CHAPTER 6. MOTIVIC INTEGRATION

Contents

In this chapter we introduce the motivic measure and motivic integrals on smooth varieties, following an introductory section describing the Grothendieck ring of varieties and universal Euler characteristics. We prove the Change of Variables Formula, and use it to deduce the fact that K-equivalent varieties have the same Hodge numbers (all of these results are due to Kontsevich). The main references I will use are the lecture notes [Bl] and [La]. Other useful sources that I will use at times are [Cr], [Lo], [Mu] and [Ve].

1. Grothendieck ring and generalized Euler characteristics

Kontsevich's idea was to replace the Haar measure from the case of p-adic integration with a measure taking values in the Grothendieck ring of all varieties over the complex numbers.

Definition 1.1. The *Grothendieck group* $K_0(\mathrm{Var}_\mathbf{C})$ of complex algebraic varieties is the group generated by classes $[X]$ associated to each such variety, subject to the relations

- $[X] = [Y]$ if X and Y are isomorphic

- $[X] = [Z] + [X - Z]$ if $Z \subset X$ is a closed subset.

It is also called the *Grothendieck ring* if in addition one introduces the multiplication operation

$$[X] \cdot [Y] = [X \times Y].$$

The unit element for addition is $0 = [\emptyset]$, and for multiplication $1 = [\mathrm{pt}]$. It is also convenient to formally introduce the symbol

$$\mathbb{L} := [\mathbf{A}^1].$$

Example 1.2. By inductively using the usual affine cover of $\mathbf{P}^n$, we get

$$[\mathbf{P}^n] = 1 + [\mathbf{A}^1] + \ldots + [\mathbf{A}^n] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n.$$

(Note that this is analogous to the formula $|\mathbf{P}^n(\mathbf{F}_q)| = 1 + q + \ldots + q^n$, analogy which will be developed further.)

Example 1.3 (Constructible sets)**.** Every constructible subset C of an algebraic variety X defines a class in $K_0(\mathrm{Var}_{\mathbf{C}})$. Indeed, write

$$C = \coprod_{i=1}^{k} S_i$$

where S_i are disjoint locally closed subsets of X. We can the write each S_i as $S_i = Y_i - Z_i$ with $Y_i, Z_i \subset X$ closed. This means we can define

$$[S_i] = [Y_i] - [Z_i], \quad i = 1, \ldots, k$$

and then

$$[C] = \sum_{i=1}^{k} [S_i] = \sum_{i=1}^{k} [Y_i] - \sum_{i=1}^{k} [Z_i].$$

Exercise 1.4. Show that $K_0(\mathrm{Var}_{\mathbf{C}})$ is generated by the classes of smooth varieties. Even better, show that it is generated by the classes of smooth quasi-projective varieties. (See Theorem 1.18 below for a stronger statement.)

Exercise 1.5. Let $f : X \to Y$ be a piecewise trivial fibration with fiber F. Show that $[X] = [Y] \cdot [F]$.

The ring $K_0(\mathrm{Var}_{\mathbf{C}})$ is quite hard to understand and use directly; for instance, it is very difficult to know when two classes are equal. The main way it is used is via homomorphisms

$$K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow R$$

where R is some more familiar ring. In the rest of this section we will study a few important such homomorphisms, which come from the topology or Hodge theory of algebraic varieties and are sometimes called *generalized Euler characteristics*. The main difficulty is that by virtue of the very definition of the Grothendieck ring, one needs to deal with the behavior of natural invariants on varieties which are not necessarily compact.

Euler characteristics. Let X be an algebraic variety over $\mathbf{C}$. We denote by $H_c^*(X, \mathbf{Q})$ the rational singular cohomology of X with compact support. Some properties of this cohomology theory are as follows (see [Ha] Ch.3 or [Fu] §22.c and §24.c):

• If X is compact, then $H_c^*(X, \mathbf{Q}) = H^*(X, \mathbf{Q})$.

• For each i one has an isomorphism

$$H_c^i(X, \mathbf{Q}) \simeq \varinjlim_{K} H^i(X, X - K; \mathbf{Q}),$$

where the limit is taken over all compact subsets $K \subset X$.

- If X is smooth, than the real version $H_c^*(X, \mathbf{R})$ can be computed as the de Rham cohomology defined by smooth forms with compact support.

- If $Y \subset X$ is a Zariski closed subset and $U = X - Y$, then there exists a long exact sequence

(1) $$\ldots \longrightarrow H_c^i(U, \mathbf{Q}) \longrightarrow H_c^i(X, \mathbf{Q}) \longrightarrow H_c^i(Y, \mathbf{Q}) \longrightarrow H_c^{i+1}(U, \mathbf{Q}) \longrightarrow \ldots$$

- (Künneth formula.) For any two varieties X, Y and any i one has a natural isomorphism

(2) $$H_c^i(X \times Y, \mathbf{Q}) \simeq \bigoplus_{k+l=i} H_c^k(X, \mathbf{Q}) \otimes H_c^l(Y, \mathbf{Q}).$$

Definition 1.6. The *compactly supported Euler characteristic* of X is
$$\chi_c(X) := \sum_{i \geq 0} (-1)^i \dim H_c^i(X, \mathbf{Q}).$$

This Euler characteristic provides the first example of a ring homomorphism of the type mentioned above.

