CHAPTER 4. K-EQUIVALENT VARIETIES AND BETTI NUMBERS

Contents

In this chapter we will prove one of the main results in this course, namely Batyrev theorem [Ba] on the equality of Betti numbers for birational Calabi-Yau varieties, and more generally for K-equivalent varieties (cf. also [It], [Wa]). This is done by means of a reduction mod p and lifting to a p-adic field procedure, which allows one to use the p-adic integration and Weil conjecture methods studied in the previous chapters. I will take the opportunity to recall some important notions from birational geometry. Later in the course we will encounter the motivic integration methods introduced by Kontsevich, which allow one to improve the theorem mentioned above and obtain equality for all Hodge numbers.

1. Generalities on K-equivalent varieties

In the minimal model program, starting with a smooth variety X (of non-negative Kodaira dimension), one tries to produce a minimal model of X. Roughly speaking this is a variety Y with mild singularities, birational to X, but whose canonical class K_Y is in some technical sense "smallest" among all such varieties birational to X (the correct notion is that K_X is *nef*). Minimal models are not necessarily unique; this and other considerations lead to trying to look for ways of putting some sort of ordering on the elements of the birational equivalence class of X by means of comparing canonical bundles. The discussion below is a first step towards this in the case of smooth varieties, which is the relevant case for our questions here; a full discussion from the point of view of birational geometry would necessarily have to involve singularities.

Definition 1.1. Let X and Y be smooth projective varieties over the complex numbers. They are called *K-equivalent* if there exsists a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$.

An important thing to keep in mind: due to the general resolution of singularities package, any two birational X and Y are dominated by common smooth birational models

Z as in the definition above. However, in general the relationship between $f^*\omega_X$ and $g^*\omega_Y$ is more complicated. Here is though a simple but important example where any common model Z does the job.

Definition 1.2. Let X be a smooth projective (or compact complex) manifold. Then X is called (weakly[1]) *Calabi-Yau* if $\omega_X \simeq \mathcal{O}_X$.

Example 1.3. Any two birational Calabi-Yau varieties are K-equivalent.

Note that any two birational smooth projective curves are isomorphic. The K-equivalence condition does not produce anything interesting in the case of surfaces either.

Proposition 1.4. *If X and Y are K-equivalent surfaces, then $X \simeq Y$.*

Proof. Consider a smooth projective birational model Z as in Definition 1.1, so that $f^*\omega_X \simeq g^*\omega_Y$. The key point is that in the case of surfaces, any birational morphism can be factored into a finite sequence of standard one-point blow-ups (see e.g. [Ha] Ch.V Corollary 5.4). Consider therefore a blow-up $\pi : Z = \mathrm{Bl}_p(Z') \to Z'$ at a point p of a smooth projective surface Z', such that say g is obtained as a composition

$$Z \xrightarrow{\pi} Z' \xrightarrow{g'} Y.$$

Denote by E the exceptional divisor of π. Then clearly $(g^*\omega_Y)_{|E} \simeq \mathcal{O}_E$, and so also $(f^*\omega_X)_{|E} \simeq \mathcal{O}_E$. This implies that E is contracted by f, so that f factors through π as well. Indeed, denoting by $E_i, \ldots, E_k$ the components of the exceptional locus of f, we have

$$K_Z = f^*K_X + E_1 + \ldots + E_k.$$

Since E is a (-1)-curve, by the genus formula we have $K_Z \cdot E = -1$. Combining this with the triviality above, we obtain

$$E \cdot (E_i + \ldots + E_k) = -1,$$

from which it follows that E must be one of the E_i. Consequently, one can replace Z by Z', with birational maps to f' to X and g' to Y, such that the K-equivalence condition still holds. But then one can inductively continue this argument along the finite sequence of blow-ups which factor g, until one necessarily reaches $X \simeq Y$. $\square$

In the rest of the section I will explain a few geometric consequences of K-equivalence.

