

# Kodaira dimension of algebraic varieties

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**Introduction**

These are notes for the topics course Math 271Z I am currently teaching at Harvard (Fall semester 2022). They are based in part on older notes for a course I taught at Northwestern in Spring 2015.

The intention is to cover results around Iitaka's subadditivity conjecture for the Kodaira dimension, as well as more recent related topics.

## CHAPTER 1

### Preliminaries

This chapter is devoted to the basic definitions and examples, and to some fundamental positivity tools.

#### 1.1. Plurigenera

Let  $X$  be a smooth projective variety of dimension  $n$  over an algebraically closed field  $k$ . The crucial invariant of  $X$  we will repeatedly refer to is its *canonical bundle*

$$\omega_X := \bigwedge^n \Omega_X^1.$$

**DEFINITION 1.1.1.** The *plurigenera* of  $X$  are the non-negative integers

$$P_m(X) = h^0(X, \omega_X^{\otimes m}) := \dim_k H^0(X, \omega_X^{\otimes m}), \quad \forall m \geq 1.$$

**EXAMPLE 1.1.2 (Projective space).** We have  $\omega_{\mathbf{P}^n} \simeq \mathcal{O}_{\mathbf{P}^n}(-n-1)$ , and so  $P_m(\mathbf{P}^n) = 0$  for all  $m \geq 1$ .

**EXAMPLE 1.1.3 (Curves).** If  $C$  is a smooth projective curve of genus  $g$ , by definition  $P_1(C) = g$ . Moreover:

- If  $C = \mathbf{P}^1$ , i.e.  $g = 0$ , then  $\omega_C = \mathcal{O}_{\mathbf{P}^1}(-2)$ , and so  $P_m(C) = 0$  for all  $m \geq 1$ .
- If  $C$  is elliptic, i.e.  $g = 1$ , then  $\omega_C \simeq \mathcal{O}_C$ , and in particular  $P_m(C) = 1$  for all  $m \geq 1$ .
- If  $g \geq 2$ , then

$$\deg \omega_C^{\otimes m} = m(2g - 2) > 2g - 1, \quad \forall m \geq 2,$$

so  $H^1(C, \omega_C^{\otimes m}) = 0$ , and so by Riemann-Roch

$$P_m(C) = m(2g - 2) - g + 1 = (2m - 1)(g - 1), \quad \forall m \geq 2.$$

**EXAMPLE 1.1.4 (Hypersurfaces).** Let  $X \subset \mathbf{P}^n$  be a smooth hypersurface of degree  $d$ . If  $\mathcal{O}_X(1)$  is the restriction of  $\mathcal{O}_{\mathbf{P}^n}(1)$  to  $X$ , we have

$$\omega_X \simeq \mathcal{O}_X(d - n - 1).$$

- If  $d \leq n$ , then  $P_m(X) = 0$  for all  $m \geq 0$ .
- If  $d = n + 1$ , then  $\omega_X \simeq \mathcal{O}_X$ , and in particular  $P_m(X) = 1$  for all  $m \geq 0$ .
- If  $d \geq n + 2$ , then  $\omega_X$  is a very ample line bundle. Using Serre Vanishing and the basic properties of the Hilbert polynomial of  $X$ , we have

$$P_m(X) = \chi(X, \omega_X^{\otimes m}) = \frac{d(d - n - 1)}{(n - 1)!} \cdot m^{n-1} + O(m^{n-2}) \quad \text{for } m \gg 0.$$

Recall also that if  $n \geq 3$ , then

$$H^i(X, \mathcal{O}_X) = 0, \quad \forall 0 < i < n - 1.$$

**EXERCISE 1.1.5.** Recall that for a smooth complete intersection  $X \subset \mathbf{P}^N$  of hypersurfaces of degrees  $d_1, \dots, d_k$ , we have

$$\omega_X \simeq \mathcal{O}_X(d_1 + \dots + d_k - n - 1).$$

Use this in order to do calculations similar to the case of hypersurfaces.

**EXAMPLE 1.1.6 (Abelian varieties.)** Let  $X$  be an abelian variety. Then  $T_X$  is trivial, and in particular  $\omega_X \simeq \mathcal{O}_X$ . Thus this is another example where

$$P_m(X) = 1, \quad \forall m \geq 0.$$

However, note that unlike in the case of hypersurfaces

$$H^i(X, \mathcal{O}_X) \simeq \bigwedge^i H^1(X, \mathcal{O}_X) \neq 0, \quad \forall i \geq 0.$$

**DEFINITION 1.1.7 (Calabi-Yau's).** (1) We will call a *weak Calabi-Yau* variety a smooth projective variety  $X$  with  $\omega_X \simeq \mathcal{O}_X$ . If in addition

$$H^i(X, \mathcal{O}_X) = 0, \quad \forall 0 < i < \dim X,$$

we will say that  $X$  is *Calabi-Yau*.<sup>1</sup>

Thus a hypersurface of degree  $d = n + 1$  in  $\mathbf{P}^n$  is Calabi-Yau, while an abelian variety is weak Calabi-Yau, but not Calabi-Yau.

(2) A *K3 surface* is a Calabi-Yau variety  $X$  of dimension 2. In other words,  $\omega_X \simeq \mathcal{O}_X$ , and  $H^1(X, \mathcal{O}_X) = 0$ .

**EXAMPLE 1.1.8.** According to the examples above, a hypersurface of degree 4 (a *quartic surface*) in  $\mathbf{P}^3$  is a K3 surface. So is a complete intersection of type (2, 3), i.e. of a general quadric and a cubic in  $\mathbf{P}^4$ , and one of type (2, 2, 2), i.e. of three general quadrics in  $\mathbf{P}^5$ . For simple numerical reasons, there are no other complete intersection K3 surfaces (check this!).

**EXAMPLE 1.1.9.** If  $X$  and  $Y$  are smooth projective varieties, show that

$$P_m(X \times Y) = P_m(X) \cdot P_m(Y).$$

This is an immediate consequence of the fact that  $\omega_{X \times Y} \simeq \omega_X \boxtimes \omega_Y$ .

**EXAMPLE 1.1.10 (Projective bundles).** Here is a basic example of a fibration that is not usually a product. Let  $E$  be a locally free sheaf on  $X$ , of rank  $r \geq 2$ . For the associated projective bundle  $\mathbf{P}(E) \rightarrow X$  (see [Ha, II, §7]) we have

$$P_m(\mathbf{P}(E)) = 0 \quad \text{for all } m \geq 1.$$

Indeed, the canonical bundle of  $\mathbf{P}(E)$  is given by the formula

$$\omega_{\mathbf{P}(E)} \simeq \pi^*(\det E \otimes \omega_X) \otimes \mathcal{O}_{\mathbf{P}(E)}(-r),$$

<sup>1</sup>Often even this is not enough for the proper definition: one should also require, at least over  $\mathbf{C}$ , that  $X$  be simply connected, but we will ignore this here.

see e.g. [Ha, III, Exercise 8.4]. Therefore the projection formula gives

$$P_m(\mathbf{P}(E)) = h^0(X, (\det E \otimes \omega_X)^{\otimes m} \otimes \pi_* \mathcal{O}_{\mathbf{P}(E)}(-rm)),$$

and using base change it is immediate to check that  $\pi_* \mathcal{O}_{\mathbf{P}(E)}(-k) = 0$  for  $k \geq 1$ .

A *ruled surface* is a projective bundle  $\pi : \mathbf{P}(E) \rightarrow C$ , where  $C$  is a smooth projective curve over an algebraically closed field, and  $E$  is a locally free sheaf of rank 2 on  $C$ . (The definition is often stated differently, namely as a surjective morphism  $\pi : X \rightarrow C$  with fibers isomorphic to  $\mathbf{P}^1$ , but it is standard to see that it is equivalent to the one given here.) Thus ruled surfaces provide us with new examples of smooth projective surfaces with vanishing plurigenera (and same for projective bundles in arbitrary dimension).

We will see more examples of fibrations later on. They usually have to be rather complicated; here is one possible reason why this is the case.

**EXERCISE 1.1.11.** Let  $\pi : S \rightarrow \mathbf{P}^1$  be a surjective morphism, with  $S$  a smooth projective surface. Assume that  $P_m(S) \neq 0$  for some  $m \geq 1$ . Show that:

- (1)  $P_m(F) \neq 0$ , where  $F$  is the general fiber of  $\pi$ .
- (2)  $\pi$  is not a smooth morphism.

Finally, one of the most important properties for us is the following:

**EXERCISE 1.1.12.** The plurigenera are birational invariants. In other words, if  $X$  and  $Y$  are birational smooth projective varieties, then

$$P_m(X) = P_m(Y) \quad \text{for all } m \geq 0.$$

(You can check that the approach for  $P_1$  in [Ha, Theorem II.8.19] works in fact for all  $m$ .)