Lemma 1.7. *There is a ring homomorphism*
$$\chi_c : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}, \quad [X] \mapsto \chi_c(X).$$

Proof. The fact that χ_c is well defined on $K_0(\mathrm{Var}_{\mathbf{C}})$, i.e. that
$$\chi_c(X) = \chi_c(Y) + \chi_c(U)$$
whenever one has a decomposition $X = Y \cup U$ as above, follows from (1). The fact that it is a ring homomorphism follows from the Künneth formula (2). $\square$

We are however mostly interested in the usual Euler characteristic
$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathbf{Q}) = \sum_{i \geq 0} (-1)^i b_i(X),$$
even in the non-compact case. It turns out though that this is the same as the compactly supported one; this is a slightly deeper result.

Theorem 1.8. *If X is a complex algebraic variety, then $\chi_c(X) = \chi(X)$.*

Proof. (Sketch.) In general, if X is an oriented real manifold of dimension m, Poincaré duality in the non-compact case states that
$$H_c^i(X, \mathbf{Q}) \simeq H_{m-i}(X, \mathbf{Q}).$$
In particular, this quickly takes care of the case of smooth complex varieties.

Let now X be an arbitrary complex algebraic variety, and denote by Y its singular locus. We can take a resolution of singularities $\pi : X' \to X$ such that π is an isomorphism over $U = X - Y$, and let $E = \pi^{-1}(Y)$. From the smooth case we know
$$\chi(X') = \chi_c(X') \quad \text{and} \quad \chi(U) = \chi_c(U).$$

By induction on dimension we can also assume $\chi(Y) = \chi_c(Y)$ and $\chi(E) = \chi_c(E)$. Therefore we have
$$\chi_c(X) = \chi_c(U) + \chi_c(Y) = \chi(U) + \chi(Y),$$
so it is enough to show that this last sum is equal to $\chi(X)$. Note also that the above implies that we know the identity

(3) $$\chi(X') = \chi(U) + \chi(E).$$

To show the required identity, one uses the existence of a small classical smooth neighborhood of $Y \subset N$ such that

• N deformation retracts onto Y

• $\pi^{-1}(N)$ deformation retracts onto E.[1]

Now an application of the Mayer-Vietoris sequence on X gives
$$\chi(X) = \chi(U) + \chi(N) - \chi(N - Y),$$
and $\chi(N) = \chi(Y)$, so it is enough to prove that $\chi(N - Y) = 0$. This can be checked on X': note that $N - Y \simeq \pi^{-1}(N) - E$, while the Mayer-Vietoris sequence on X' gives
$$\chi(X') = \chi(U) + \chi(\pi^{-1}(N)) - \chi(\pi^{-1}(N) - E).$$
Using (3) and the fact that $\pi^{-1}(N)$ retracts to E, one obtains the desired conclusion. $\qquad\square$

Corollary 1.9. *There is a ring homomorphism*
$$\chi : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}, \ \ [X] \mapsto \chi(X).$$

Example 1.10. This result is not true if we leave the world of complex varieties. For instance, take $X = S^1$, so that $X = \mathbf{R} \cup \mathrm{pt}$. We have $\chi(S^1) = 0$, while $\chi(\mathbf{R}) = \chi(\mathrm{pt}) = 1$.

Virtual Poincaré polynomial. Using work of Deligne on Hodge theory, one can go one step further with respect to the previous example, and take into account the behavior of the individual Betti numbers with respect to the Grothendieck ring.

Theorem 1.11 (Deligne). *To every quasi-projective complex algebraic variety X, one can associate a* virtual Poincaré *polynomial*
$$P_X(t) \in \mathbf{Z}[t]$$
satisfying the following properties:

(i) If X is smooth and projective, then
$$P_X(t) = \sum_{i \geq 0} b_i(X) \cdot t^i.$$

(ii) If $Y \subset X$ is a closed subset, and $U = X - Y$, then
$$P_X(t) = P_Y(t) + P_U(t).$$

(iii) $P_{X \times Y}(t) = P_X(t) \cdot P_Y(t)$.

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[1]The first condition follows since one can triangulate X using Y as a subcomplex. The second follows from the first and the properness of π.

Corollary 1.12. *There is a ring homomorphism*

$$P : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}[t], \ \ [X] \mapsto P_X(t).$$

Proof. We use the fact that by Exercise 1.4 it is enough to define P on quasi-projective varieties, in which case we can use (ii) and (iii) in the Theorem above. $\square$

Remark 1.13. (1) Properties (i) and (ii) in the theorem uniquely determine the polynomial $P_X(t)$ for each X. Indeed, by induction on dimension, it is enough to characterize $P_U(t)$ with U smooth. General resolution of singularities shows that we can embed U as an open set in a smooth projective variety X, such that

$$X - U = \sum E_i \text{ is an SNC divisor.}$$

Using (ii) have

$$P_U(t) = P_X(t) - \sum P_{E_i}(t) + \sum P_{E_i \cap E_j}(t) - \ldots$$

which determines P_U by (i).

