. Corollary 6.2.3 implies that the nontriviality part of Theorem 6.2.4 holds for arbitrary singular points. Indeed, if $x \in D$ is a point of multiplicity m , the corollary implies that $I_k(D)$ becomes nontrivial at x when $k \geq \frac{n+1-m}{m}$, or equivalently

$$I_k(D)_x \subseteq \mathfrak{m}_x \quad \text{for } k \geq \left\lceil \frac{n}{m} \right\rceil.$$

The last statement in Corollary 6.2.3 immediately implies the fact that the smoothness of D is precisely characterized by the triviality of all Hodge ideals, or equivalently by the equality between the Hodge and pole order filtrations.

COROLLARY 6.2.6. *Let X be a smooth complex variety of dimension n , and D a reduced effective divisor on X . Then the following are equivalent:*

- (1) D is smooth.
- (2) the Hodge filtration and pole order filtration on $\mathcal{O}_X(*D)$ coincide.
- (3) $I_k(D) = \mathcal{O}_X$ for all $k \geq 0$.
- (4) $I_k(D) = \mathcal{O}_X$ for some $k \geq \frac{n-1}{2}$.

PROOF. We know that if D is smooth, then $I_k(D) = \mathcal{O}_X$ for all k . On the other hand, it follows from Corollary 6.2.3 that if D is singular, then $I_k(D) \neq \mathcal{O}_X$ for all $k \geq \frac{n-1}{2}$. \square

6.3. Topological vanishing

In this section D continues to be a reduced effective divisor in the smooth n -dimensional variety X , but we assume in addition that X is projective. It is often of interest to understand the Hodge theory of the complement $U = X \setminus D$, in particular to study the Hodge filtration on $H^\bullet(U, \mathbf{C})$. Saito showed that there is a close relationship between this Hodge filtration and $F_k \mathcal{O}_X(*D)$, or equivalently the ideals $I_k(D)$; see for instance [Sa3].

LEMMA 6.3.1 ([Sa3, 4.6(ii)]). *For every integer i there is a natural morphism*

$$\varphi_i: H^i(X, \mathrm{DR}(\mathcal{O}_X(*D))) \longrightarrow H^{i+n}(U, \mathbf{C}),$$

*which is a filtered isomorphism. Here the right hand side is endowed with Deligne's Hodge filtration, and the left hand side with Saito's filtration given by the image of $H^i(X, F_\bullet \mathrm{DR}(\mathcal{O}_X(*D)))$.*

The left-hand side in the isomorphism above is the cohomology

$$H^i a_{X*}(\mathcal{O}_X(*D), F)$$

of the filtered direct image, where $a_X: X \rightarrow \mathrm{pt}$. This filtration is strict, and we saw in Example 1.6.3 that this is equivalent to the degeneration at E_1 of the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = \mathbf{H}^{p+q}(X, \mathrm{gr}_{-q}^F \mathrm{DR}(\mathcal{O}_X(*D))) \implies H^{p+q}(X, \mathrm{DR}(\mathcal{O}_X(*D))).$$

As a consequence one obtains

COROLLARY 6.3.2. *For every integer k , the hypercohomology $\mathbf{H}^i(X, \mathrm{gr}_{-q}^F \mathrm{DR}(\mathcal{O}_X(*D)))$ is a subquotient of the singular cohomology $H^{i+n}(U, \mathbf{C})$.*

If we now assume in addition that D is an ample divisor, then U is an affine variety, hence by the Andreotti-Frenkel theorem (see e.g. [La, Theorem 3.1.1]) we have that

$$H^j(U, \mathbf{Z}) = 0 \quad \text{for all } j > n.$$

We deduce:

COROLLARY 6.3.3. *If D is ample, for every $k \in \mathbf{Z}$ we have*

$$\mathbf{H}^i(X, \mathrm{gr}_k^F \mathrm{DR}(\mathcal{O}_X(*D))) = 0 \quad \text{for all } i > 0.$$

For example, we have

$$\mathrm{gr}_{-n}^F \mathrm{DR}(\mathcal{O}_X(*D)) \simeq \omega_X \otimes F_0 \mathcal{O}_X(*D) = \omega_X(D) \otimes I_0(D),$$

hence when D is ample this recovers a special case of the Nadel Vanishing, Theorem 4.5.4, namely

$$(6.3.1) \quad H^i(X, \omega_X(D) \otimes I_0(D)) = 0 \quad \text{for all } i > 0.$$

6.4. Vanishing theorem for Hodge ideals

Multiplier ideals on smooth projective varieties satisfy the celebrated Nadel vanishing theorem; see [La, Theorem 9.4.8]. Recall from Theorem 4.5.4, together with Proposition 6.1.18, that for the ideal $I_0(D)$ this says that given any ample line bundle L , one has

$$H^i(X, \omega_X(D) \otimes L \otimes I_0(D)) = 0 \quad \text{for all } i \geq 1.$$

General vanishing. We next obtain an analogous result for the entire sequence of Hodge ideals $I_k(D)$. Things however necessarily get more complicated; in brief, in order to have full vanishing, higher log-canonicity conditions and borderline Nakano vanishing type properties need to be satisfied.