## 1.2. Iitaka dimension

Let  $X$  be a projective variety, and let  $L$  be a line bundle on  $X$ . For each  $m \geq 0$  such that  $h^0(X, L^{\otimes m}) \neq 0$ , the linear system  $|L^{\otimes m}|$  induces a rational map from  $X$  to a projective space, and more precisely a morphism

$$\varphi_m : X \setminus B_m \rightarrow \mathbf{P}^{N_m}, \quad N_m = h^0(X, L^{\otimes m}) - 1,$$

where  $B_m = \text{Bs}(L^{\otimes m})$  is its base locus. We denote by  $\varphi_m(X)$  the closure of the image of  $\varphi_m$  in  $\mathbf{P}^{N_m}$ .

**DEFINITION 1.2.1.** (1) If  $X$  is normal, the *Iitaka dimension* of  $L$  is

$$\kappa(L) = \kappa(X, L) = \max_{m \geq 1} \dim \varphi_m(X)$$

if  $\varphi_m(X) \neq \emptyset$  for some  $m$ . We set  $\kappa(L) = -\infty$  otherwise (i.e. when  $h^0(X, L^{\otimes m}) = 0$  for all  $m \geq 0$ ). Clearly

$$\kappa(L) \in \{-\infty, 0, 1, \dots, \dim X\}.$$

In general, we consider the normalization  $f : \tilde{X} \rightarrow X$ , and define

$$\kappa(L) := \kappa(\tilde{X}, f^*L).$$

(2) For a Cartier divisor  $D$  on  $X$ , we define  $\kappa(D) := \kappa(\mathcal{O}_X(D))$ , and more generally for a  $\mathbb{Q}$ -Cartier divisor  $D$  such that  $kD$  is Cartier, we define  $\kappa(D) := \kappa(\mathcal{O}_X(kD))$ .

(3) The line bundle  $L$  is called *big* if  $\kappa(L) = \dim X$ . More generally, a  $\mathbb{Q}$ -Cartier divisor  $D$  is big if  $\kappa(D) = \dim X$ .

EXAMPLE 1.2.2. The purpose of considering the normalization in the definition is to obtain a birational theory. Here is a simple example showing that without passing to normalization one may obtain a different answer.

Let  $C$  be plane nodal rational curve, and  $f: \tilde{C} \rightarrow C$  its normalization, so that  $\tilde{C} \simeq \mathbf{P}^1$ . Then  $\text{Pic}^0(C) \simeq \mathbf{C}^*$  (given by the scalar used to glue the stalks of  $\mathcal{O}_{\tilde{C}}$  at the two preimages of the node); therefore we can pick a non-torsion line bundle  $L \in \text{Pic}^0(C)$ , for which  $H^0(C, L^{\otimes m}) = 0$  for all  $m \geq 1$ . On the other hand, we have  $f^*L \simeq \mathcal{O}_{\tilde{C}}$ , hence  $\kappa(L) = \kappa(\tilde{C}, \mathcal{O}_{\tilde{C}}) = 1$ .

EXAMPLE 1.2.3. If  $L$  is ample, then it is big. Indeed, for  $m \gg 0$  we have that  $L$  is very ample, and so  $\varphi_m$  is an embedding.

**Equivalent interpretation.** We consider the *semigroup* of  $L$ , namely

$$N(L) := \{m \in \mathbb{N} \mid H^0(X, L^{\otimes m}) \neq 0\}.$$

Indeed, this is a semigroup with respect to addition, because of the existence of multiplication maps

$$(1.2.1) \quad H^0(X, L^{\otimes k}) \otimes H^0(X, L^{\otimes l}) \longrightarrow H^0(X, L^{\otimes k+l}).$$

We can also consider

$$e(L) := \gcd \{m \mid m \in N(L)\} \geq 1.$$

All sufficiently large elements of  $N(L)$  are multiples of  $e(L)$ , and all sufficiently large multiples of  $e(L)$  are in  $N(L)$ . The number  $e(L)$  is the largest with this property, and is called the *exponent* of  $L$ . In the definition of the Iitaka dimension, the only relevant integers  $m$  are those in  $N(L)$ ; in this context, by sufficiently large and divisible integers we will mean sufficiently large multiples of  $e(L)$ .

PROPOSITION 1.2.4. *In the setting above, let  $\kappa = \kappa(X, L)$ . Then there exist constants  $a, b > 0$  such that*

$$a \cdot m^\kappa \leq h^0(X, L^{\otimes m}) \leq b \cdot m^\kappa,$$

for sufficiently large and divisible  $m$ .

PROOF. The lower bound is a quite direct calculation; check it as an exercise! The more interesting part is the upper bound, and we concentrate on this.

We have  $\dim \varphi_m(X) = \kappa(L)$ , for all  $m \in N(L)$  sufficiently large. Let's first assume that  $L$  is big, i.e.  $\kappa(L) = \dim X$ . Consider an ample line bundle  $A$  on  $X$  such that

$$H^0(X, A \otimes L^{-1}) \neq 0.$$

(Note that this is always possible, since by Serre's theorem  $L^{-1}$  twisted by any large power of an ample line bundle is globally generated.) This gives the following sequence of inequalities, for some constant  $C > 0$ .

$$h^0(X, L^{\otimes m}) \leq h^0(X, A^{\otimes m}) \leq C \cdot m^{\dim X} = C \cdot m^{\kappa(L)},$$

where the second inequality is given by the Hilbert polynomial of  $A$ . Note that this shows in fact that the inequality

$$(1.2.2) \quad h^0(X, L^{\otimes m}) \leq C \cdot m^{\dim X}$$

holds for *any* line bundle  $L$ , for some constant  $C > 0$  depending on  $L$ .

Assume now that  $\kappa(L) < \dim X$ ; we reduce this case to the previous one. To this end, pick  $H_1, \dots, H_p$  general very ample divisors on  $X$ , with  $p = \dim X - \kappa(L)$ , and denote  $X' = H_1 \cap \dots \cap H_p$ . It is a standard fact that each  $H_i$  dominates  $\varphi_m(X)$ , i.e. its image is dense. (Idea: one can reduce to the case when the map is defined everywhere, by considering the closure of the graph; if  $H$  didn't map surjectively, then it would not meet a general fiber of the map, which is positive dimensional because  $\kappa(L) < \dim X$ . But very ample divisors have to meet a general positive dimensional subvariety of a family sweeping  $X$ . We will understand this type of argument better once we study positivity in more detail.) We can do this in such a way that  $X'$  dominates  $\varphi_m(X)$  for all sufficiently large  $m$ ,<sup>2</sup> and of course  $\dim X' = \kappa(L)$ . Based on (1.2.2), we have

$$h^0(X', L_{|X'}^{\otimes m}) \leq C \cdot m^{\dim X'} = C \cdot m^{\kappa(L)},$$

for some constant  $C > 0$ . (It is also the case that  $L_{|X'}$  is a big line bundle, though strictly speaking this is not needed here.) At this stage we are done, since we in fact have that the restriction map

$$H^0(X, L^{\otimes m}) \longrightarrow H^0(X', L_{|X'}^{\otimes m})$$

is injective for all  $m \in N(L)$  sufficiently large. Indeed, by the definition of the map induced by  $L^{\otimes m}$ , the sections in  $H^0(X, L^{\otimes m})$  correspond to the hyperplanes in  $\mathbf{P}^{N_m}$ . If the restriction map in question weren't injective, it would mean that there is a hyperplane in  $\mathbf{P}^{N_m}$  containing the image of  $X'$ . But this image is  $\varphi_m(X)$ , a contradiction.  $\square$

**EXERCISE 1.2.5.** Let  $f: X \rightarrow Y$  be a finite surjective morphism of normal projective varieties, and  $L$  a line bundle on  $Y$ . Then

$$\kappa(Y, L) = \kappa(X, f^*L).$$

Further interesting results and equivalent interpretations regarding the Iitaka dimension (or  $D$ -dimension, as originally named by Iitaka), can be found in the comprehensive [Mo, §1]. Here is for instance another useful interpretation, whose proof I will skip. We assume that  $X$  is defined over an algebraically closed field  $k$ .

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<sup>2</sup>For this we need to be over an uncountable field, since we have to do this for countably many  $m$ . Note however that we can assume this: if you take the base change  $X_K = X \times_k K$  for any field extension  $k \subset K$ , the numerical invariants in the statement are preserved, and so it suffices to prove the statement for  $X_K$ .