(2) As with the Euler characteristic, no such polynomial can exist for real varieties. Again consider the case of $S^1 = \mathbf{R} \cup \mathrm{pt}$. If P existed, we'd have

$$P_{S^1}(t) = 1 + t \ \text{ and } \ P_{\mathrm{pt}}(t) = 1,$$

which would then force $P_{\mathbf{R}}(t) = t$. But we can also consider two points $p_1, p_2 \in S^1$ (think the north and south pole), and the new decomposition $S^1 = \mathbf{R} \cup \mathbf{R} \cup p_1 \cup p_2$ then leads to a contradiction.

Remark 1.14. The proof of Theorem 1.11 uses Deligne's construction of a *weight filtration* on $H^*(X, \mathbf{Q})$, a first step towards mixed Hodge theory. I am planning to include this at some point.

Virtual Hodge polynomial. Further work of Deligne shows that we can refine the picture above to take into account the Hodge numbers as well. His result is the following:

Theorem 1.15 (Deligne). *To every quasi-projective complex algebraic variety X, one can associate a* virtual Hodge polynomial

$$H_X(u, v) \in \mathbf{Z}[u, v]$$

satisfying the following properties:

(i) If X is smooth and projective, then

$$H_X(u, v) = \sum_{p,q \geq 0} (-1)^{p+q} \cdot h^{p,q}(X) \cdot u^p v^q.$$

(ii) If $Y \subset X$ is a closed subset, and $U = X - Y$, then

$$H_X(u, v) = H_Y(u, v) + H_U(u, v).$$

(iii) $H_{X \times Y}(u, v) = H_X(u, v) \cdot H_Y(u, v).$

Exactly as in the case of the virtual Poincaré polynomial, we obtain:

Corollary 1.16. *There is a ring homomorphism*

$$H : K_0(\mathrm{Var}_\mathbf{C}) \longrightarrow \mathbf{Z}[u, v], \quad [X] \mapsto H_X(u, v).$$

Remark 1.17. Continuing Remark 1.14, the theorem above follows Deligne's result saying that one can endow the complex cohomology with compact support of every complex quasi-projective variety with a natural mixed Hodge structure (extending the usual Hodge theory for smooth projective varieties): the associated graded pieces of the weight filtration on $H^*(X, \mathbf{Q})$ carry after complexification natural Hodge filtrations. Again, I am planning to include a discussion at some point.

There is an alternative approach to the existence of the virtual Poincaré and Hodge polynomials, which avoids mixed Hodge theory, but is based on another deep result, namely the Weak Factorization Theorem for birational maps [AKMW]. Using this result, Bittner [Bi] was able to give the following simpler description of the Grothendieck ring of varieties.

Theorem 1.18 (Bittner)**.** *Let $K_0'(\mathrm{Var}_\mathbf{C})$ be the ring generated by isomorphism classes of smooth projective varieties, subject to the following relations: if $Y \subset X$ is a smooth projective subvariety, then*

$$[X] - [Y] = [\mathrm{Bl}_Y(X)] - [E],$$

where $\mathrm{Bl}_Y(X)$ is the blow-up of X along Y and E is its exceptional divisor. Then the natural ring homomorphism

$$K_0'(\mathrm{Var}_\mathbf{C}) \longrightarrow K_0(\mathrm{Var}_\mathbf{C}), \quad [X] \mapsto [X]$$

is an isomorphism.

Exercise 1.19. Using the theorem above, define P_X and H_X by reducing the arbitrary case to the smooth projective case, where they are well understood.

2. Motivic measure and integrals on smooth varieties

To understand the origins of the motivic measure, let's continue the parallel with the p-adic situation started at the beginning of Ch.V.

(1) We have $\mathbf{A}^n(\mathbf{Z}_p)(= \mathbf{Z}_p^n)$, together with a reduction mod p^m map

$$\psi_m : \mathbf{A}^n(\mathbf{Z}_p) \longrightarrow \mathbf{A}^n(\mathbf{Z}/p^m\mathbf{Z}).$$

On $\mathbf{A}^n(\mathbf{Z}_p)$ we have the Haar measure μ, normalized such that

$$\mu(\mathbf{A}^n(\mathbf{Z}_p)) = 1 = \frac{p^n}{|\mathbf{A}^n(\mathbf{Z}/p\mathbf{Z})|}.$$

(2) For any set $C_m \subset \mathbf{A}^n(\mathbf{Z}/p^{m+1}\mathbf{Z})$ we have

$$(4) \qquad \mu(\psi_m^{-1}(C_m)) = \frac{|C_m|}{p^{n(m+1)}} = \frac{|C_m|}{|\mathbf{A}^n(\mathbf{Z}/p^{m+1}\mathbf{Z})|}.$$

In practice we used only the measure of subsets $C \subset \mathbf{A}^n(\mathbf{Z}_p)$ of the form $C = \psi_m^{-1}(C_m)$ as above. (This, assuming we know things are well defined, means that we could have even forgotten about the Haar measure and just used formula (4) directly.)