Exercise 1.5. Let Z be a common smooth birational model for X and Y as in Definition 1.1. Then

$$f^*\omega_X \simeq g^*\omega_Y \iff K_{Z/X} = K_{Z/Y}.$$

(Hint: use the fact that for any birational morphism $f : Z \to X$ with Z and X smooth, and any exceptional divisor $E \subset Z$, one has $f_*\mathcal{O}_Z(E) \simeq f_*\mathcal{O}_Z \simeq \mathcal{O}_X$; the second isomorphism is standard, and the first is a well-known result of Fujita.)

[1] For the full Calabi-Yau condition one usually also requires that X be simply connected and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Lemma 1.6. *Let X and Y be K-equivalent varieties. Then there exist Zariski open subsets $U \subset X$ and $V \in Y$ such that*

$$U \simeq V, \quad \text{codim}_X(X - U) \geq 2 \quad \text{and} \quad \text{codim}_Y(Y - V) \geq 2.$$

In other words, X and Y are isomorphic in codimension 1.

Proof. By what we have seen in Ch.3 §3, the exceptional locus of f is precisely the support of the relative canonical divisor $K_{Z/X}$. Exercise 1.5 says that this is the same as the exceptional locus of g. Denoting this locus by E, this immediately implies that X and Y are isomorphic outside of the images $f(E)$ and $g(E)$, which obviously have codimension at least 2 in X and Y respectively. □

We can put this discussion in a somewhat broader context, starting with an explanation of the terminology *minimal model* from the point of view of the "size" of the canonical bundle.

Definition 1.7. (1) A line bundle L on a projective variety X is called *nef* if $L \cdot C \geq 0$ for every irreducible curve $C \subset X$.

(2) A smooth projective variety X is called *minimal* if ω_X is nef.

(3) Let X and Y be smooth projective birational varieties. We say that $K_X \leq K_Y$ if there exist a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $g^*K_Y - f^*K_X$ is linearly equivalent to an effective divisor (in other words $H^0(Z, g^*\omega_Y \otimes f^*\omega_X^{-1}) \neq 0$).

Example 1.8. If $f : Y \to X$ is the blow-up of X along a smooth subvariety, then $K_X \leq K_Y$.

Proposition 1.9. *Let X and Y be smooth projective birational varieties, with K_X nef. Then $K_X \leq K_Y$. In particular, any two birational minimal varieties are K-equivalent.*[2]

Proof. Let Z be a common birational model for X and Y, with birational morphisms $f : Z \to X$ and $g : Y \to Z$. We can write

$$K_{Z/X} - A = K_{Z/Y} - B,$$

where A and B are effective divisors with no common components, and $\text{codim}_Y(\text{Supp}(B)) \geq 2$. The result follows if we show that $B = 0$, so let's assume that $B \neq 0$ and derive a contradiction.

Let's denote $n = \dim X = \dim Y = \dim Z$ and $d = \dim g(\text{Supp}(B))$. Let H and M be very ample divisors on Y and Z respectively. Given general divisors $H_1, \ldots, H_d \in |H|$ and $M_1, \ldots, M_{n-d-2} \in |M|$, we can consider the generic surface section

$$S := g^*H_1 \cap \ldots \cap g^*H_d \cap M_1 \cap \ldots \cap M_{n-d-2}$$

[2] Let me repeat that in general one needs to work with singular varieties; besides special situations like the Calabi-Yau case, it is rare that one can reach a minimal model which is non-singular, let alone more than one.

of Z, which is smooth by Bertini's theorem. Note now that on Z we have

$$g^*H^d \cdot M^{n-d-2} \cdot B^2 \geq 0.$$

Indeed, note that $B \equiv f^*K_X + A - g^*K_Y$, so this follows from the following three facts:

- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot f^*K_X \geq 0$ since K_X is nef.
- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot A \geq 0$ since A and B have no common components.
- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot g^*K_Y = 0$ since B is contracted by g.