DEFINITION 1.2.6. The *section ring* of  $L$  is the ring

$$R(L) := \bigoplus_{m \geq 0} H^0(X, L^{\otimes m}).$$

This is a graded integral  $k$ -algebra due to the multiplication maps in (1.2.1).<sup>3</sup>

PROPOSITION 1.2.7 ([Mo, Proposition 1.4]). *If  $\kappa(L) \geq 0$ , and  $Q(R(L))$  is the quotient field of  $R(L)$ , then*

$$\kappa(L) = \text{trdeg}_k Q(R(L)) - 1.$$

Mori establishes the theory of the Iitaka dimension in *loc. cit.* largely based on this interpretation.

**Iitaka fibration.** We start by recalling a standard concept:

DEFINITION 1.2.8. A morphism of normal varieties  $f: X \rightarrow Y$  is called an *algebraic fiber space* if it is surjective, projective, with connected fibers.

REMARK 1.2.9. (1) Note that for an arbitrary projective morphism  $f: X \rightarrow Y$ , the assumption  $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$  implies that  $f$  is surjective, with connected fibers. This condition is also equivalent to saying that if  $X \rightarrow Z \rightarrow Y$  is the Stein factorization of  $f$ , then the finite part  $Z \rightarrow Y$  is in fact the identity map.

(2) Moreover, by a variant Zariski's Main Theorem, under the normality assumption the connectedness of all the fibers is in fact equivalent to  $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ . Imposing the latter interpretation allows us then to extend the definition even when  $X$  and  $Y$  are not normal.

EXERCISE 1.2.10. Show that a surjective projective morphism of normal varieties is an algebraic fiber space if and only if the field of rational functions  $k(Y)$  is algebraically closed in  $k(X)$  via the corresponding field extension  $k(Y) \subseteq k(X)$ .

An important technical tool in the study of line bundles of non-negative Kodaira dimension is the following:

THEOREM 1.2.11. *Let  $X$  be a normal projective variety, and let  $L$  be a line bundle on  $X$  such that  $\kappa(L) \geq 0$ . Then there exists an algebraic fiber space of normal varieties  $f: X' \rightarrow Y$  such that:*

- (1)  $\dim Y = \kappa(L)$ .
- (2) *There is a birational morphism  $\varphi: X' \rightarrow X$ , such that if  $L' := \varphi^*L$  and  $F$  is a very general fiber of  $f$ , then  $\kappa(F, L'|_F) = 0$ .*

*Up to birational equivalence,  $f$  is the unique morphism with such properties.*

For a more precise statement, and proof, see [La, Theorem 2.1.23]. It turns out, and this is important in various arguments, that the morphism  $f$  in the statement is birational to the mapping  $\varphi_m: X \cdots \rightarrow Y_m = \overline{\varphi_m(X)} \subseteq \mathbf{P}^{N_m}$  induced by the linear system  $|L^{\otimes m}|$ , for any  $m$  sufficiently large and divisible.

<sup>3</sup>In general this is not finitely generated, though it is so when  $L = \omega_X$  by a celebrated theorem of Birkar-Cascini-Hacon-McKernan.

DEFINITION 1.2.12. The morphism  $f$  in Theorem 1.2.11 is called the *Iitaka fibration* of the line bundle  $L$ . This notion is defined only up to birational equivalence, hence at least in characteristic zero one may assume that  $X'$  and  $Y$  are nonsingular. Similarly, one can define the Iitaka fibration of a ( $\mathbb{Q}$ -)Cartier divisor  $D$ .

REMARK 1.2.13. Note that Proposition 1.2.4 can also be deduced as a simple consequence of the existence of the Iitaka fibration; see the proof of [La, Corollary 2.1.38].

The discussion above also leads to a statement that we will use repeatedly:

PROPOSITION 1.2.14 (**Easy Addition**). *Let  $f: X \rightarrow Y$  be an algebraic fiber space of normal projective varieties, and let  $L$  be a line bundle on  $X$ . Then we have*

$$\kappa(L) \leq \kappa(F, L|_F) + \dim Y,$$

where  $F$  is the general fiber of  $f$ .

PROOF. If  $\kappa(L) = -\infty$  the statement is clear. Therefore we may assume  $\kappa(L) \geq 0$ , and consider a sufficiently large and divisible  $m$  such that  $\varphi_m: X \cdots \rightarrow \mathbf{P}^{N_m}$  induced by  $|L^{\otimes m}|$  gives the Iitaka fibration, so that if  $Z := \overline{\varphi_m(X)}$ , then  $\dim Z = \kappa(L)$ .

Moreover, we can take  $m$  such that for the general fiber  $F$  we have

$$\dim \overline{\varphi_m(F)} \leq \kappa(F, L|_F).$$

Indeed, note that the restriction morphism

$$H^0(X, L^{\otimes m}) \rightarrow H^0(F, L|_F^{\otimes m})$$

induces a rational map

$$\mathbf{P}(H^0(F, L|_F^{\otimes m})) \cdots \rightarrow \mathbf{P}(H^0(X, L^{\otimes m}))$$

which is compatible with the inclusion  $F \hookrightarrow X$  via the two maps given by the complete linear systems  $|L|_F^{\otimes m}|$  and  $|L^{\otimes m}|$ .

Finally, if  $W := \overline{(f, \varphi_m)(X)} \subseteq Y \times Z$ , we have  $\dim W \geq \dim Z$  via the second projection, while on the other hand

$$\dim W = \dim Y + \dim p^{-1}(y)$$

for  $y \in Y$  general, where  $p_1$  is the first projection. But by the previous paragraph  $\dim p^{-1}(y) \leq \kappa(F, L|_F)$ , and we are done.  $\square$

### 1.3. Kodaira dimension

We now discuss the main character studied in this course, the fundamental (numerical) birational invariant of an algebraic variety.

DEFINITION 1.3.1. (1) If  $X$  is a smooth projective variety, the *Kodaira dimension* of  $X$  is

$$\kappa(X) := \kappa(X, \omega_X).$$

Due to the stronger statement in Exercise 1.1.12, the Kodaira dimension is a birational invariant. In particular, if  $X$  is a singular projective variety, and  $\tilde{X} \rightarrow X$  is any resolution of singularities, we set

$$\kappa(X) := \kappa(\tilde{X}),$$

and this is a well-defined invariant.

(2) We say that  $X$  is of *general type* if  $\kappa(X) = \dim X$ .

In other words, the rough interpretation for the Kodaira dimension is that

$$P_m(X) \sim m^{\kappa(X)}$$

for  $m$  sufficiently large and divisible.

EXAMPLE 1.3.2.  $\kappa(\mathbf{P}^n) = -\infty$ .

EXAMPLE 1.3.3. Example 1.1.3 gives us the following classification of smooth projective curves  $C$  of genus  $g$  in terms of Kodaira dimension:

- $\kappa(C) = -\infty \iff g = 0$ , i.e. if  $C \simeq \mathbf{P}^1$ .
- $\kappa(C) = 0 \iff g = 1$ , i.e. if  $C$  is elliptic.
- $\kappa(C) = 1 \iff g \geq 2$ . These are the curves of general type.

EXAMPLE 1.3.4. If  $X$  is a (weak) Calabi-Yau variety (like a  $K3$  surface, an abelian variety, or a hypersurface of degree  $n + 1$  in  $\mathbf{P}^n$ ), then  $\kappa(X) = 0$ .

EXAMPLE 1.3.5. Example 1.1.4 gives us the Kodaira dimension of a hypersurface  $X \subset \mathbf{P}^n$  of degree  $d$ .

- $d \leq n \iff \kappa(X) = -\infty$ .
- $d = n + 1 \iff \kappa(X) = 0$
- $d \geq n + 2 \iff \kappa(X) = n - 1 = \dim X$ .

EXAMPLE 1.3.6. If  $X$  and  $Y$  are smooth projective varieties, then

$$\kappa(X \times Y) = \kappa(X) + \kappa(Y).$$

This follows from Example 1.1.9.

EXAMPLE 1.3.7. The example above shows that one can produce examples of varieties of any allowed Kodaira dimension. Let's see this for surfaces: say  $E$  is an elliptic curve, and  $C$  is a curve of genus  $g \geq 2$ , and  $D$  is any curve. Then:

- $\kappa(\mathbf{P}^1 \times D) = -\infty$ .
- $\kappa(E \times E) = 0$ .
- $\kappa(E \times C) = 1$ .
- $\kappa(C \times C) = 2$ .

This can easily be extended to arbitrary dimension.

EXAMPLE 1.3.8. Let  $X$  be a smooth projective variety, and  $E$  a locally free sheaf of rank  $r \geq 2$ . Then for the associated projective bundle we have  $\kappa(\mathbf{P}(E)) = 0$ . In particular this is the case for ruled surfaces. See Example 1.1.10.