Extrapolating from the case of affine space, in the arc space situation we have

(1') For any $\mathbf{C}$-scheme X, the arc space X_∞ and jet schemes X_m, with truncation maps

$$\psi_m : X_\infty \longrightarrow X_m.$$

We look for a measure μ on X_∞, normalized such that $\mu(\mathbf{A}^n_\infty) = 1$.

(2') The role of sets C_m as above is played by constructible subsets $C_m \subset X_m$, while that of C by the cylinders $C = \psi_m^{-1}(C_m)$. By analogy we look for a formula of the type

$$\mu(C) = \frac{|C_m|}{|(\mathbf{A}^n)_{m+1}|},$$

only this time the "number of points" does not make sense any more, and the symbol $|\cdot|$ will have to stand for something else (namely the class in the Grothendieck ring). This in particular leads to considering the following:

Definition 2.1. The ring $\mathcal{M}_{\mathbf{C}}$ is the localization of $K_0(\mathrm{Var}_{\mathbf{C}})$ in the class $\mathbb{L}$, i.e.

$$\mathcal{M}_{\mathbf{C}} := K_0(\mathrm{Var}_{\mathbf{C}})[\mathbb{L}^{-1}].$$

Let now X be a smooth complex variety of dimension n. As a preliminary step, we define the measure of a cylinder set in X_∞ as an element in $\mathcal{M}_{\mathbf{C}}$.[2]

Definition 2.2. Let $C \subset X_\infty$ be a cylinder, written as $C = \psi_m^{-1}(C_m)$, with $C_m \subset X_m$ a constructible set. The *motivic measure* of C is

$$\mu(C) := \frac{[C_m]}{\mathbb{L}^{n(m+1)}} \in \mathcal{M}_{\mathbf{C}}.$$

This is well defined: indeed, say $C = \psi_k^{-1}(C_k)$ as well, with $k > m$, and $C_k = (\pi_m^k)^{-1}(C_m)$. Then via π_m^k, C_k is an $\mathbf{A}^{n(k-m)}$-bundle over C_m, which in view of Exercise 1.5 gives

$$[C_k] = [C_m] \cdot \mathbb{L}^{n(k-m)}.$$

We also define the motivic measure of X as

$$\mu(X) := \mu(X_\infty).$$

Since $X_\infty = \psi_0^{-1}(X)$, we have

$$\mu(X) = \frac{[X]}{\mathbb{L}^n}.$$

In particular, we have $\mu(\mathbf{A}^n) = 1$.

The ring $\mathcal{M}_{\mathbf{C}}$ is a first step, but it is not yet the suitable answer for fully defining a measure. The problem is that one cannot take limits in it. To understand the convergence problem a bit better, and motivate the definition of motivic integrals at the same time,

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[2]so something that typically looks like $[V]/\mathbb{L}^k$, where V is a variety.

let's go through a calculation analogous to that for p-adic integrals. Given a polynomial $f \in \mathbf{Z}_p[X_1, \ldots, X_n]$, we have computed

$$\int_{\mathbf{Z}_p^n} |f| d\mu = \sum_{m=0}^{\infty} \mu(\{x \mid |f(x)|_p = p^{-m}\}) \cdot p^{-m} = \sum_{m=0}^{\infty} \mu(\mathrm{ord}_f^{-1}(m)) \cdot \frac{1}{|\mathbf{A}^m(\mathbf{Z}/p\mathbf{Z})|}.^{[3]}$$

Now given $f \in \mathbf{C}[X_1, \ldots, X_n]$, or the divisor $D = (f = 0) \subset \mathbf{A}^n$, we can try to define completely analogously

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_f} := \sum_{m=0}^{\infty} \mu(\mathrm{ord}_f^{-1}(m)) \cdot \frac{1}{\mathbb{L}^m}.$$

However the sum on the right cannot usually make sense in the ring $\mathcal{M}_{\mathbf{C}}$. We need to complete this ring in order to be able to talk about the (possible) convergence of such an infinite sum. Let's be even more concrete: we have seen that essentially by definition

$$\mathrm{Cont}^m(D) = \mathrm{ord}_f^{-1}(m) = \psi_{m-1}^{-1}(D_{m-1}) - \psi_m^{-1}(D_m).$$

Using the definition above, we then must have

$$\mu\left(\mathrm{ord}_f^{-1}(m)\right) = \frac{[D_{m-1}]}{\mathbb{L}^{nm}} - \frac{[D_m]}{\mathbb{L}^{n(m+1)}}.$$