Consequently, denoting $D = B_{|S}$, we have $D^2 \geq 0$ on S. But note that by construction D consists of a union of exceptional curves on X (with respect to the restriction of g to S), and so by a standard consequence of the Hodge index theorem[3] $D^2 < 0$, a contradiction. □

Remark 1.10. A slight refinement of this argument shows that if $f : X \cdots \to Y$ is a birational map between smooth varieties such that K_X is nef along the exceptional locus Z of f, then $K_X \leq K_Y$, and Z has codimension at least 2.

We are aiming towards relating topological and holomorphic invariants of K-equivalent varieties. While the full results are quite deep, part of this follows from the structural results covered in this section.

Proposition 1.11. *Let $f : X \cdots \to Y$ be a birational (rational) map between smooth projective complex varieties, which is an isomorphism outside of codimension at least 2 closed subsets. Then there are natural isomorphisms*

$$H^i(X, \mathbf{Z}) \simeq H^i(Y, \mathbf{Z}) \quad \text{for } i \leq 2,$$

compatible (after complexification) with the Hodge decompositions on the two sides.

Proof. Poincaré duality implies that equivalently we can aim for natural isomorphisms

$$H_{2n-i}(X, \mathbf{Z}) \simeq H_{2n-i}(Y, \mathbf{Z}),$$

where n is the dimension of X, and $i \leq 2$. Now X and Y are diffeomorphic as real manifolds outside closed subsets of real codimension at least 4, and therefore this diffeomorphism sees all $(2n-i)$-cycles on X and Y with $i \leq 2$, inducing the desired natural isomorphism.

On the other hand, we have seen in Chapter 1 that an isomorphism of open sets that are complements of closed analytic subvarietes induces via Hartogs' theorem isomorphisms

$$H^{0,i}(X) \simeq H^{0,i}(Y) \quad \text{and} \quad H^{i,0}(X) \simeq H^{i,0}(Y) \quad \text{for all } i.$$

Since for $i \leq 2$, $H^i(X, \mathbf{C})$ and $H^i(Y, \mathbf{C})$ have at most one component in their Hodge decomposition which is not of this form, it is clear that the complexification of the isomorphisms above respects the Hodge structures. □

Remark 1.12. One can similarly show the same result at the level of homotopy groups:

$$\pi_i(X) \simeq \pi_i(Y) \quad \text{for } i \leq 2.$$

[3] See for example [Ha] Ch.V, Exercise 5.4(a).

2. Reduction mod p and lifting to the p-adics

Let's briefly go through two standard arithmetic reduction procedures:

Reduction mod p. Let X be a scheme of finite type over $\mathbf{C}$. Then there exists a finitely generated $\mathbf{Z}$-algebra R and X_R a scheme over Spec R such that

$$X \simeq X_R \times_{\text{Spec } R} \text{Spec } \mathbf{C}.$$

This is obtained in general by gluing the affine case. In the affine case, X is given in some $\mathbf{A}^n$ as the zero locus of a finite number of polynomials. We can take $R = \mathbf{Z}[a_1, \ldots, a_N]$, where the a_i are all the coefficients of these polynomials; X can clearly be lifted to a scheme X_R over R by considering the exact same equations. Note that X_R does not depend only on R, but also on the choice of defining equations.

Now let $\mathfrak{p} \subset R$ be a maximal ideal, lying over $(p) \subset \mathbf{Z}$ with p a prime number. Then we have that $R/\mathfrak{p}$ is a finite extension of $\mathbf{Z}/p\mathbf{Z}$, so

$$R/\mathfrak{p} \simeq \mathbf{F}_q, \text{ with } q = p^r, r > 0.$$

The fiber of X_R over the corresponding point in Spec R,

$$X_{\mathfrak{p}} := X_R \times_{\text{Spec} R} \text{Spec } R/\mathfrak{p},$$

is a scheme of finite type over $\mathbf{F}_q$, called the *reduction mod p of X*.