EXAMPLE 1.3.9. Let  $f: X \rightarrow Y$  be a generically finite surjective morphism of smooth projective varieties. Then:

- (1)  $\kappa(X) \geq \kappa(Y)$ .
- (2) If moreover  $f$  is étale, then  $\kappa(X) = \kappa(Y)$ .

The first item is due to the fact that we have

$$\omega_X \simeq f^* \omega_Y \otimes \mathcal{O}_X(R),$$

where  $R$  is the (effective) ramification divisor, combined with the projection formula (recall that we always have an injection  $\mathcal{O}_Y \hookrightarrow f_* \mathcal{O}_X$ ). For the second item we use in addition the fact that  $R = 0$ , and Exercise 1.2.5.

EXAMPLE 1.3.10 (**Iitaka fibration**). When  $L = \omega_X$ , the fibration in Theorem 1.2.11 is called the *Iitaka fibration of  $X$* . Up to birational equivalence, we can think of it as a morphism  $X' \rightarrow Y$ , with  $X'$  birational to  $X$ ,  $\dim Y = \kappa(X)$ , and  $\kappa(F) = 0$  for its very general fiber  $F$ .

EXAMPLE 1.3.11 (**Easy Addition**). When  $f: X \rightarrow Y$  is an algebraic fiber space of smooth projective varieties and general fiber  $F$ , Proposition 1.2.14 gives

$$\kappa(F) + \dim Y \geq \kappa(X).$$

In particular, if  $\kappa(X) \geq 0$ , then  $\kappa(F) \geq 0$ .

## 1.4. Big and nef line bundles

**Nef divisors and line bundles.** The next definition introduces the most famous semi-positivity notion in algebraic geometry.

DEFINITION 1.4.1. Let  $X$  be a projective (or proper) scheme, and  $D$  a Cartier (or  $\mathbb{Q}$ -Cartier) divisor on  $X$ . Then  $D$  is *nef* if  $D \cdot C \geq 0$  for every irreducible curve  $C \subseteq X$ . We can of course make the same definition for line bundles.

The definition is numerical, and therefore we can actually talk about nef classes in  $N^1(X)$  or  $N^1(X)_{\mathbb{Q}}$ .

EXAMPLE 1.4.2. (1) Ample divisors are nef by the Nakai-Moishezon criterion, which I recall here; see [La, Theorem 1.2.23] for a proof.

THEOREM 1.4.3 (**Nakai-Moishezon ampleness criterion**). *Let  $X$  be a proper variety over a field, and  $L$  a line bundle on  $X$ . Then  $L$  is ample if and only if  $L^{\dim V} \cdot V > 0$  for any subvariety  $V \subseteq X$ .*

(2) More generally, *semiample* (i.e. such that a multiple is basepoint-free) divisors are nef. Even more generally:

EXERCISE 1.4.4. Let  $f : X \rightarrow Y$  be a proper and surjective morphism, and  $L$  a line bundle on  $Y$ . Then  $L$  is nef if and only if  $f^*L$  is nef.

(3) Any effective divisor on a homogeneous variety is nef (as we can translate it so it does not contain any given curve).

EXERCISE 1.4.5. Let  $D$  and  $E$  be Cartier divisors on  $X$  such that  $mD + E$  is nef for all  $m \geq 1$  (or sufficiently large). Then  $D$  is nef.

Note however that Nakai-Moishezon suggests a stronger notion of semi-positivity. This is in fact equivalent to nefness by the following important result of Kleiman:

THEOREM 1.4.6. *Let  $X$  be a proper variety (or scheme), and  $D$  a  $\mathbb{Q}$ -divisor on  $X$ . Then  $D$  is nef if and only if*

$$D^{\dim V} \cdot V \geq 0$$

for every irreducible subvariety  $V \subseteq X$ .

PROOF. For simplicity we assume that  $X$  is projective.<sup>4</sup> One implication is clear. Assume now that  $D$  is nef. We do induction on  $n = \dim X$ ; the case  $n = 1$  is clear, and we assume that we know the statement for all proper varieties of dimension at most  $n - 1$ . Thus we know inductively that

$$D^{\dim V} \cdot V \geq 0, \quad \forall V \subset X \text{ of dimension } \leq n - 1,$$

and so we only need to show that  $D^n \geq 0$ .

To this end, fix an ample divisor  $H$  on  $X$ , and consider the polynomial in  $t \in \mathbb{R}$ :

$$P(t) := (D + tH)^n = \sum_{k=0}^n \binom{n}{k} \cdot t^{n-k} \cdot (D^k \cdot H^{n-k}).$$

Note that we can formally do this, even though we haven't talked about  $\mathbb{R}$ -divisors in detail;  $P$  takes values in  $\mathbb{R}$ . Assuming that  $P(0) < 0$ , we want to obtain a contradiction.

By the inductive hypothesis, for  $k < n$  we have  $D^k \cdot H^{n-k} \geq 0$ , since  $H^{n-k}$  is represented by an effective cycle class of dimension  $k$ . Thus the coefficients of  $t^{n-k}$  in  $P(t)$  are non-negative for all  $k < n$ , while the coefficient of  $t^n$  is positive. This implies that  $P'(t) > 0$  for all  $t > 0$ , in which range  $P$  is therefore increasing. Since in addition we are assuming  $P(0) < 0$ , it follows that  $P$  has only one real root  $a > 0$ .

We now show that  $D + tH$  is ample for any rational number  $t > a$ . Using Nakai-Moishezon, we need to check that

$$(D + tH)^{\dim V} \cdot V > 0, \quad \forall V \subseteq X.$$

If  $V = X$ , then this simply says that

$$P(t) > P(a) = 0 \quad \text{for } t > a.$$

---

<sup>4</sup>One can easily reduce to this case using *Chow's Lemma*, which says that if  $X$  is a proper scheme, then there exists a projective scheme  $Y$  and a surjective morphism  $f : Y \rightarrow X$  which is an isomorphism over a dense open set in  $X$ .

If  $V \subsetneq X$ , then in the term by term expansion of  $(D + tH)^{\dim V} \cdot V$  we have as above that  $D^k \cdot H^{\dim V - k} \cdot V \geq 0$  for  $k > 0$ , while  $H^{\dim V} \cdot V > 0$  since  $H$  is ample. The claim follows.

Finally, write

$$P(t) = Q(t) + R(t), \quad Q(t) = D \cdot (D + tH)^{n-1}, \quad R(t) = tH \cdot (D + tH)^{n-1}.$$

We know that if  $t > a$ , then  $D + tH$  is ample, and so  $Q(t) \geq 0$  since  $(D + tH)^{n-1}$  is represented by an effective class of dimension 1. By continuity we have that  $Q(a) \geq 0$ . On the other hand, let's note that  $R(a) > 0$ , which gives our contradiction since then  $P(a) > 0$  as well. Indeed, yet again by the same argument as above, all the terms involving both  $D$  and  $H$  are non-negative, while  $H^n > 0$ .  $\square$

One of the most important consequences of this theorem is that nef divisors are limits of ample ones. More precisely:

**COROLLARY 1.4.7.** *Let  $X$  be a projective scheme, and  $D$  and  $H$   $\mathbb{Q}$ -divisors on  $X$ . Then:*

- (1) *If  $D$  is nef and  $H$  is ample, then  $D + H$  is ample.*
- (2) *If  $D + \varepsilon H$  is ample for all  $\varepsilon \in \mathbb{Q}_{>0}$ , then  $D$  is nef.*

**PROOF.** For (1) we apply Nakai-Moishezon; for every  $V \subseteq X$  we have

$$(D + H)^{\dim V} \cdot V = \sum_{k=0}^{\dim V} \binom{\dim V}{k} \cdot (D^{\dim V - k} \cdot H^k \cdot V).$$

Since  $H$  is ample,  $H^k \cdot V$  is represented by an effective class of dimension  $\dim V - k$  on  $V$ , and therefore the nefness of  $D$  implies by Kleiman's theorem above that all the terms are non-negative. Moreover,  $H^{\dim V} \cdot V > 0$ , and so the total intersection number is  $> 0$ .

For (2), consider any irreducible curve  $C$  in  $X$ . Since  $D + \varepsilon H$  is ample for all  $\varepsilon > 0$ , we know that

$$(D + \varepsilon H) \cdot C > 0, \quad \forall \varepsilon > 0.$$

Passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain  $D \cdot C \geq 0$ .  $\square$

**REMARK 1.4.8.** The study of cones of ample and nef numerical divisor classes inside the Néron-Severi space is of great importance in higher dimensional geometry. See [La, §1.4 and 1.5] for an extensive introduction.