THEOREM 6.4.1. *Let X be a smooth projective variety of dimension n , D a reduced effective divisor, and L a line bundle on X . Fix $k \geq 1$, and assume that the pair (X, D) is $(k-1)$ -log-canonical, i.e. $I_{k-1}(D) = \mathcal{O}_X$.² Then we have:*

(1) *If $k \leq \frac{n}{2}$, and L is a line bundle such that $L(pD)$ is ample for all $0 \leq p \leq k$, then*

$$H^i(X, \omega_X((k+1)D) \otimes L \otimes I_k(D)) = 0$$

for all $i \geq 2$. Moreover,

$$H^1(X, \omega_X((k+1)D) \otimes L \otimes I_k(D)) = 0$$

holds if $H^j(X, \Omega_X^{n-j} \otimes L((k-j+1)D)) = 0$ for all $1 \leq j \leq k$.

²Note that by Proposition 6.1.17 this is equivalent to $I_0(D) = I_1(D) = \dots = I_{k-1}(D) = \mathcal{O}_X$.

(2) If $k \geq \frac{n+1}{2}$, then D is smooth by Corollary 6.2.6, and so $I_k(D) = \mathcal{O}_X$. In this case, if L is a line bundle such that $L(kD)$ is ample, then

$$H^i(X, \omega_X((k+1)D) \otimes L) = 0 \quad \text{for all } i > 0.$$

(3) If D is ample, then (1) and (2) also hold with $L = \mathcal{O}_X$.

PROOF. The statement for $k \geq \frac{n+1}{2}$, i.e. part (2), is simply an application of Kodaira vanishing. Indeed, D is smooth, and we have a short exact sequence

$$0 \longrightarrow \omega_X \otimes L(kD) \longrightarrow \omega_X \otimes L((k+1)D) \longrightarrow \omega_D \otimes L(kD)|_D \longrightarrow 0.$$

Passing to cohomology, by assumption Kodaira vanishing applies to the two extremes, which implies vanishing for the term in the middle.

We thus concentrate on the case $k \leq \frac{n}{2}$, i.e. part (1). Note first that since $I_{k-1}(D) = \mathcal{O}_X$, we have a short exact sequence

$$0 \rightarrow \omega_X \otimes L(kD) \rightarrow \omega_X \otimes L((k+1)D) \otimes I_k(D) \rightarrow \omega_X \otimes L \otimes \text{gr}_k^F \mathcal{O}_X(*D) \rightarrow 0.$$

Passing to cohomology and using Kodaira vanishing, this implies immediately that the vanishing statements we are aiming for are equivalent to the same vanishing statements for

$$H^i(X, \omega_X \otimes L \otimes \text{gr}_k^F \mathcal{O}_X(*D)).$$

We now consider the complex

$$C^\bullet := (\text{gr}_{-n+k}^F \text{DR}(\mathcal{O}_X(*D)) \otimes L)[-k].$$

Given the hypothesis on the ideals $I_p(D)$, this can be identified with a complex of the form

$$\begin{aligned} & [\Omega_X^{n-k} \otimes L(D) \longrightarrow \Omega_X^{n-k+1} \otimes L \otimes \mathcal{O}_D(2D) \longrightarrow \dots \\ & \dots \longrightarrow \Omega_X^{n-1} \otimes L \otimes \mathcal{O}_D(kD) \longrightarrow \omega_X \otimes L \otimes \text{gr}_k^F \mathcal{O}_X(*D)] \end{aligned}$$

concentrated in degrees 0 up to k . Theorem 4.5.1 implies that

$$(6.4.1) \quad \mathbf{H}^j(X, C^\bullet) = 0 \quad \text{for all } j \geq k+1.$$

We use the spectral sequence

$$E_1^{p,q} = H^q(X, C^p) \implies \mathbf{H}^{p+q}(X, C^\bullet).$$

The vanishing statements we are interested in are for the terms $E_1^{k,i}$ with $i \geq 1$. Note to begin with that $E_1^{k+1,i} = 0$ since $C^{k+1} = 0$. On the other hand,

$$E_1^{k-1,i} = H^i(X, \Omega_X^{n-1} \otimes L \otimes \mathcal{O}_D(kD)),$$

and so we have an exact sequence

$$H^i(X, \Omega_X^{n-1} \otimes L(kD)) \longrightarrow E_1^{k-1,i} \longrightarrow H^{i+1}(X, \Omega_X^{n-1} \otimes L((k-1)D)).$$

Now there are two cases:

(1) If $i \geq 2$, we deduce that $E_1^{k-1,i} = 0$ by Nakano vanishing, and so $E_1^{k,i} = E_2^{k,i}$.

(2) If $i = 1$, using Nakano vanishing we obtain a surjective morphism

$$H^1(X, \Omega_X^{n-1} \otimes L(kD)) \longrightarrow E_1^{k-1,1}.$$

If the extra vanishing hypothesis on the term on the left holds, then we draw the same conclusion as in (1).

We need to analyze in a similar way the terms $E_r^{k,i}$ with $r \geq 2$. On one hand, we always have $E_r^{k+r,i-r+1} = 0$ because of the length of the complex. On the other hand, we will show that under our hypothesis we have $E_1^{k-r,i+r-1} = 0$, from which we infer that $E_r^{k-r,i+r-1} = 0$ as well. Granting this, we obtain

$$E_r^{k,i} = E_{r+1}^{k,i}.$$

Repeating this argument for each r , we finally obtain

$$E_1^{k,i} = E_\infty^{k,i} = 0,$$

where the vanishing follows from (6.4.1), as $i \geq 1$.