Assume now that X is a smooth variety over $\mathbf{C}$. Then there exists a nonempty open set $U \subset \text{Spec } R$, containing (0), such that the structural map $\pi : X_R \to \text{Spec } R$ is smooth over U. This implies that all the reductions mod p corresponding to points in U are smooth varieties over the corresponding $\mathbf{F}_q$. (In such a situation we say that X has *good reduction* at those primes.)

Completely analogously, reduction mod p can be done simultaneously for a finite collection of schemes of finite type, coherent sheaves on them, and morphisms between them. Indeed, thinking again locally, the schemes are given by a finite number of equations, the morphisms are determined by their graphs, which are again schemes of finite type, while the coherent sheaves are locally finitely presented modules, so again given by a finite number of equations.

We will in fact only use the following:

Proposition 2.1. *Let $f : X \to Y$ be a morphism between varieties over $\mathbf{C}$. Then there exists a finitely generated $\mathbf{Z}$-algebra R and schemes X_R and Y_R over* Spec R*, together with a morphism $f_R : X_R \to Y_R$ over* Spec R*, such that $X_R \times_{\text{Spec } R} \text{Spec } \mathbf{C} \simeq X$, $X_R \times_{\text{Spec } R} \text{Spec } \mathbf{C} \simeq Y$ and $f_R \times_{\text{Spec} R} \text{Spec } \mathbf{C} = f$. In addition:*

- *if X and Y are smooth (respectively projective), then there exists a Zariski open set $U \subset \text{Spec } R$ such that $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are smooth (respectively projective) for every $\mathfrak{p} \in U$.*

- *if f is birational (respectively proper), then there exists a Zariski open set $U \subset \text{Spec } R$ such that the induced $f_{\mathfrak{p}} : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ is birational (respectively proper) for every $\mathfrak{p} \in U$.*

Exercise 2.2. Prove the assertions in the Proposition that do not immediately follow from the discussion above.

Lifting to the p-adics. Let R be a finitely generated $\mathbf{Z}$-algebra as above, and X_R a scheme of finite type over Spec R. Let $\mathfrak{p} \subset R$ be a maximal ideal, with $X_{\mathfrak{p}}$ the fiber of X_R over $\mathfrak{p}$; as noted above, this is a scheme of finite type over $\mathbf{F}_q$, the residue field of R at $\mathfrak{p}$.

Proposition 2.3. *With the notation above, there exists a p-adic field K with ring of integers $(\mathcal{O}_K, \mathfrak{m}_K)$ such that $\mathcal{O}_K/\mathfrak{m}_K \simeq \boldsymbol{F}_q$, and a scheme of finite type $\mathcal{X}$ over* Spec $\mathcal{O}_K$ *which is a* lifting *of $X_{\mathfrak{p}}$ over $\mathcal{O}_K$, i.e. such that*

$$X_{\mathfrak{p}} \simeq \mathcal{X} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \boldsymbol{F}_q.$$

If $X_{\mathfrak{p}}$ is smooth over $\boldsymbol{F}_q$, then $\mathcal{X}$ can be taken to be smooth over $\mathcal{O}_K$.

Proof. Let $L = Q(R)$ be the quotient field of R. Since R is a finitely generated algebra over $\mathbf{Z}$, we have a finite field extension $\mathbf{Q} \subset L$ (L is a *number field*). By restricting to a general complete intersection curve through the origin in Spec R, we can assume that $\dim R = 1$, and that for a general non-zero prime ideal $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is regular, i.e. a DVR (in fact by further restriction to a suitable affine open subset, we can assume that R is a Dedekind domain).