In particular, we need to deal with the convergence of infinite sums whose general term is of the form $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$. Consider the following notion: the *virtual dimension* of a class $\frac{[V]}{\mathbb{L}^k} \in \mathcal{M}_{\mathbf{C}}$ is $\dim V - i$. We would like to arrange that classes in $\mathcal{M}_{\mathbf{C}}$ become "small" when their virtual dimension becomes very negative. This is motivated in part by the following general fact, which we assume for now:

Theorem 2.3. *Let $Z \subset X$ be any closed subscheme. Then there exists a constant $C = C_Z > 0$ such that*

$$\mathrm{codim}_{X_{m-1}} Z_{m-1} \geq C \cdot m \quad \text{for} \quad m \gg 0.$$

Using this fact, note that the virtual dimension of $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$ is at most $-C \cdot m$, tending to $-\infty$ as $m \to \infty$.

The solution to this convergence problem is to pass to a completion of $\mathcal{M}_{\mathbf{C}}$ with respect to the following filtration induced by the virtual dimension. Define for each k the subgroup $F^k \mathcal{M}_{\mathbf{C}}$ spanned by the classes $\frac{[V]}{\mathbb{L}^i}$ with $\dim V - i \leq -k$. We have

$$\cdots \subset F^k \mathcal{M}_{\mathbf{C}} \subset F^{k-1} \mathcal{M}_{\mathbf{C}} \subset \cdots \quad \text{and} \quad F^k \mathcal{M}_{\mathbf{C}} \cdot F^l \mathcal{M}_{\mathbf{C}} \subset F^{k+l} \mathcal{M}_{\mathbf{C}}$$

so as usual this filtration defines a topology on $\mathcal{M}_{\mathbf{C}}$.

Definition 2.4. We define

$$\widehat{\mathcal{M}_{\mathbf{C}}} = \varprojlim_m \mathcal{M}_{\mathbf{C}}/F^k \mathcal{M}_{\mathbf{C}}$$

i.e. the completion of $\mathcal{M}_{\mathbf{C}}$ with respect to the filtration $F^\bullet \mathcal{M}_{\mathbf{C}}$.

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[3]Recall that $|f(x)|_p = p^{-m}$ is equivalent to $\mathrm{ord}_p(f(x)) = m$. I denoted by ord_f the function $\mathrm{ord}_f(x) := \mathrm{ord}_p(f(x))$, to make it look like the order function in the arc case.

In concrete terms, given a sequence of elements

$$\alpha_p = \frac{[V_p]}{\mathbb{L}^{i_p}} \in \mathcal{M}_{\mathbf{C}},$$

in $\widehat{\mathcal{M}_{\mathbf{C}}}$ we have

$$\alpha_p \to 0 \iff \dim V_p - i_p \to -\infty.$$

Exercise 2.5. Show that $\sum_n \alpha_n$ converges in $\widehat{\mathcal{M}_{\mathbf{C}}}$ if and only if $\alpha_n \to 0$ when $n \to \infty$.

For instance, by what we said earlier, the sum with terms $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$ considered above makes sense in $\widehat{\mathcal{M}_{\mathbf{C}}}$.

Remark 2.6. It is not known at the moment whether the natural map $\mathcal{M}_{\mathbf{C}} \to \widehat{\mathcal{M}_{\mathbf{C}}}$ is injective. Note that

$$\mathrm{Ker}\left(\mathcal{M}_{\mathbf{C}} \longrightarrow \widehat{\mathcal{M}_{\mathbf{C}}}\right) = \bigcap_{m \geq 0} F^m \mathcal{M}_{\mathbf{C}}.$$

Exercise 2.7. Show that in $\widehat{\mathcal{M}_{\mathbf{C}}}$ one has for each p:

$$\sum_{i=1}^{\infty} \mathbb{L}^{-pi} = \frac{1}{1 - \mathbb{L}^{-p}}.$$

We are now ready to give the definition of a measurable function with respect to the motivic measure, and of the associated motivic integral.

Definition 2.8. Let X be a smooth complex variety. Let

$$F : X_\infty \longrightarrow \mathbf{N} \cup \{\infty\}$$

be a function such that

• $F^{-1}(m)$ is a cylinder for every $m \in \mathbf{N}$.

• $F^{-1}(\infty)$ has measure 0 (i.e. it is the intersection of a decreasing family of cylinder sets whose measure tends to 0 in $\widehat{\mathcal{M}_{\mathbf{C}}}$).

Such a function is called *measurable*. The *motivic integral* of a measurable function F is defined as

$$\int_{X_\infty} \mathbb{L}^{-F} = \sum_{m=0}^{\infty} \mu\left(F^{-1}(m)\right) \cdot \frac{1}{\mathbb{L}^m},$$

provided the right-hand-side converges in $\widehat{\mathcal{M}_{\mathbf{C}}}$. In particular, if $Z \subset X$ is a closed subscheme we can consider $F = \mathrm{ord}_Z$, and

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_Z} = \sum_{m=0}^{\infty} \mu\left(\mathrm{Cont}^m(Z)\right) \cdot \frac{1}{\mathbb{L}^m}.$$

Exercise 2.9. Let $Z \subset X$ be a locally closed subvariety of a smooth variety. Show that Z_∞ has measure 0 with respect to the motivic measure on X_∞. This implies in particular that the function $F = \mathrm{ord}_Z$ defined above is indeed measurable.