We are thus left with proving that $E_1^{k-r,i+r-1} = 0$. If $r > k$ this is clear, since the complex C^\bullet starts in degree 0. If $k = r$, we have

$$E_1^{0,i+k-1} = H^{i+k-1}(X, \Omega_X^{n-k} \otimes L(D)).$$

If $i \geq 2$ this is 0 by Nakano vanishing, while if $i = 1$ it is 0 because of our hypothesis. Finally, if $k \geq r + 1$, we have

$$E_1^{k-r,i+r-1} = H^{i+r-1}(X, \Omega_X^{n-r} \otimes L \otimes \mathcal{O}_D((k-r+1)D)),$$

which sits in an exact sequence

$$\begin{aligned} H^{i+r-1}(X, \Omega_X^{n-r} \otimes L((k-r+1)D)) &\longrightarrow E_1^{k-r,i+r-1} \longrightarrow \\ &\longrightarrow H^{i+r}(X, \Omega_X^{n-r} \otimes L((k-r)D)). \end{aligned}$$

We again have two cases:

- (1) If $i \geq 2$, we deduce that $E_1^{k-r,i+r-1} = 0$ by Nakano vanishing.
- (2) If $i = 1$, using Nakano vanishing we obtain a surjective morphism

$$H^r(X, \Omega_X^{n-r} \otimes L((k-r+1)D)) \longrightarrow E_1^{k-r,i+r-1},$$

and if the extra hypothesis on the term on the left holds, then we draw the same conclusion as in (1).

The proof of (3) is identical, replacing the application of Theorem 4.5.1 by that of Corollary 6.3.3. \square

Vanishing on \mathbf{P}^n and toric varieties. Theorem 6.4.1 takes a nice form on toric varieties, due to the fact that the extra condition on bundles of holomorphic forms is automatically satisfied by the Bott-Danilov-Steenbrink vanishing theorem; this says that for any ample line bundle A on a smooth projective toric variety X one has

$$H^i(X, \Omega_X^j \otimes A) = 0 \quad \text{for all } j \geq 0 \text{ and } i > 0.$$

COROLLARY 6.4.2. *Let D be a reduced effective $(k-1)$ -log-canonical divisor on a smooth projective toric variety X , and let L be a line bundle on X such that $L(pD)$ is ample for all $0 \leq p \leq k$. Then*

$$H^i(X, \omega_X((k+1)D) \otimes L \otimes I_k(D)) = 0 \quad \text{for all } i > 0.$$

If D is ample, the same holds with $L = \mathcal{O}_X$.

REMARK 6.4.3. An even stronger vanishing theorem on toric varieties has been recently proved by Y. Dutta. Will include.

On \mathbf{P}^n however the situation is even better, since one can eliminate the log-canonicity assumption as well. This is due to the existence, for any $j \geq 1$, of the Koszul resolution

$$(6.4.2) \quad 0 \rightarrow \bigoplus \mathcal{O}_{\mathbf{P}^n}(-n-1) \rightarrow \bigoplus \mathcal{O}_{\mathbf{P}^n}(-n) \rightarrow \cdots \rightarrow \bigoplus \mathcal{O}_{\mathbf{P}^n}(-j-1) \rightarrow \Omega_{\mathbf{P}^n}^j \rightarrow 0.$$

THEOREM 6.4.4. *Let D be a reduced hypersurface of degree d in \mathbf{P}^n . If $\ell \geq (k+1)d - n - 1$, then*

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell) \otimes I_k(D)) = 0 \quad \text{for all } i \geq 1.$$

PROOF. If D is $(k-1)$ -log canonical, then this is a special example of Corollary 6.4.2. To see that this condition is not needed, we have to return to the proof of Theorem 6.4.1, and see what happens if we do not assume the triviality of the Hodge ideals up to $I_{k-1}(D)$.

First, for each $k \geq 1$ we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow \omega_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(kd) \otimes I_{k-1}(D) &\longrightarrow \omega_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}((k+1)d) \otimes I_k(D) \longrightarrow \\ &\longrightarrow \omega_{\mathbf{P}^n} \otimes \mathrm{gr}_k^F \mathcal{O}_{\mathbf{P}^n}(*D) \rightarrow 0. \end{aligned}$$

We can then proceed by induction: after twisting by any $L = \mathcal{O}_{\mathbf{P}^n}(a)$ with $a \geq 0$, assuming the vanishing in the statement for the term on the left, if we also have it for the term on the right, we obtain it for the term in the middle. The process can indeed be started, since for $k=1$ vanishing for the term on the left is simply Nadel vanishing.