Now the completion $\widehat{R_{\mathfrak{p}}}$ with respect to the $\mathfrak{p}R_{\mathfrak{p}}$-adic topology is a complete DVR, and hence the ring of integers $\mathcal{O}_K$ of a p-adic field K. It has the same residue field, since

$$\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}} \simeq R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \simeq \mathbf{F}_q.$$

Note that we could have gone to the reduction mod p by passing first to a scheme over the DVR $R_{\mathfrak{p}}$, namely $\bar{X}_{\mathfrak{p}} := X_R \times_{\text{Spec} R} \text{Spec } R_{\mathfrak{p}}$, so that

$$X_{\mathfrak{p}} \simeq \bar{X}_{\mathfrak{p}} \times_{\text{Spec} R_{\mathfrak{p}}} \text{Spec } \mathbf{F}_q.$$

But we have a natural injective homomorphism $R_{\mathfrak{p}} \hookrightarrow \widehat{R_{\mathfrak{p}}} = \mathcal{O}_K$, and so we can define

$$\mathcal{X} := \bar{X}_{\mathfrak{p}} \times_{\text{Spec} R_{\mathfrak{p}}} \text{Spec } \mathcal{O}_K,$$

which is a scheme of finite type over Spec $\mathcal{O}_K$. This is a lifting of $X_{\mathfrak{p}}$, since

$$\mathcal{X} \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathbf{F}_q \simeq (\bar{X}_{\mathfrak{p}} \times_{\text{Spec} R_{\mathfrak{p}}} \text{Spec } \mathcal{O}_K) \times_{\text{Spec } \mathcal{O}_K} \text{Spec } \mathbf{F}_q \simeq X_{\mathfrak{p}}.$$

Finally, it is well known that passing to the completion preserves smoothness (over the completed ring), so if $X_{\mathfrak{p}}$ is smooth then so it $\mathcal{X}$ over Spec $\mathcal{O}_K$.

□

Remark 2.4. One can also approach this type of result using the following theorem of independent interest (see e.g. [Ca] Ch.V):

Theorem 2.5 ("Embedding theorem")**.** *Let F be a number field, and $C \subset F$ a finite subset of non-zero elements. Then there exist infinitely many primes p for which there is an embedding $\alpha : F \hookrightarrow \mathbf{Q}_p$, which can be chosen such that $|\alpha(c)|_p = 1$ for all $c \in C$ (in other words C is mapped into the invertible elements of $\mathbf{Z}_p$).*

Let now $F = Q(R)$, the quotient field of R. Since R is a finitely generated $\mathbf{Z}$-algebra, F is a number field. Note that $\mathcal{X}$ is defined over F. According to the Embedding Theorem, for an infinite number of primes p there exist embeddings $\alpha : F \hookrightarrow \mathbf{Q}_p$, which can be chosen such that any finite set in R can be mapped to the p-adic units. In particular we can apply this to the generators of R as an algebra over $\mathbf{Z}$ (i.e. basically the coefficients of the polynomials defining X), so that these have images in $\mathbf{Z}_p$.

Using a further finite extension $\mathbf{Q}_p \subset K$, so that we have a local homomorphism $R \subset \mathcal{O}_K$, the defining equations of X can be lifted to equations with coefficients in $\mathcal{O}_K$, and this gives the desired lifting $\mathcal{X}$ over Spec $\mathcal{O}_K$.

3. Batyrev's theorem on the invariance of Betti numbers

The main goal of this section, and in fact of most of the course up to this point, is to prove the following result due to Batyrev [Ba].

Theorem 3.1. *Let X and Y be birational complex Calabi-Yau varieties. Then*

$$b_i(X) = b_i(Y) \text{ for all } i.$$

More generally, the same statement holds for any two complex smooth projective K-equivalent varieties.

Proof. Assuming that X and Y are K-equivalent varieties of dimension n, we consider a smooth Z together with birational maps $f : Z \to X$ and $g : Z \to Y$ as in Definition 1.1. We first use reduction mod p and lifting to the p-adics to obtain models for X, Y, Z, f and g over the ring of integers of a p-adic field.