**Big line bundles.** Recall the following important notion regarding line bundles on a normal projective variety  $X$ .

**DEFINITION 1.4.9.** A line bundle  $L$  on  $X$  is called *big* if  $\kappa(L) = \dim X$ . Equivalently, there exists a constant  $C > 0$  such that

$$h^0(X, L^{\otimes m}) \geq C \cdot m^{\dim X}$$

for all  $m$  sufficiently large and divisible. We can make the same definition for a Cartier divisor  $D$ , by considering  $L = \mathcal{O}_X(D)$ . Moreover, since it is easy to see that the definition

depends only on sufficiently large and divisible multiples, we can similarly define bigness if  $D$  is a  $\mathbb{Q}$ -divisor.

EXAMPLE 1.4.10. (1) An ample divisor is big. More generally, if  $A$  is an ample divisor and  $E$  is an effective divisor, then  $A + E$  is big. Indeed, note that the number of sections of  $\mathcal{O}_X(m(A + E))$  is larger than that of  $\mathcal{O}_X(mA)$ , and so the statement follows from asymptotic Riemann-Roch.

(2) If  $f : X \rightarrow Y$  is a generically finite surjective morphism of normal projective varieties, and  $L$  is a big line bundle on  $Y$ , then  $f^*L$  is big on  $X$ . The most important instance is when  $f$  is birational and  $L$  is ample; for this reason bigness is sometimes called the birational version of ampleness.

For the proof, note that we have

$$H^0(X, f^*L^{\otimes m}) \simeq H^0(Y, f_*f^*L^{\otimes m}) \simeq H^0(Y, L^{\otimes m} \otimes f_*\mathcal{O}_X),$$

where for the second equality we applied the projection formula. But since  $f$  is surjective, we have an inclusion  $\mathcal{O}_Y \hookrightarrow f_*\mathcal{O}_X$ , and therefore

$$H^0(Y, L^{\otimes m}) \subset H^0(X, f^*L^{\otimes m}), \quad \text{for all } m.$$

Note also that  $\dim X = \dim Y$ , since  $f$  is generically finite. This immediately implies what we want.

LEMMA 1.4.11. *Let  $L$  be a big line bundle on  $X$ , and  $M$  any other line bundle. Then*

$$H^0(X, L^{\otimes m} \otimes M) \neq 0$$

*for  $m$  sufficiently large and divisible.*

PROOF. It is standard that we can write  $M \simeq \mathcal{O}_X(D - E)$ , with  $D$  and  $E$  very ample divisors. It obviously suffices to replace  $M$  by  $M(-D)$ , i.e. by  $\mathcal{O}_X(-E)$ . Consider now the exact sequence

$$0 \longrightarrow H^0(X, L^{\otimes m} \otimes \mathcal{O}_X(-E)) \longrightarrow H^0(X, L^{\otimes m}) \longrightarrow H^0(E, L|_E^{\otimes m}).$$

We know that there exists  $C > 0$  such that  $h^0(X, L^{\otimes m}) \geq C \cdot m^n$  for  $m$  sufficiently large and divisible, where  $n = \dim X$ . On the other hand, since  $E$  is a divisor,  $h^0(E, L|_E^{\otimes m})$  grows at most like  $m^{n-1}$ , and so for such  $m$  we get the conclusion.  $\square$

The following is Kodaira's important characterization of big divisors; colloquially, it says that "big is ample plus effective".

PROPOSITION 1.4.12 (**Kodaira's Lemma**). *If  $D$  is a big divisor, then for any ample divisor  $A$ , there exists  $m > 0$  and an effective divisor  $E$  such that  $mD \sim A + E$ . Conversely, if there exists an ample divisor  $A$ , an effective divisor  $E$ , and  $m > 0$  such that*

$$mD \equiv A + E,$$

*then  $D$  is big.*

PROOF. If  $D$  is big, we can simply apply Lemma 1.4.11 with  $L = \mathcal{O}_X(D)$  and  $M = \mathcal{O}_X(-A)$ .

If  $mD \equiv A + E$ , then  $B = mD - E$  is numerically equivalent to an ample divisor, and hence ample. We then obviously have

$$\kappa(D) \geq \kappa(B) = \dim X.$$

□

COROLLARY 1.4.13. *Bigness is a numerical property: if  $D_1 \equiv D_2$ , then  $D_1$  is big if and only if  $D_2$  is big.*

EXERCISE 1.4.14. (1) If  $L$  is a big line bundle on  $X$ , then show that there is a proper closed subset  $Z \subset X$  such that  $L|_V$  is big for every subvariety  $V$  of  $X$  such that  $V \not\subseteq Z$ .

(2) Give an example of a big line bundle  $L$  and a subvariety  $V \subset X$  such that  $L|_V$  is not big.

Finally we record the following very useful technical statement involving big line bundles and the Iitaka dimension, for later use.

LEMMA 1.4.15. *Let  $f : X \rightarrow Y$  be an algebraic fiber space with  $X$  and  $Y$  normal and projective, and with general fiber  $F$ , and let  $N$  a line bundle on  $X$ . Then there exists a big line bundle  $L$  on  $Y$  and an integer  $m > 0$  with  $f^*L \hookrightarrow N^{\otimes m}$  if and only if*

$$\kappa(X, N) = \kappa(F, N|_F) + \dim Y.$$

PROOF. We only prove the “only if” statement (which will be used in the sequel), and leave the other implication as an exercise.<sup>5</sup> Note that by Easy Addition we always have the inequality

$$\kappa(X, N) \leq \kappa(F, N|_F) + \dim Y,$$

hence the point is to prove the opposite inequality given our hypothesis.

By Kodaira’s Lemma, Proposition 1.4.12, we may assume that  $L$  is an ample, or even very ample, line bundle. By possibly replacing  $N$  by a multiple, we may also assume that:

- $f^*L \hookrightarrow N$ .
- $\varphi_{|N|} : X \cdots \rightarrow \mathbf{P}^n = \mathbf{P}(H^0(X, N))$  gives the Iitaka fibration of  $N$ .

Since  $f$  is a fiber space, the first item gives an inclusion  $H^0(Y, L) \subseteq H^0(X, N)$ , which in turn induces a rational projection map

$$\mathbf{P}(H^0(X, N)) \cdots \rightarrow \mathbf{P}(H^0(X, L)).$$

Denoting by  $X_N$  the closure of the image of  $\varphi_{|N|}$ , since we are assuming that  $Y$  is isomorphic to its image via  $\varphi_{|L|}$ , this projection induces a rational map  $X_N \cdots \rightarrow Y$ .

<sup>5</sup>See also [Mo, Proposition 1.14] for a more complete statement due to Fujita, saying that both conditions are equivalent to the fact that  $f$  factors through the Iitaka fibration of  $N$ .

Since passing to further resolutions does not affect our problem, we may consider smooth projective varieties  $\tilde{X}$  and  $\tilde{X}_N$  with proper birational morphisms to  $X$  and  $X_N$  respectively, such that we have a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{h} & \tilde{X}_N \\ & \searrow \tilde{f} & \downarrow g \\ & & Y \end{array}$$

where  $\tilde{f}$ ,  $h$  and  $g$  replace  $f$ ,  $\varphi_{|N|}$  and the projection map respectively. Let's denote their respective general fibers by  $\tilde{F}$ ,  $H$  and  $G$ , so that we have an induced algebraic fiber space

$$\tilde{F} \xrightarrow{H} G.$$

Note the following points:

- Since  $\tilde{F}$  and  $F$  are birational, we have  $\kappa(\tilde{F}, \tilde{N}_{|\tilde{F}}) = \kappa(F, N_{|F})$ , where  $\tilde{N}$  is the pullback of  $N$  to  $\tilde{X}$ .
- Since  $\varphi_{|N|}$  gives the Iitaka fibration, and  $H$  is birational to its fiber, we have  $\kappa(\tilde{N}_{|H}) = 0$ .
- $\dim G = \dim X_N - \dim Y = \kappa(N) - \dim Y$ , again since  $\varphi_{|N|}$  gives the Iitaka fibration.

We are left with observing that the Easy Addition formula applied to  $\tilde{N}_{|\tilde{F}}$  gives

$$\kappa(\tilde{N}_{|H}) + \dim G \geq \kappa(\tilde{F}, \tilde{N}_{|\tilde{F}}),$$

which given the three items above is equivalent to the desired inequality.  $\square$

**Nef and big divisors.** The combination of these two notions is particularly powerful, and unlike bigness by itself, it can be characterized by a simple numerical condition. Note for motivation that the pullback of an ample divisor by a birational (or even generically finite) morphism is big and nef.