We therefore need to prove vanishing of the type

$$H^i(\mathbf{P}^n, \omega_{\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(a) \otimes \mathrm{gr}_k^F \mathcal{O}_{\mathbf{P}^n}(*D)) = 0$$

for $i > 0$ and $a \geq 0$. Note first that Saito vanishing applies in the exact same way as in Theorem 6.4.1, in the form of vanishing for $\mathbf{H}^j(\mathbf{P}^n, C^\bullet)$ for $j \geq k+1$. Without any assumptions on the Hodge ideals however, the complex C^\bullet now looks as follows:

$$\begin{aligned} &[\Omega_{\mathbf{P}^n}^{n-k} \otimes \mathcal{O}_{\mathbf{P}^n}(d) \otimes I_0(D) \longrightarrow \Omega_{\mathbf{P}^n}^{n-k+1} \otimes \mathrm{gr}_1^F \mathcal{O}_{\mathbf{P}^n}(*D) \longrightarrow \cdots \\ &\cdots \longrightarrow \Omega_{\mathbf{P}^n}^{n-1} \otimes \mathrm{gr}_{k-1}^F \mathcal{O}_{\mathbf{P}^n}(*D) \longrightarrow \omega_{\mathbf{P}^n} \otimes \mathrm{gr}_k^F \mathcal{O}_X(*D)] \otimes \mathcal{O}_{\mathbf{P}^n}(a) \end{aligned}$$

concentrated in degrees 0 up to k . We use the spectral sequence in the proof of Theorem 6.4.1, and recall that we are interested in the vanishing of the terms $E_1^{k,i}$ with $i \geq 1$. Again, this term is isomorphic to $E_2^{k,i}$ if we have vanishing for the term $E_1^{k-1,i}$, which by definition sits in an exact sequence

$$\begin{aligned} H^i(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^{n-1} \otimes \mathcal{O}_{\mathbf{P}^n}(kd+a) \otimes I_{k-1}(D)) &\longrightarrow E_1^{k-1,i} \longrightarrow \\ &\longrightarrow H^{i+1}(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^{n-1} \otimes \mathcal{O}_{\mathbf{P}^n}((k-1)d+a) \otimes I_{k-2}(D)). \end{aligned}$$

We claim that, using the inductive hypothesis, both extremal terms are equal to 0, which gives what we want. For this we use the short exact sequence

$$0 \rightarrow \bigoplus \mathcal{O}_{\mathbf{P}^n}(-n-1) \rightarrow \bigoplus \mathcal{O}_{\mathbf{P}^n}(-n) \rightarrow \Omega_{\mathbf{P}^n}^{n-1} \rightarrow 0$$

given by the Koszul complex. It shows that to have the vanishing of the term on the left, it is enough to have

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell) \otimes I_{k-1}(D)) = 0 \quad \text{for all } i \geq 1,$$

with $\ell \geq kd - n - 1$, i.e. exactly the inductive hypothesis for $k - 1$. Similarly, it shows that for the vanishing of the term on the right, it is enough to have

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell) \otimes I_{k-2}(D)) = 0 \quad \text{for all } i \geq 1,$$

with $\ell \geq (k-1)d - n - 1$, i.e. the inductive hypothesis for $k - 2$.

Now one needs to analyze the terms $E_r^{k,i}$ with $r \geq 2$. Just as in the proof of Theorem 6.4.1, to show that they are all isomorphic to each other, which leads to the statement of the theorem, it is enough to show that $E_1^{k-r,i+r-1} = 0$ for all such r . When $r > k$ this is clear, while when $r = k$ we have

$$E_1^{0,i+k-1} = H^{i+k-1}(X, \Omega_X^{n-k} \otimes \mathcal{O}_{\mathbf{P}^n}(d+a) \otimes I_0(D)).$$

We again use the Koszul complex (6.4.2) for $j = n - k$. This gives by a simple calculation that it suffices to have

$$H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell) \otimes I_0(D)) = 0 \quad \text{for all } i > 0$$

and all $\ell \geq d - n - 1$, which is Nadel vanishing. When $r < k$, just as before we have an exact sequence

$$\begin{aligned} H^{i+r-1}(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^{n-r} \otimes \mathcal{O}_{\mathbf{P}^n}((k-r+1)d+a) \otimes I_{k-r}(D)) &\longrightarrow E_1^{k-r,i-r+1} \longrightarrow \\ &\longrightarrow H^{i+r}(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^{n-r} \otimes \mathcal{O}_{\mathbf{P}^n}((k-r)d+a) \otimes I_{k-r-1}(D)). \end{aligned}$$

A completely similar use of the Koszul complex (6.4.2), this time for $j = n - r$, together with the inductive hypothesis, implies that $E_1^{k-r,i-r+1} = 0$. \square

Vanishing on abelian varieties. On abelian varieties we can obtain stronger vanishing statements than those in the previous sections. In this paragraph X will always be a complex abelian variety of dimension g , and D a reduced effective ample divisor on X .

LEMMA 6.4.5. *We have*

$$H^i(X, \text{DR}(\mathcal{O}_X(*D)) \otimes \mathbf{C}_\rho) = 0 \quad \text{for all } i > 0,$$

where \mathbf{C}_ρ denotes the rank one local system associated to any $\rho \in \text{Char}(X)$.