More precisely, by the reduction mod p procedure (especially Proposition 2.1), there exist a finitely generated $\mathbf{Z}$-algebra R and liftings X_R, Y_R, Z_R, $f_R : Z_R \to X_R$ and $g_R : Z_R \to Y_R$ over Spec R. We can assume (by passing to an open subset if necessary), that X_R, Y_R and Z_R are smooth and projective over Spec R, f_R and g_R are birational, and

$$f_R^* \, \Omega^n_{X_R/R} \simeq g_R^* \, \Omega^n_{Y_R/R}$$

on Z_R.

Next, by the lifting to the p-adics procedure (see Proposition 2.3), there exists a p-adic field K, with ring of integers $\mathcal{O}_K$ whose residue field is $\mathbf{F}_q$, and smooth schemes $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ over $S = \text{Spec } \mathcal{O}_K$ with special fibers $X_{\mathfrak{p}}$, $Y_{\mathfrak{p}}$, $Z_{\mathfrak{p}}$ over the closed point corresponding to $\mathfrak{m}_K$ (i.e. $\bar{X} = \mathcal{X} \times_S \text{ Spec } \mathbf{F}_q$, etc.) which are isomorphic to the fibers of X_R, Y_R, Z_R over the prime $\mathfrak{p}$. In addition, there are proper birational morphisms over S

$$\tilde{f} : \mathcal{Z} \to \mathcal{X}, \;\; \tilde{g} : \mathcal{Z} \to \mathcal{Y}$$

extending f_R and g_R, such that on $\mathcal{Z}$ we have

$$\tilde{f}^* \, \Omega^n_{\mathcal{X}/S} \simeq \tilde{g}^* \, \Omega^n_{\mathcal{Y}/S}. \tag{1}$$

Under this assumption, the claim is that

$$|\mathcal{X}(\mathbf{F}_q)| = |\mathcal{Y}(\mathbf{F}_q)|.$$

Using Corollary 5.4 in Chapter 3, i.e. Weil's result on the interpretation of the number of points over $\mathbf{F}_q$ as a normalized volume with respect to the canonical p-adic measure, it is enough then to show that

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\text{can}} = \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\text{can}}.$$

Recall now that the canonical measure is defined by gluing the p-adic measures defined by any local generators ω of $\Omega^n_{\mathcal{X}/S}$ on open sets where this line bundle is trivial. By pull-pack by f and g, these induce local generators for $\tilde{f}^* \, \Omega^n_{\mathcal{X}/S}$ and $\tilde{g}^* \, \Omega^n_{\mathcal{Y}/S}$ respectively. Using the change of variable formula, Theorem 3.8 in Ch.III, on both sides, we obtain our equality via

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\text{can}} = \int_{\mathcal{Z}(\mathcal{O}_K)} |\tilde{f}^* \, \Omega^n_{\mathcal{X}/S}| d\mu = \int_{\mathcal{Z}(\mathcal{O}_K)} |\tilde{g}^* \, \Omega^n_{\mathcal{Y}/S}| d\mu = \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\text{can}}.$$

Note that in the middle terms we have used the notation $|\tilde{f}^* \, \Omega^n_{\mathcal{X}/S}| d\mu$ for the measure on $\mathcal{Z}$ coming from gluing the local measures defined by the generators $f^*\omega$ (and similarly for $\mathcal{Y}$). By (1), these coincide.

Finally, for any $m \geq 2$ we can replace the finitely generated $\mathbf{Z}$-algebra R by a cyclotomic extension $R_m \subset \mathbf{C}$, which is obtained by adjoining to R all complex $(q^m - 1)$-roots of unity. It is easy to check that R_m has a maximal ideal $\mathfrak{p}_m$ lying over $\mathfrak{p} \subset R$ such that $R_m/\mathfrak{p}_m \simeq \mathbf{F}_{q^m}$, the degree m extension of $\mathbf{F}_q \simeq R/\mathfrak{p}$. Over $\mathbf{C}$, this still reduces to the initial X. We can repeat the exact same arguments as above, the conclusion being that

$$|\mathcal{X}(\mathbf{F}_{q^m})| = |\mathcal{Y}(\mathbf{F}_{q^m})| \quad \text{for all } m \geq 1.$$