Example 2.10. If $F \equiv 0$, then

$$\int_{X_\infty} \mathbb{L}^{-0} = \mu(X_\infty) = \frac{[X]}{\mathbb{L}^n}.$$

Example 2.11. Let $Z = \{0\} \subset X = \mathbf{A}^1$. We have

$$Z_m = \{0\} \subset X_m = \mathbf{A}^{m+1}.$$

Recall that we have

$$\mathrm{Cont}^m(Z) = \psi_{m-1}^{-1}(Z_{m-1}) - \psi_m^{-1}(Z_m) = \psi_{m-1}^{-1}(0) - \psi_m^{-1}(0),$$

which gives

$$\mu\left(\mathrm{Cont}^m(Z)\right) = \frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}.$$

We obtain

$$\int_{\mathbf{A}_\infty^1} \mathbb{L}^{-\mathrm{ord}_{\{0\}}} = \sum_{m\geq 0} \left(\frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}\right) \cdot \frac{1}{\mathbb{L}^m} = \left(1 - \frac{1}{\mathbb{L}}\right) \cdot \sum_{m\geq 0} \frac{1}{\mathbb{L}^{2m}} =$$

$$= \frac{\mathbb{L}-1}{\mathbb{L}} \cdot \frac{1}{1 - \mathbb{L}^{-2}} = \frac{\mathbb{L}}{\mathbb{L}+1}.^{4}$$

Example 2.12. Generalizing the example above, let's consider a smooth divisor D in a smooth variety X, and compute $\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_D}$. We have

$$\mathrm{Cont}^m(D) = \psi_{m-1}^{-1}(D_{m-1}) - \psi_m^{-1}(D_m)$$

and therefore, for $m \geq 1$,

$$\mu(\mathrm{Cont}^m(D)) = \frac{[D_{m-1}]}{\mathbb{L}^{nm}} - \frac{[D_m]}{\mathbb{L}^{n(m+1)}} =$$

$$= [D] \cdot \mathbb{L}^{(n-1)(m-1)} \cdot \frac{1}{\mathbb{L}^{nm}} - [D] \cdot \mathbb{L}^{(n-1)m} \cdot \frac{1}{\mathbb{L}^{n(m+1)}} = [D] \cdot \left(\frac{1}{\mathbb{L}^{n+m-1}} - \frac{1}{\mathbb{L}^{n+m}}\right),$$

where the next to last identity uses the fact that D_k is a locally trivial $\mathbf{A}^{(n-1)k}$-bundle over D for each k. This finally gives

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_D} = \sum_{m=0}^{\infty} \mu\left(\mathrm{Cont}^m(D)\right) \cdot \frac{1}{\mathbb{L}^m} = \frac{[X-D]}{\mathbb{L}^n} + [D] \cdot \sum_{m=1}^{\infty} \left(\frac{1}{\mathbb{L}^{n+m-1}} - \frac{1}{\mathbb{L}^{n+m}}\right) \cdot \frac{1}{\mathbb{L}^m} =$$

$$= \frac{[X-D]}{\mathbb{L}^n} + \frac{[D]}{\mathbb{L}^{n+1}} \cdot \sum_{m=1}^{\infty} \left(\frac{1}{\mathbb{L}^{m-1}} - \frac{1}{\mathbb{L}^m}\right) \cdot \frac{1}{\mathbb{L}^{m-1}} = \frac{[X-D]}{\mathbb{L}^n} + \frac{[D]}{\mathbb{L}^n \cdot (\mathbb{L}+1)}.$$

This example is generalized further by the following two exercises.

Exercise 2.13. Compute $\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_Z}$, where $Z \subset X$ is a smooth subvariety of codimension c.

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[4]Compare this with the p-adic integral calculation $\int_{\mathbf{Z}_p} |x| d\mu = \frac{p}{p+1}$ in Ch.III.

Exercise 2.14. Let X be a smooth complex variety, and $D = \sum_{i=1}^{k} a_i D_i$ a divisor with simple normal crossings support on X. Show that $F = \operatorname{ord}_D$ is integrable, and

$$\int_{X_\infty} \mathbb{L}^{-\operatorname{ord}_D} = \sum_{J \subset \{1,\dots,k\}} [D_J^0] \cdot \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1},$$

where $D_J^0 := \cap_{j \in J} D_j - \cap_{j \notin J} D_j$. (This is combinatorially quite intricate – see e.g. [Cr] Theorem 1.17 or the exercises at the end of §2 in [Bl] – so start for instance with the cases $k = 1, 2$.)