PROOF. Denote as always by $j: U \hookrightarrow X$ the inclusion, where $U = X \setminus D$. If we denote by P the perverse sheaf $\text{DR}(\mathcal{O}_X(*D))$, then

$$P \simeq j_* j^* P.$$

The projection formula then gives

$$H^i(X, \text{DR}(\mathcal{O}_X(*D)) \otimes \mathbf{C}_\rho) \simeq H^i(U, j^*(P \otimes \mathbf{C}_\rho)) = 0$$

for all $i > 0$, where the last equality follows by Artin vanishing (see e.g. [Di, Corollary 5.2.18]) since U is affine. \square

THEOREM 6.4.6. *For all $k \geq 0$ the following are true, and equivalent:*

- (1) $H^i(X, \mathcal{O}_X((k+1)D) \otimes I_k(D) \otimes \alpha) = 0$ for all $i > 0$ and all $\alpha \in \text{Pic}^0(X)$.
- (2) $H^i(X, \text{gr}_k^F \mathcal{O}_X(*D) \otimes \alpha) = 0$ for all $i > 0$ and all $\alpha \in \text{Pic}^0(X)$.

PROOF. We proceed by induction on k . In the case $k = 0$, the equivalence is obvious. On the other hand, the result is true by Nadel vanishing, Theorem 4.5.4.

Now for any k we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(kD) \otimes I_{k-1}(D) \otimes \alpha &\longrightarrow \mathcal{O}_X((k+1)D) \otimes I_k(D) \otimes \alpha \longrightarrow \\ &\longrightarrow \text{gr}_k^F \mathcal{O}_X(*D) \otimes \alpha \longrightarrow 0. \end{aligned}$$

Assuming that the result holds for $k-1$, passing to cohomology gives the equivalence between (1) and (2) for k .

We denote by \mathbf{C}_α the unitary rank one local system associated to α . Considering the spectral sequence

$$E_1^{p,q} = \mathbf{H}^{p+q}(X, \text{gr}_{-q}^F \text{DR}(\mathcal{O}_X(*D)) \otimes \alpha) \implies H^{p+q}(X, \text{DR}(\mathcal{O}_X(*D)) \otimes \mathbf{C}_\alpha),$$

precisely as in the proof of Corollary 6.3.3 we obtain

$$\mathbf{H}^i(X, \text{gr}_\ell^F \text{DR}(\mathcal{O}_X(*D)) \otimes \alpha) = 0$$

for all $i > 0$, all ℓ , and all $\alpha \in \text{Pic}^0(X)$, by virtue of Lemma 6.4.5.

We use this with $\ell = -g + k$. More precisely, we look at the complex

$$C^\bullet := (\text{gr}_{-g+k}^F \text{DR}(\mathcal{O}_X(*D)) \otimes \alpha)[-k].$$

Given that Ω_X^1 is trivial, this can be identified with a complex of the form

$$[\bigoplus \text{gr}_0^F \mathcal{O}_X(*D) \otimes \alpha \rightarrow \bigoplus \text{gr}_1^F \mathcal{O}_X(*D) \otimes \alpha \rightarrow \cdots \rightarrow \text{gr}_k^F \mathcal{O}_X(*D) \otimes \alpha]$$

concentrated in degrees 0 up to k . The vanishing above says that

$$\mathbf{H}^j(X, C^\bullet) = 0 \quad \text{for all } j \geq k+1.$$

We use the spectral sequence

$$E_1^{p,q} = H^q(X, C^p) \implies \mathbf{H}^{p+q}(X, C^\bullet).$$

Note that for $i \geq 1$, $E_1^{k+1,i} = 0$ since $C^{k+1} = 0$, while $E_1^{k-1,i} = 0$ by the inductive hypothesis. It follows that

$$E_1^{k,i} \simeq E_2^{k,i}.$$

Continuing this way, for any $r \geq 2$, we have that for $E_r^{k,i}$ the outgoing term is 0 because of the length of the complex, while the incoming term is 0 by induction. The conclusion is that

$$E_1^{k,i} \simeq E_\infty^{k,i}.$$

Since $H^{k+i}(X, C^\bullet) = 0$, we obtain that

$$E_1^{k,i} = H^i(X, \text{gr}_k^F \mathcal{O}_X(*D) \otimes \alpha) = 0.$$

□

REMARK 6.4.7. A similar inductive argument as in Theorem 6.4.6 shows that when D is arbitrary and L is an ample line bundle, one has

$$H^i(X, \mathcal{O}_X((k+1)D) \otimes L \otimes I_k(D)) = 0,$$

and that this is equivalent to the statement

$$H^i(X, \mathrm{gr}_k^F \mathcal{O}_X(*D) \otimes L) = 0,$$

both for all $i > 0$ and all k . However this last statement is already a special case of [PS, §2.3, Lemma 1].

6.5. Applications to hypersurfaces in \mathbf{P}^n

We use Theorem 6.4.4 to give a numerical criterion for the triviality of the ideals $I_k(D)$ when D is a hypersurface in projective space, or to impose restrictions on the corresponding closed subschemes $Z_k \subseteq \mathbf{P}^n$. To put things in context, recall that the log-canonical threshold of a hypersurface D of degree d with isolated singularities in \mathbf{P}^n satisfies

$$\mathrm{lct}(D) \geq \min \left\{ \frac{n}{d}, 1 \right\}$$

(see, for example, [dFEM, Corollary 3.6]). Consequently $I_0(D) = \mathcal{O}_X$, i.e. the pair (X, D) is log-canonical, when $n+1 > d$. More generally, for an arbitrary hypersurface $D \subset \mathbf{P}^n$, a standard application of Nadel vanishing implies that if (X, D) is not log-canonical, then $\dim \mathrm{Sing}(D) \geq n - d + 1$.