This implies that the local Weil zeta functions of $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ studied in Ch.II satisfy

$$Z(X_{\mathfrak{p}}; t) = Z(Y_{\mathfrak{p}}; t).$$

Since $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are reductions mod p of the complex smooth projective varieties X and Y, the Betti numbers component of the Weil conjectures, Theorem 3.4 in Ch.II, implies that $b_i(X) = b_i(Y)$ for all i. □

Remark 3.2. (1) Although with what we have covered until now we are not yet able to show this, a stronger statement holds, namely and two K-equivalent varieties have the same *Hodge numbers*. To show this, inspired by p-adic integration Kontsevich introduced the technique of *motivic integration*; we will see this, together with the proof of the more general statement, in the next chapter. Note however that it is now known how to recover the Hodge numbers using p-adic techniques as well, using the above methods plus results from p-adic Hodge theory (see [It]).

(2) The proof above shows in fact slightly more than what we have stated (see [Wa] Theorem 3.1): if X and Y are not necessarily K-equivalent, but satisfy $K_X \leq K_Y$, then

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\text{can}} \; \leq \; \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\text{can}}.$$

This implies, in the notation above, that for each $m \geq 1$

$$|X_{\mathfrak{p}}(\mathbf{F}_{q^m})| \leq |Y_{\mathfrak{p}}(\mathbf{F}_{q^m})|.$$

(3) It is interesting to note that, unlike in the motivic integration approach, here it is not crucial to use the change of variables formula (except somewhat trivially via isomorphisms). Let's see this in the case of Calabi-Yau manifolds; the argument can be quickly adapted to the general case. Assume therefore that there are gauge forms $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{Y}}$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively. We know that there are isomorphic open subsets

$$\alpha : \mathcal{U} \subset \mathcal{X} \xrightarrow{\simeq} \mathcal{V} \subset \mathcal{Y}$$

over S, with complements of codimension at least 2. The restriction $\omega_{\mathcal{U}}$ of $\omega_{\mathcal{X}}$ is a gauge form on $\mathcal{U}$, and via the isomorphism α so is $\alpha^*\omega_{\mathcal{V}}$. This means that there exists a nowhere vanishing regular function $f \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*)$ such that

$$\alpha^*\omega_{\mathcal{V}} = f \cdot \omega_{\mathcal{U}}.$$

But since $\operatorname{codim}_{\mathcal{X}}(\mathcal{X} - \mathcal{U}) \geq 2$, it follows that f extends to an invertible function on $\mathcal{X}$, i.e. an element in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$, so that $|f(x)| = 1$ for all $x \in \mathcal{X}(K)$. Consequently, the Weil p-adic measures on $\mathcal{U}(K)$ given by $\alpha^*\omega_{\mathcal{V}}$ and $\omega_{\mathcal{U}}$ are the same. This implies the equality of integrals

$$\int_{\mathcal{U}(K)} d\mu_{\omega_{\mathcal{X}}} = \int_{\mathcal{V}(K)} d\mu_{\omega_{\mathcal{Y}}}.$$

But we have seen in Proposition 5.5 in Ch. III that the complements of $\mathcal{U}$ and $\mathcal{V}$ are sets of measure zero with respect to the p-adic measure, so this is the same as

$$\int_{\mathcal{X}(K)} d\mu_{\omega_{\mathcal{X}}} = \int_{\mathcal{Y}(K)} d\mu_{\omega_{\mathcal{Y}}}.$$

Finally, since $\mathcal{X}$ and $\mathcal{Y}$ are projective over S, we have that $\mathcal{X}(K) = \mathcal{X}(\mathcal{O}_K)$ and $\mathcal{Y}(K) = \mathcal{Y}(\mathcal{O}_K)$ (see Exercise 5.1 in Ch.II).

(4) It is an open problem whether the analogue of Theorem 3.1 is true for compact Kähler manifolds.

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