3. Change of variables formula

The main result making the theory work is the following *Change of Variables Formula* for motivic integrals due to Kontsevich, and based on the Birational Transformation Theorem in the previous chapter.

Theorem 3.1. *Let $f : X \to Y$ be a proper birational morphism between smooth complex varieties, and let $F : Y_\infty \to \mathbf{N} \cup \{\infty\}$ be an integrable function with respect to the motivic measure. Then*

$$\int_{Y_\infty} \mathbb{L}^{-F} = \int_{X_\infty} \mathbb{L}^{-(F \circ f_\infty + \operatorname{ord}_{K_{X/Y}})}.$$

In particular, if $D \subset X$ is an effective divisor, then

$$\int_{Y_\infty} \mathbb{L}^{-\operatorname{ord}_D} = \int_{X_\infty} \mathbb{L}^{-\operatorname{ord}_{f^*D + K_{X/Y}}}.$$

Proof. Recall from Exercise 3.4 in Ch.IV that $f_\infty : X_\infty \to Y_\infty$ is surjective. For every $p \geq 0$, consider the cylinders

- $C_p = F^{-1}(p) \subset Y_\infty$.

- $D_p = (F \circ f_\infty)^{-1}(p) \subset X_\infty$.

- $D_{p,e} = D_p \cap \operatorname{Cont}^e(K_{X/Y}) \subset X_\infty$.

We have

$$\int_{Y_\infty} \mathbb{L}^{-F} = \sum_{p=0}^{\infty} \mu(C_p) \cdot \frac{1}{\mathbb{L}^p}$$

while

$$\int_{X_\infty} \mathbb{L}^{-(F \circ f_\infty + \operatorname{ord}_{K_{X/Y}})} = \sum_{l=0}^{\infty} \mu\left((F \circ f_\infty + \operatorname{ord}_{K_{X/Y}})^{-1}(l)\right) \cdot \frac{1}{\mathbb{L}^l} =$$

$$= \sum_{p=0}^{\infty} \sum_{e=0}^{\infty} \mu\left((F \circ f_\infty)^{-1}(p) \cap \operatorname{ord}_{K_{X/Y}}^{-1}(e)\right) \cdot \frac{1}{\mathbb{L}^{p+e}} = \sum_{p=0}^{\infty} \sum_{e=0}^{\infty} \mu(D_{p,e}) \cdot \frac{1}{\mathbb{L}^{p+e}}.$$

Claim: the set

$$C_{p,e} := f_\infty(D_{p,e}) \subset Y_\infty$$

is a cylinder set, and the induced map $D_{p,e} \to C_{p,e}$ is an $\mathbf{A}^e$-piecewise trivial fibration in the following sense: there exists some integer m, and locally closed subsets $C_{p,e}^m \subset Y_m$ and $D_{p,e}^m \subset X_m$ such that

$$C_{p,e} = \psi_p^{-1}(C_{p,e}^m) \quad \text{and} \quad D_{p,e} = \psi_p^{-1}(D_{p,e}^m),$$

and the restriction

$$f_m : D_{p,e}^m \to C_{p,e}^m$$

is an $\mathbf{A}^e$-piecewise trivial fibration.

Assuming the Claim for the moment, let's conclude the proof of the Theorem. First, the $\mathbf{A}^e$-fibration assertion implies

$$\mu(D_{p,e}) = \mu(C_{p,e}) \cdot \mathbb{L}^e.$$

Going back to the formulas above, note then that

$$\sum_{e=0}^{\infty} \mu(D_{p,e}) \cdot \frac{1}{\mathbb{L}^{p+e}} = \sum_{e=0}^{\infty} \mu(C_{p,e}) \cdot \frac{1}{\mathbb{L}^p}.$$

To finish the proof, it remains to note that

$$\mu(C_p) = \sum_{e=0}^{\infty} \mu(C_{p,e}).$$

But this is clear, since we have

$$C_p = \left(\coprod_{e \geq 0} C_{p,e} \right) \cup C_{p,\infty}$$

where $C_{p,\infty}$ is a subset of E_∞ (where $E \subset X$ is the exceptional locus), hence a set of measure 0.

We are left with proving the Claim. By the definition of a measurable function, each C_p is a cylinder, so for say $m \gg e$ there exists constructible

$$C_p^m \subset X_m \quad \text{such that} \quad C_p = \psi_m^{-1}(C_p^m).$$

By definition this implies also that

$$D_p = \psi_m^{-1}(D_p^m) \quad \text{with} \quad D_p^m = f_m^{-1}(C_p^m)$$

and

$$D_{p,e} = \psi_m^{-1}(D_{p,e}^m) \quad \text{with} \quad D_{p,e}^m = D_p^m \cap \psi_m \left(\mathrm{Cont}^e(K_{X/Y}) \right).$$

Assume in particular that $m \geq 2e$. Then by the Birational Transformation Rule, Theorem 3.5 in Ch.V, we know that $\psi_m \left(\mathrm{Cont}^e(K_{X/Y}) \right)$ is a union of fibers of f_m, as is D_p^m. This implies that

$$D_{p,e}^m = f_m^{-1}(C_{p,e}^m) \quad \text{with} \quad C_{p,e}^m = f_m(D_{p,e}^m).$$

Now as $C_{p,e} = \psi_m^{-1}(C_{p,e}^m)$, we see that $C_{p,e}$ is a cylinder. Moreover, by the same Birational Transformation Rule, the induced map

$$f_m : D_{p,e}^m \to C_{p,e}^m$$

is an $\mathbf{A}^e$-piecewise trivial fibration. $\square$

Note in particular that after a log-resolution, in the order function case the right-hand-side of the Change of Variables formula can be computed explicitly as in Exercise 2.14.