The next result generalizes this to $I_k(D)$ with $k \geq 1$. It also extends a result of Deligne, see the remarks after [Sa3, Theorem 0.11], which gives a numerical criterion for the triviality of $I_k(D)$ for hypersurfaces with isolated singularities.

THEOREM 6.5.1. *Let D be a reduced hypersurface of degree d in \mathbf{P}^n , and for each k denote by Z_k the subscheme associated to the ideal $I_k(D)$, and by z_k its dimension. Then:*

- (1) *If $z_k < n - (k+1)d + 1$, then in fact $Z_k = \emptyset$, i.e. (X, D) is k -log-canonical. The converse is of course true if $n - (k+1)d + 1 \geq 0$.*
- (2) *If $z_k \geq n - (k+1)d + 1$, then*

$$\deg Z_k \leq \binom{(k+1)d - 1}{n - z_k},$$

with the convention that $\dim \emptyset$ and $\deg \emptyset$ are -1 .

- (3) *The dimension 0 part of Z_k imposes independent conditions on hypersurfaces of degree at least $(k+1)d - n - 1$.*

PROOF. If Z_k is non-empty, then by intersecting D with a general linear subspace L of \mathbf{P}^n of dimension $n - z_k$, we obtain a reduced hypersurface $D_L \subset L = \mathbf{P}^{n-z_k}$ such that subscheme associated to $I_k(D_L)$ is non-empty and 0-dimensional. Indeed, by the generic

restriction theorem for Hodge ideals,³ Theorem ??, we have that $I_k(D_L) = I_k(D) \cdot \mathcal{O}_L$. Denoting by B the line bundle $\omega_L \otimes \mathcal{O}_L((k+1)D_L)$, there is a short exact sequence

$$0 \longrightarrow B \otimes I_k(D_L) \longrightarrow B \longrightarrow B \otimes \mathcal{O}_{Z(I_k(D_L))} \longrightarrow 0.$$

For (1), the condition $z_k < n - (k+1)d + 1$ is precisely equivalent to

$$H^0(L, B) = 0.$$

On the other hand, we have

$$H^1(L, B \otimes I_k(D_L)) = 0.$$

as a special case of Theorem 6.4.4, and so by passing to cohomology in the exact sequence above we deduce that $Z(I_k(D_L)) = \emptyset$, a contradiction.

For (2), the statement is trivial if $Z_k = \emptyset$, so we can assume that this is not the case. We then have that $Z(I_k(D_L))$ is a 0-dimensional scheme of length $\deg Z_k$. The same argument shows that we have a surjection

$$H^0(L, B) \longrightarrow B \otimes \mathcal{O}_{Z(I_k(D_L))} \longrightarrow 0,$$

and this time the space on the left has dimension $\binom{(k+1)d-1}{n-z_k}$.

Part (3) follows from the fact that, due to the same vanishing theorem, if Z'_k is the 0-dimensional part of Z_k , then for every $\ell \geq (k+1)d - n - 1$, there is a surjection

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell)) \rightarrow \mathcal{O}_{Z'_k}.$$

□

REMARK 6.5.2. Part (1) in the theorem says in particular that if D has isolated singularities, then $I_k(D) = \mathcal{O}_X$ whenever $n - (k+1)d + 1 > 0$; this is a result of Deligne, see [Sa3, 4.6(iii)], that was originally phrased in terms of the equality $F_k H^i(U, \mathbf{C}) = P_k H^i(U, \mathbf{C})$, where $U = \mathbf{P}^n \setminus D$. A related local result was proved by Saito [Sa3, Theorem 0.11] in terms of the roots of the Bernstein-Sato polynomial of D ; in the case of hypersurfaces in \mathbf{P}^n whose singularities are isolated and non-degenerate with respect to the corresponding Newton polyhedra, Saito's result discussed in Remark ?? implies Deligne's theorem mentioned above.

We now exploit part (3) in Theorem 6.5.1, i.e. the fact that the isolated points of Z_k impose independent conditions on hypersurfaces of degree at least $(k+1)d - n - 1$ in \mathbf{P}^n , in conjunction with the nontriviality criteria in §6.2. We assume that $n \geq 3$, when some singularities are naturally detected by appropriate Hodge ideals $I_k(D)$ with $k \geq 1$; the method applies in \mathbf{P}^2 as well, but in this case it is known that the type of results we are aiming for can already be obtained by considering the multiplier ideal $I_0(D)$ or the adjoint ideal $\text{adj}(D)$.

As motivation, recall that when $X \subset \mathbf{P}^3$ is a reduced surface of degree d whose only singularities are nodes, i.e ordinary double points, a classical result of Severi says that

³In class we looked only at the case of isolated singularities, i.e. $z_k \leq 0$, in which there is no need to appeal to this theorem.

the set of nodes on X imposes independent conditions on hypersurfaces of degree at least $2d - 5$ in \mathbf{P}^3 . Park and Woo [PW] showed that in fact this holds replacing the set of nodes by that of all singular points, and gave similar bounds for isolated singular points on hypersurfaces in arbitrary \mathbf{P}^n . The use of the Hodge ideals $I_k(D)$ for suitable k leads to further results in any dimension.