**THEOREM 1.4.16.** *Let  $X$  be a projective variety of dimension  $n$ , and  $D$  and  $E$  nef  $\mathbb{Q}$ -divisors on  $X$ . If  $D^n > n \cdot (D^{n-1} \cdot E)$ , then  $D - E$  is big.*

**PROOF.** Note first that the inequality in the hypothesis still holds if we replace  $D$  and  $E$  by  $D + \varepsilon A$  and  $E + \varepsilon A$ , where  $A$  is an ample divisor and  $0 < \varepsilon \ll 1$ . These are both ample, so after passing then to large multiples, we can assume that both  $D$  and  $E$  are very ample integral divisors.

Fix now  $m > 0$ , and a divisor  $E_m \in |mE|$ . Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(m(D - E)) \xrightarrow{\cdot E_m} \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_{E_m}(mD) \longrightarrow 0.$$

Since  $D$  is very ample, we know that  $h^0(X, \mathcal{O}_X(mD))$  grows like  $\frac{D^n}{n!} \cdot m^n$ , and so to conclude it suffices to prove that

$$h^0(E_m, \mathcal{O}_{E_m}(mD)) \leq n \cdot \frac{D^{n-1} \cdot E}{n!} \cdot m^n + O(m^{n-1}).$$

We are allowed to choose  $E_m$  as we like, and so we can take  $E_m = B_1 + \cdots + B_m$ , with  $B_i \in |E|$  general members. A simple argument then shows that it suffices to have

$$h^0(B_i, \mathcal{O}_{B_i}(mD)) \leq n \cdot \frac{D^{n-1} \cdot E}{n!} \cdot m^{n-1} + O(m^{n-2})$$

for all  $i$ . But this is true, with equality, by Riemann-Roch.  $\square$

**COROLLARY 1.4.17.** *Let  $X$  be a projective variety of dimension  $n$ , and  $D$  a nef  $\mathbb{Q}$ -divisor on  $X$ . Then  $D$  is big if and only if  $D^n > 0$ .*

**PROOF.** If  $D^n > 0$ , then Theorem 1.4.16 applies with  $E = 0$ . Conversely, assume that  $D$  is nef and big. By bigness, for every very ample divisor  $H$ , there exist  $m > 0$  and an effective divisor  $E$  such that

$$mD \sim H + E.$$

Now Kleiman's theorem implies that  $D^{n-1} \cdot E \geq 0$ , and so

$$m \cdot D^n = D^{n-1} \cdot (H + E) \geq D^{n-1} \cdot H.$$

We can however choose  $H$  general enough so that  $D|_H$  is big (using for instance Exercise 1.4.14), and so reasoning inductively on dimension we get  $D^{n-1} \cdot H = D|_H^{n-1} > 0$ .  $\square$

## 1.5. Birational classification of surfaces

I would like to motivate the problem we will be studying by first (mostly) understanding it in the case of smooth projective surfaces, where a good classification theory is available. Let's first collect some basic facts.

**Note.** Much of the general material in this section can be found in [Ha] Ch.V. The more refined results, including the main classification theorem, are the subject of Beauville's book [Be]. The proof of (2) and some of (1) in Theorem 1.5.10 is the crux of the matter, and takes a good part of [Be].

The following result is crucial in treating birational isomorphism classes of smooth projective surfaces over algebraically closed fields. Its proof is a classical application of Zariski's Main Theorem; see [Ha] V.5 for a detailed discussion. Unless otherwise specified, blow-up means blow-up at one point.

**THEOREM 1.5.1.** *A birational morphism of smooth projective surfaces factors as a finite sequence of blow-ups. A birational map between smooth projective surfaces factors as a finite sequence of blow-ups followed by a finite sequence of blow-downs; more precisely, if  $X_1$  and  $X_2$  are birational, then there exists a smooth projective surface  $Y$  and morphisms  $f_1 : Y \rightarrow X_1$  and  $f_2 : Y \rightarrow X_2$  that are both finite compositions of blow-ups.*

Recall now that if  $X$  and  $Y$  are birational, then  $\kappa(X) = \kappa(Y)$ . Hence when studying the behavior of the Kodaira dimension, it will often suffice to look for special representatives in a birational equivalence class. With this in mind, we have:

DEFINITION 1.5.2. (1) A  $(-1)$ -curve on a smooth projective surface  $X$  is a curve  $E \subset X$  such that  $E \simeq \mathbf{P}^1$  and  $E^2 = -1$ .

(2) A smooth projective surface  $X$  is *minimal* if it contains no  $(-1)$ -curves.

REMARK 1.5.3. The reason for this definition is Castelnuovo's contractibility criterion (see [Ha, Theorem 5.7]), which says that if  $X$  contains a  $(-1)$ -curve  $E$ , then there exists a smooth projective surface  $Y$  and a birational morphism  $f : X \rightarrow Y$  which contracts precisely  $E$  (the map is in fact the blow-up of a point on  $Y$ ). Thus  $(-1)$ -curves can only appear as exceptional divisors of blow-ups at smooth points. Minimality in this formulation is then equivalent to the more intuitive notion:  $X$  is minimal if and only if any birational morphism  $X \rightarrow Y$  to a smooth projective surface  $Y$  is in fact an isomorphism.

In more modern language, the crucial notion here is that of nefness.

PROPOSITION 1.5.4. *Let  $X$  be a smooth projective surface with  $\kappa(X) \geq 0$ . Then  $X$  is minimal if and only if  $K_X$  is nef.*

PROOF. Since a  $(-1)$ -curve is a rational curve  $E$  such that  $E^2 = -1$ , and consequently  $K_X \cdot E = -1$ . Therefore it is clear that if  $K_X$  is nef, then such a curve cannot exist.

Assume now that  $X$  is minimal. Since  $\kappa(X) \geq 0$ , we can find an effective divisor

$$D = \sum a_i C_i \in |mK_X|, \quad \text{some } m > 0,$$

with  $a_i > 0$  and  $C_i$  irreducible curves. If  $K_X$  were not nef, then there would exist  $C \subset X$  irreducible curve such that  $K_X \cdot C < 0$ , and so then clearly  $C = C_i$  for some  $i$ . We then have

$$0 > D \cdot C \geq a_i \cdot (C_i \cdot C)$$

and so  $C^2 < 0$ . But now the adjunction formula says

$$2p_a(C) - 2 = C^2 + K_X \cdot C$$

and the left hand side is at least  $-2$ , while both numbers on the right hand side are negative. This implies immediately that  $p_a(C) = 0$ , hence  $C$  is smooth, and in fact a  $(-1)$ -curve.  $\square$

REMARK 1.5.5. In dimension at least 3, this is taken to be the definition of minimality in a birational equivalence class: a variety with  $\kappa(X) \geq 0$  is *minimal* if  $K_X$  is nef. However, in this case it soon becomes clear that one cannot stay inside the world of smooth varieties; in fact  $X$  should be allowed to be a projective normal variety with  $K_X$   $\mathbb{Q}$ -Cartier, and having terminal singularities. Whether such varieties exist in every birational equivalence class is the one of the main topics of the minimal model program.

It is well known that, starting with any smooth projective surface, one can arrive at a minimal one after contracting a finite number of  $(-1)$ -curves; see for instance [Ha, Theorem 5.8]. Therefore if we want to classify surfaces according to Kodaira dimension, it is enough to focus on minimal models.

EXAMPLE 1.5.6. Among the surfaces we've discussed until now,  $\mathbf{P}^2$  is minimal since all curves are very ample, while abelian surfaces and  $K3$  surfaces are minimal since  $\omega_X \simeq \mathcal{O}_X$ . Surfaces of degree at least 5 in  $\mathbf{P}^3$  (i.e. those that are hypersurfaces of general type) are minimal since  $\omega_X$  is ample. Products  $E \times C$ , with  $E$  and  $C$  smooth projective curves of genus 1 and  $\geq 2$  respectively, are minimal surfaces with Kodaira dimension 1.

Let's see that almost all ruled surfaces are minimal as well. The first claim is that every ruled surface  $\pi : X \rightarrow C$  with  $C$  a curve of genus  $g(C) \geq 1$  is minimal. Indeed, assume that  $E$  is a rational curve in  $X$ ; the induced map  $E \rightarrow C$  cannot be surjective since  $g(E) < g(C)$  (a consequence of the Riemann-Hurwitz formula), and so  $\pi(E)$  is a point. This means that  $E$  is a fiber of  $\pi$ , and so  $E^2 = 0$ . Note that if we fix  $C$ , then all of these minimal models are birational to  $C \times \mathbf{P}^1$ . One can show that these are all the minimal models in this birational equivalence class.

Ruled surfaces over  $\mathbf{P}^1$  are treated in the next example.

EXAMPLE 1.5.7 (**Rational ruled surfaces**). Now consider a rational ruled surface  $\pi : X \rightarrow \mathbf{P}^1$ . Note that these are all birational to  $\mathbf{P}^2$ . We know that  $X = \mathbf{P}(E)$  for some rank 2 vector bundle  $E$  on  $\mathbf{P}^1$ , and since these are always direct sums of line bundles by Grothendieck's theorem, after normalization it follows that  $X$  is isomorphic to one of

$$F_n := \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(n)), \quad \text{for some } n \geq 0.$$

These are sometimes called *Hirzebruch surfaces*.