4. Kontsevich's theorem on K-equivalent varieties

Recall that in §1 we defined a few "motivic" invariants on $K_0(\mathrm{Var}_\mathbf{C})$, namely the Euler characteristic χ, the virtual Poincaré polynomial P, and the virtual Hodge polynomial H. Note that:

- $\chi(\mathbb{L}) = \chi(\mathbf{A}^1) = 1$.

- $P_\mathbb{L} = P_{\mathbf{A}^1} = P_{\mathbf{P}^1} - P_\mathrm{pt} = t^2$.

- $H_\mathbb{L} = H_{\mathbf{A}^1} = H_{\mathbf{P}^1} - H_\mathrm{pt} = uv$.

As a consequence, passing to the localization in $\mathcal{M}_\mathbf{C} = K_0(\mathrm{Var}_\mathbf{C})[\mathbb{L}^{-1}]$, we obtain ring homomorphisms

- $\chi : \mathcal{M}_\mathbf{C} \longrightarrow \mathbf{Z}$.

- $P : \mathcal{M}_\mathbf{C} \longrightarrow \mathbf{Z}[t, \frac{1}{t}]$.

- $H : \mathcal{M}_\mathbf{C} \longrightarrow \mathbf{Z}[u, v, \frac{1}{uv}]$.

Lemma 4.1. *Let $\overline{\mathcal{M}}_\mathbf{C} := \mathrm{Im}(\mathcal{M}_\mathbf{C} \to \widehat{\mathcal{M}_\mathbf{C}})$. Then χ, P and H factor through $\overline{\mathcal{M}}_\mathbf{C}$, i.e. we have an induced*

$$H : \overline{\mathcal{M}}_\mathbf{C} \longrightarrow \mathbf{Z}[u, v, \frac{1}{uv}],$$

and analogous statements for χ and P.

Proof. We show the statement for H; the others are analogous. We need to show that if

$$\alpha \in \mathrm{Ker}(\mathcal{M}_\mathbf{C} \longrightarrow \widehat{\mathcal{M}_\mathbf{C}}) = \bigcap_{m \geq 0} F^m \mathcal{M}_\mathbf{C},$$

then $H(\alpha) = 0$. Now by definition $F^m \mathcal{M}_\mathbf{C}$ is generated by classes $[V]/\mathbb{L}^i$, with $\dim V - i \leq -m$. For each such class, the associated virtual Hodge polynomial has degree $2 \dim V - 2i \leq -2m$, and so we obtain that

$$\deg H(\alpha) \leq -2m \quad \text{for all } m,$$

which implies that $H(\alpha) = 0$. $\qquad\square$

Remark 4.2. One can extend H to

$$H : \widehat{\mathcal{M}_\mathbf{C}} \longrightarrow \mathbf{Z}[\![u, v, \frac{1}{uv}]\!]$$

as well, and similarly for the other invariants.

We are finally able to prove the main result of the notes, improving Batyrev's theorem on Betti numbers discussed in Ch.IV.

Theorem 4.3 (Kontsevich). *Let X and Y be smooth projective complex K-equivalent varieties. Then*

$$h^{p,q}(X) = h^{p,q}(Y) \ \text{ for all } p, q.$$

Proof. By Lemma 4.1, it is enough to show that

$$[X] = [Y] \in \widehat{\mathcal{M}_\mathbf{C}}$$

since then they have the same Hodge polynomial. We will in fact show $\mu(X) = \mu(Y)$. Recall that the fact that X and Y are K-equivalent means that there exists a smooth projective complex variety Z, and a diagram

with f and g birational and $K_{Z/X} = K_{Z/Y}$. We apply the Change of Variables Formula, Theorem 3.1, to the function $F \equiv 0$ with respect to both f and g. Denoting $\dim X = \dim Y = n$, this gives

$$\frac{[X]}{\mathbb{L}^n} = \mu(X) = \int_{X_\infty} \mathbb{L}^{-0} = \int_{Z_\infty} \mathbb{L}^{-\mathrm{ord}_{K_{Z/X}}} = \int_{Z_\infty} \mathbb{L}^{-\mathrm{ord}_{K_{Z/Y}}} = \int_{Y_\infty} \mathbb{L}^{-0} = \mu(Y) = \frac{[Y]}{\mathbb{L}^n}.$$

$\square$

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