COROLLARY 6.5.3. *Let D be a reduced hypersurface of degree d in \mathbf{P}^n , with $n \geq 3$, and denote by S_m the set of isolated singular points on D of multiplicity $m \geq 2$. Then S_m imposes independent conditions on hypersurfaces of degree at least $(\lfloor \frac{n}{m} \rfloor + 1)d - n - 1$.*

PROOF. We know from Corollary 6.2.3 (see also Remark 6.2.5) that

$$I_k(D) \subseteq I_S,$$

where $k = \lfloor \frac{n}{m} \rfloor$. Since S is a set of isolated points, the result then follows from Theorem 6.5.1 (3), which says that there is a surjection

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}((k+1)d - n - 1)) \rightarrow \mathcal{O}_{Z'_k} \rightarrow 0,$$

where Z'_k is the 0-dimensional part of Z_k . \square

When $n = 3$ and $m = 2$ for instance, the bound is one worse than the Severi bound, but at least when $n \geq 5$ and $m \geq 3$, in many instances this seems to lead to the best known bounds.

6.6. Applications to singularities of theta divisors

A well-known result of Kollár [Ko4, Theorem 17.3], revisited by Ein-Lazarsfeld [EL], states that if (A, Θ) is a principally polarized abelian variety (ppav) of dimension g , then the pair (A, Θ) is log-canonical, and in particular $\text{mult}_x(\Theta) \leq g$ for any $x \in \Theta$. By a result of Smith-Varley, it is also known that when the multiplicity is equal to g , the ppav must be reducible; for this and related results, it is useful to consult [EL] and the references therein.

For irreducible ppav's it is believed however that the situation should be substantially better. One has the following folklore:

CONJECTURE 6.6.1. *Let (A, Θ) be an irreducible ppav of dimension g . Then*

$$\text{mult}_x(\Theta) \leq \frac{g+1}{2}, \quad \text{for all } x \in \Theta.$$

Following [MP1, Theorem 29.2], as an application of the study of Hodge ideals we show that the conjecture holds for theta divisors with isolated singularities. One thing to note is that while the result of [Ko4], and the main tool in [EL], is the triviality of the multiplier ideal $I_0(\Theta)$, here we rely in turn on the *non-triviality* of $I_1(\Theta)$.

THEOREM 6.6.2. *Let (X, Θ) be an irreducible ppav of dimension g , such that Θ has isolated singularities. Then:*

- (1) *For every $x \in \Theta$ we have $\text{mult}_x(\Theta) \leq \frac{g+1}{2}$.*

(2) Moreover, there can be at most one $x \in \Theta$ such that $\text{mult}_x(\Theta) = \frac{g+1}{2}$.

PROOF. Note in passing that for $g \geq 3$ irreducibility follows automatically from the assumption on isolated singularities. Indeed, if (A, Θ) splits as a product of ppav's, then we have $\dim \text{Sing}(\Theta) = g - 2$. For $g = 2$ it is necessary to assume it, as the bound fails for a product of elliptic curves.

Assuming that $\text{mult}_x(\Theta) \geq \frac{g+2}{2}$, by Theorem 6.2.1 (see also Example 6.2.2) we obtain that

$$I_1(\Theta) \subseteq \mathfrak{m}_x^2.$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(2\Theta) \otimes I_1(\Theta) \longrightarrow \mathcal{O}_X(2\Theta) \otimes \mathfrak{m}_x^2 \longrightarrow \mathcal{O}_X(2\Theta) \otimes \mathfrak{m}_x^2 / I_1(\Theta) \longrightarrow 0.$$

For every $\alpha \in \text{Pic}^0(X)$, by tensoring the exact sequence with α and using (1) in Theorem 6.4.6 and the fact that $I_1(\Theta)$ has finite co-support, we obtain

$$H^i(X, \mathcal{O}_X(2\Theta) \otimes \mathfrak{m}_x^2 \otimes \alpha) = 0 \quad \text{for all } i > 0.$$

In particular, as $\mathcal{O}_X(2\Theta)$ is globally generated, the vanishing of H^1 implies that the linear system $|2\Theta|$ separates tangent vectors at each point of X . To see this, note that the collection of line bundles $\mathcal{O}_X(2\Theta) \otimes \alpha$ is, as α varies in $\text{Pic}^0(X)$, the same as the collection of line bundles $t_a^* \mathcal{O}_X(2\Theta)$ as a varies in X , where t_a denotes translation by a . But this is a contradiction; indeed, it is well known that when Θ is irreducible this linear system provides a $2 : 1$ map which is ramified at the 2-torsion points (and more precisely factors through the Kummer variety of X). This proves (1).

For (2), assume that there are two distinct points $x, y \in \Theta$ having multiplicity $\frac{g+1}{2}$. According again to Example 6.2.2, it follows that

$$I_1(\Theta) \subseteq \mathfrak{m}_x \otimes \mathfrak{m}_y.$$

Using the same argument as in (1), we obtain

$$H^i(X, \mathcal{O}_X(2\Theta) \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y \otimes \alpha) = 0 \quad \text{for all } i > 0,$$

and therefore conclude that the linear system $|2\Theta|$ separates all points of the form $x - a$ and $y - a$ with $a \in A$. Note however that the equation

$$x - a = a - y$$

does have solutions, which contradicts the fact that $|2\Theta|$ does not separate nonzero points of the form z and $-z$ (both mapping to the same point on the Kummer variety). \square

REMARK 6.6.3. (1) The conjecture is fully known in dimension up to five, and more generally for Prym varieties associated to double covers of irreducible stable curves; see [Ca, Theorem 3].