EXERCISE 1.5.8. With the notation above, show the following statements:

- (1)  $F_n \simeq F_m$  if and only if  $n = m$ .
- (2)  $F_n$  has a section over the base  $\mathbf{P}^1$  with self-intersection  $-n$ .
- (3)  $F_n$  is minimal if and only if  $n \neq 1$ .
- (4)  $F_1 \simeq \text{Bl}_p(\mathbf{P}^2)$  for some  $p \in \mathbf{P}^2$ .

It can be shown that every minimal rational surface is either  $\mathbf{P}^2$  or one of the  $F_n$  with  $n \neq 1$ . Thus overall for surfaces with  $\kappa(X) = -\infty$ , minimal models are not unique, but they are completely classified. On the other hand, one can show the following result:

THEOREM 1.5.9. *Let  $X_1$  and  $X_2$  be non-ruled minimal surfaces. Then every birational map from  $X_1$  to  $X_2$  is an isomorphism. In particular, every non-ruled minimal surface admits a unique minimal model.*

The main theorem in the birational classification of surfaces is the following list of minimal models. I will only state it in characteristic 0; small modifications have to be made when  $\text{char } k = p > 0$ . We use the following standard notation:

$$p_g(X) = P_1(X) = h^0(X, \omega_X) \quad \text{and} \quad q(X) = h^1(X, \mathcal{O}_X) = h^1(X, \omega_X).$$

THEOREM 1.5.10. *Let  $X$  be a minimal surface. Then one of the following holds:*

- (1) *If  $\kappa(X) = -\infty$ , then  $X$  is  $\mathbf{P}^2$ , a rational ruled surface different from  $F_1$ , or a ruled surface over a curve of genus at least 1.*

(2) If  $\kappa(X) = 0$ , then  $X$  belongs to one of the following four classes:

- $p_g(X) = 0$  and  $q(X) = 0$ ; in this case  $2K_X \sim 0$ , and we say that  $X$  is an Enriques surface.
- $p_g(X) = 0$  and  $q(X) = 1$ ; in this case  $S$  is a bielliptic surface:  $S \simeq E \times F/G$ , where  $E$  and  $F$  are elliptic curves, and  $G$  is a finite group of translations of  $E$  acting on  $F$  such that  $F/G \simeq \mathbf{P}^1$ .
- $p_g(X) = 1$  and  $q(X) = 0$ ; in this case  $K_X \sim 0$ , and  $X$  is a K3 surface.
- $p_g(X) = 1$  and  $q(X) = 2$ ; in this case  $X$  is an abelian surface.

(3) If  $\kappa(X) = 1$ , then there exists a smooth projective curve  $C$  and a surjective morphism  $p : X \rightarrow C$ , such that the general fiber of  $p$  is an elliptic curve. Such a surface is called an elliptic surface.<sup>6</sup>

(4) If  $\kappa(X) = 2$ , then  $X$  is by definition a surface of general type.

**EXAMPLE 1.5.11 (Enriques surfaces).** We have not discussed Enriques surfaces before, so let's establish their existence. First I recall the following general fact:

**PROPOSITION 1.5.12.** *If  $X$  be a variety, then there exists a one-to-one correspondence between étale double covers  $\pi : \tilde{X} \rightarrow X$  and 2-torsion line bundles  $L$ , i.e.  $L \neq \mathcal{O}_X$  such that  $L^{\otimes 2} \simeq \mathcal{O}_X$ . This is characterized by*

$$\pi_* \mathcal{O}_{\tilde{X}} \simeq \mathcal{O}_X \oplus L^{-1} \quad \text{and} \quad \pi^* L \simeq \mathcal{O}_{\tilde{X}}.$$

**PROOF.** I will only sketch the proof, and let you fill in the details. (We will discuss more general covering constructions later on.) Think of  $L$  as a vector bundle of rank 1 rather than an invertible sheaf, and consider its total space  $p : \mathbb{L} \rightarrow X$ . Fix an isomorphism  $\alpha : L^{\otimes 2} \simeq \mathcal{O}_X$ , so that in vector bundle language we have an isomorphism

$$\alpha : \mathbb{L} \otimes \mathbb{L} \longrightarrow X \times \mathbf{C}$$

sitting over the identity on  $X$ . We now consider

$$\tilde{X} = \{(x, u) \mid \alpha(x, u \otimes u) = (x, 1)\} \subset \mathbb{L},$$

i.e. via  $\alpha$  we put over each  $x \in X$  the two roots of unity in  $\mathbf{C}$ . The projection  $p$  induces a morphism  $\pi : \tilde{X} \rightarrow X$ , which is everywhere 2 : 1. Also, the mapping

$$\tilde{X} \longrightarrow \tilde{X} \times_X \mathbb{L} = \pi^* \mathbb{L}, \quad (x, u) \mapsto (x, (u, u))$$

gives a global section of  $\pi^* \mathbb{L}$  that does not vanish anywhere. But the existence of such a section equivalent to saying that  $\pi^* L \simeq \mathcal{O}_{\tilde{X}}$ .

On the other hand, if we start with  $\pi : \tilde{X} \rightarrow X$  as in the statement, the  $\mathbb{Z}_2$ -action on  $\pi_* \mathcal{O}_{\tilde{X}}$  decomposes it into eigenbundles, and  $L^{-1}$  is the nontrivial one.  $\square$

**PROPOSITION 1.5.13.** *Let  $X$  be an Enriques surface, and  $\pi : \tilde{X} \rightarrow X$  the étale double cover corresponding to  $\omega_X$  (recall that  $\omega_X^{\otimes 2} \simeq \mathcal{O}_X$ ). Then  $\tilde{X}$  is a K3 surface. Conversely, any quotient of a K3 surface by a fixed-point-free involution is an Enriques surface.*

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<sup>6</sup>Note however that many other types of surfaces, of different Kodaira dimension, can be elliptic surfaces.

PROOF. For the first implication, by Proposition 1.5.12 we have  $\pi^*\omega_X \simeq \mathcal{O}_{\tilde{X}}$ . On the other hand,  $\pi$  is étale, and so  $\pi^*\omega_X \simeq \omega_{\tilde{X}}$ . It follows that  $\omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}$ . Note also that since  $\pi$  is étale of degree 2, we have

$$(1.5.1) \quad 2 - q(\tilde{X}) = \chi(\mathcal{O}_{\tilde{X}}) = 2 \cdot \chi(\mathcal{O}_X) = 2$$

and so  $q(\tilde{X}) = 0$ . These are the two requirements in the definition of a  $K3$  surface.

In the opposite direction, first by the projection formula it is clear that

$$0 = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \geq h^1(X, \mathcal{O}_X),$$

hence  $q(X) = 0$ . We can then use again (1.5.1), this time in order to deduce that  $p_g(X) = 0$ .  $\square$

Here is a concrete example. Consider quadrics

$$Q_1, Q_2, Q_3 \in k[X_0, X_1, X_2] \quad \text{and} \quad Q'_1, Q'_2, Q'_3 \in k[X_3, X_4, X_5],$$

and using these build the quadrics

$$P_i = Q_i(X_0, X_1, X_2) + Q'_i(X_3, X_4, X_5) \in k[X_0, \dots, X_5], \quad i = 1, 2, 3.$$

Assuming that the  $Q_i$  and  $Q'_i$  are generic, we get three smooth quadrics in  $\mathbf{P}^5$ . Taking the complete intersection

$$X = Z(P_1) \cap Z(P_2) \cap Z(P_3) \subset \mathbf{P}^5$$

we have a complete intersection of type  $X_{2,2,2}$ , and so  $X$  is a  $K3$  surface. We now construct a fixed-point-free involution  $i$  on  $X$ ; according to Proposition 1.5.13, we will then have that  $X/i$  is an Enriques surface.

To this end, consider first the involution

$$\sigma : \mathbf{P}^5 \longrightarrow \mathbf{P}^5, \quad (x_0 : \dots : x_5) \mapsto (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5).$$

Clearly  $\sigma(X) = X$ , and so we have an induced involution  $i : X \rightarrow X$ . Now the fixed locus of  $\sigma$  is

$$F(\sigma) = Y_1 \cup Y_2, \quad Y_1 = (x_0 = x_1 = x_2 = 0) \quad \text{and} \quad Y_2 = (x_3 = x_4 = x_5 = 0).$$

But as the  $Q_i$  are generic, they do not have any points in common on  $Y_1$ , and similarly for the  $Q'_i$  on  $Y_2$  (check!). It follows that  $i$  has no fixed points.