(2) Mumford [Mum] showed, developing ideas of Andreotti-Mayer, that the locus N_0 of ppav's such that $\text{Sing}(\Theta) \neq \emptyset$ is a divisor in the moduli space of ppav's, and moreover that for the general point in every irreducible component of N_0 , Θ has isolated singularities. Thus Theorem 6.6.2 applies on a dense open set of each component of N_0 . These open

sets are in fact large: Ciliberto-van der Geer [CvdG] have shown that their complements have codimension at least two in N_0 .

(3) Equality in Conjecture 6.6.1 is known to be achieved for certain points on Jacobians of hyperelliptic curves and on the intermediate Jacobian of the cubic threefold. The latter example also shows optimality in Theorem 6.6.2: the theta divisor on the intermediate Jacobian of a smooth cubic threefold (a ppav of dimension 5) has a unique singular point, the origin, which is of multiplicity 3. Note also that in this case we have

$$I_1(\Theta)_0 = \mathfrak{m}_0 \subsetneq \mathcal{O}_{X,0}$$

according to Example ??, as the projectivized tangent cone to Θ at 0 is isomorphic to the original cubic threefold.

A similar argument involving the higher ideals $I_k(D)$ can be used to give bounds on multiplicity in terms of effective jet separation. In particular, it leads to an asymptotic bound in terms of the Seshadri constant. Given $x \in X$, we denote by $s(\ell, x)$ the largest integer s such that the linear system $|\ell\Theta|$ separates s -jets at x , i.e. such that the restriction map

$$H^0(X, \mathcal{O}_X(\ell\Theta)) \longrightarrow H^0(X, \mathcal{O}_X(\ell\Theta) \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1})$$

is surjective. It is a fundamental property of the Seshadri constant of Θ at x that

$$(6.6.1) \quad \frac{s(\ell, x)}{\ell} \leq \varepsilon(\Theta, x),$$

and that in fact $\varepsilon(\Theta, x)$ is the limit of these quotients as $\ell \rightarrow \infty$; see [La, Theorem 5.1.17] and its proof. Due to the homogeneity of X , $\varepsilon(\Theta, x)$ does not in fact depend on x , so we denote it $\varepsilon(\Theta)$. We denote also

$$s_\ell := \min \{s(\ell, x) \mid x \in X\}.$$

THEOREM 6.6.4. *Let (X, Θ) be ppav of dimension g , such that Θ has isolated singularities. Then for every $x \in \Theta$ and every $k \geq 1$ we have*

$$\text{mult}_x(\Theta) < 2 + \frac{s_{k+1}}{k+1} + \frac{g}{k+1}.$$

In particular, for every $x \in \Theta$ we have

$$\text{mult}_x(\Theta) \leq \varepsilon(\Theta) + 2 \leq \sqrt[g]{g!} + 2.$$

Hence, if $g \gg 0$, then $\text{mult}_x(\Theta)$ is at most roughly $\frac{g}{e} + 2$.

PROOF. We assume that

$$\text{mult}_x(\Theta) \geq 2 + \frac{s_{k+1}}{k+1} + \frac{g}{k+1}$$

and aim for a contradiction. Under this assumption, according to Corollary ?? it follows that

$$I_k(D) \subseteq \mathfrak{m}_x^{2+s_{k+1}}.$$

An argument identical to that in Theorem 6.6.2 then shows that for all $\alpha \in \text{Pic}^0(X)$ one has

$$H^i(X, \mathcal{O}_X((k+1)\Theta) \otimes \mathfrak{m}_x^{2+s_{k+1}} \otimes \alpha) = 0 \quad \text{for all } i > 0,$$

and so the linear system $|(k+1)\Theta|$ separates $(1+s_{k+1})$ -jets at all points of X , a contradiction.

The statement $\text{mult}_x(\Theta) \leq \varepsilon(\Theta)+2$ now follows using (6.6.1) and letting $k \rightarrow \infty$. On the other hand, the definition of the Seshadri constant automatically implies $\varepsilon(\Theta) \leq \sqrt[g]{g!}$; see [La, Proposition 5.1.9]. The last assertion follows from the well-known fact that $\lim_{g \rightarrow \infty} \frac{\sqrt[g]{g!}}{g} = \frac{1}{e}$. \square

REMARK 6.6.5. (1) The last assertion in Theorem 6.6.4 becomes better than that given by Theorem 6.6.2 for g very large. Codogni-Grushevsky-Sernesi [CGS] have shown a similar, but slightly stronger statement, using methods from intersection theory. In [CGS] it is shown that if m is the multiplicity of an isolated point on Θ , then

$$m(m-1)^{g-1} \leq g! - 4.$$

(2) There is more recent related work of Lozovanu [Lo], including some improvements. Will add.

(3) Theorem 6.6.4 holds, with the same proof, for any ample divisor D with isolated singularities on an abelian variety, replacing $g!$ with D^g .

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