REMARK 1.5.14. It can be shown that the generic Enriques surface is isomorphic to one as in the example above.

EXAMPLE 1.5.15 (**Elliptic  $K3$  surfaces**). Some  $K3$  surfaces can be written as fibrations  $f : X \rightarrow \mathbf{P}^1$ , where the general fiber is an elliptic curve.

In fact, assume that there is a smooth elliptic curve  $C \subset X$  inside a  $K3$  surface. Then by the genus formula we obtain  $C^2 = 0$ . Note also that Riemann-Roch gives us

$$h^0(X, \mathcal{O}_X(C)) - h^1(X, \mathcal{O}_X(C)) + h^2(X, \mathcal{O}_X(C)) = 2.$$

On the other hand, since  $\omega_X \simeq \mathcal{O}_X$ , Serre duality immediately implies that  $h^2(X, \mathcal{O}_X(C)) = 0$ , while  $h^1(X, \mathcal{O}_X(C)) = h^1(X, \mathcal{O}_X(-C))$ . But this last group is also 0, as it can be seen by passing to cohomology in the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

It follows that  $h^0(X, \mathcal{O}_X(C)) = 2$ , and so the linear system  $|C|$  is a pencil inducing a rational map  $f : X \rightarrow \mathbf{P}^1$ . As  $C^2 = 0$ , it follows that  $f$  is in fact a morphism; its fibers are the members of the pencil, and so the general one is a smooth elliptic curve.

Here is a concrete example when one can find such elliptic curves on a  $K3$ . Take  $X$  to be a smooth quartic surface in  $\mathbf{P}^3$  containing a line  $L$ . Take  $H$  to be a hyperplane section of  $X$  containing  $L$ , and then consider the linear system  $|H - L|$  on  $X$ . There is a 1-dimensional family of planes in  $\mathbf{P}^3$  containing  $L$ , and so this is a pencil. Now by Bézout  $H$  is a curve of degree 4, living inside a hyperplane in  $\mathbf{P}^3$ , i.e. a  $\mathbf{P}^2$ . But it contains the line  $L$ , which must then be a component, so there is a residual component  $C$  of degree 3. The linear system  $|H - L|$  is clearly basepoint-free, since the hyperplanes containing  $L$  intersect precisely in  $L$ . Thus the general such  $C$  is a smooth cubic in  $\mathbf{P}^2$ , so an elliptic curve.

**EXAMPLE 1.5.16 (Elliptic surfaces).** Let's also sketch the proof of part (3) in Theorem 1.5.10, i.e. the fact that minimal surfaces of Kodaira dimension 1 are elliptic surfaces. We begin with some preliminaries; the surfaces will always be smooth and projective.

**LEMMA 1.5.17.** *If  $X$  is a minimal surface with  $\kappa(X) = 0, 1$ , then  $K_X^2 = 0$ .*

**PROOF.** Since  $X$  is minimal,  $K_X$  is nef, and therefore  $K_X^2 \geq 0$ . But if we had  $K_X^2 > 0$ , then by Corollary 1.4.17  $K_X$  would be big, which is equivalent to  $\kappa(X) = 2$ .  $\square$

**LEMMA 1.5.18.** *Let  $X$  be a minimal surface with  $K_X^2 = 0$ , and assume that  $P_m(X) \geq 2$  for some  $m$ , so that we can write*

$$|mK_X| = |M| + F$$

with  $F$  the fixed part and  $M$  the moving part. Then

$$K_X \cdot F = K_X \cdot M = F^2 = F \cdot M = M^2 = 0.$$

**PROOF.** The hypothesis implies that  $K_X \cdot M + K_X \cdot F = 0$ . But  $K_X$  is nef, and so both summands are non-negative. It follows that

$$K_X \cdot M = K_X \cdot F = 0.$$

Now  $M$  does not have fixed components, and so  $M^2 \geq 0$  and  $M \cdot F \geq 0$ . Using that  $M \cdot K_X = M^2 + M \cdot F$  and  $F \cdot K_X = M \cdot F + F^2$ , we easily obtain the other identities.  $\square$

Now start with  $X$  minimal with  $\kappa(X) = 1$ . This last condition implies that there exists an  $m > 0$  such that  $P_m(X) \geq 2$ , i.e.  $|mK_X|$  is a positive dimensional linear system. Take its decomposition into the moving part and fixed part

$$mK_X = M + F.$$

A priori the moving part gives a rational map  $\varphi_M : X \rightarrow \mathbf{P}^N$ . By Lemma 1.5.17 and Lemma 1.5.18 we have however that  $M^2 = 0$ , which implies that  $\varphi_M$  is really a morphism (otherwise different divisors in  $M$  would intersect in the finite base locus, giving  $M^2 > 0$ ). The hypothesis also implies that we can take  $\varphi_M(X)$  to be a curve, say  $B$ .

We now consider the Stein factorization  $p : X \rightarrow C$  of the induced  $\varphi_M : X \rightarrow B$ , so that  $p$  has connected fibers and  $C$  is a normal and hence smooth curve. Denote by  $E$  the general fiber of  $p$ . Note that

$$M \simeq \varphi_M^* \mathcal{O}_{\mathbf{P}^N}(1) \simeq p^* \mathcal{O}_C(1),$$

and so  $M$  is linearly equivalent to a sum of general fibers of  $p$ . By Lemma 1.5.18 we have that  $K_X \cdot M = 0$ . On the other hand, since  $K_X$  is nef we have that  $K_X \cdot E \geq 0$ . Putting all of this together, it follows that  $K_X \cdot E = 0$ .

Note however that since  $E$  is a fiber, we also have that  $E^2 = 0$ . But the genus formula says that  $2g(E) - 2 = E^2 + K_X \cdot E$ , so we conclude that  $g(E) = 1$ .

**EXAMPLE 1.5.19.** There exist surfaces of general type with  $q(X) = 1$ . For instance one with  $p_g(X) = 1$  is the minimal resolution of a  $C_1 \times C_2/G$ , where  $C_1$  and  $C_2$  are smooth projective curves of genus 2 and 3 respectively, and  $G$  is a finite group acting on them, with  $C_1/G \simeq \mathbf{P}^1$  and  $C_2/G$  elliptic. (This takes a bit of work.) All such surfaces can be written (via the Albanese map) as fibrations  $f : X \rightarrow E$ , where  $E$  is an elliptic curve and the general fiber is a curve of genus at least 2.

**EXAMPLE 1.5.20.** Recall that there exist examples of surfaces of Kodaira dimension 1 such that the base of the elliptic fibration has genus 0 or 1.

**Behavior of Kodaira dimension with respect to morphisms.** The classification of smooth projective surfaces described earlier almost leads to the following:

**THEOREM 1.5.21.** *Let  $f : X \rightarrow C$  be an algebraic fiber space over an algebraically closed field of characteristic 0, with  $X$  and  $C$  a smooth projective surface and curve respectively. If  $F$  is the general fiber of  $f$ , then*

$$\kappa(X) \geq \kappa(F) + \kappa(C).$$

Let's see why. First, we can safely assume that  $\kappa(F)$  and  $\kappa(C)$  are nonnegative, i.e.

$$g(F) \geq 1 \quad \text{and} \quad g(C) \geq 1.$$

In this case we can also assume that  $X$  is a minimal surface, since the all rational curves must then live in the fibers of  $f$ , and the Kodaira dimension is a birational invariant.

The case  $\kappa(X) = 2$  is clear.

Since besides  $\mathbf{P}^2$  in the  $\kappa(X) = -\infty$  case we only have ruled surfaces, this case is also clear; indeed, again since rational curves cannot dominate curves of higher genus, we must have that  $f$  is the same as the ruled surface structure, but this is not allowed by the assumption  $g(F) \geq 1$ .

A minimal surface with  $\kappa(X) = 1$  has a, possibly different, fibration  $g : X \rightarrow B$ , with general fiber an elliptic curve  $E$ . Now compare  $E$  with the fibration  $f : X \rightarrow C$ .

If  $g(C) \geq 2$ , then  $E$  cannot dominate  $C$ , and so it must be a fiber of  $f$  as well. It then follows that  $f$  and  $g$  must coincide, so  $f$  is also an elliptic fibration and we are done. If  $g(C) \leq 1$ , then we are done anyway.

The remaining case  $\kappa(X) = 0$  is trickier, and what we learned earlier is not quite enough. One must show that  $g(C) = 1$  and  $g(F) = 1$ , but there is nothing in the classification list that tells us yet that this is the case. The key point is the following completely nontrivial result, first observed by Ueno in this case, and then generalized by Fujita, Kawamata, Viehweg, etc. in various ways to higher dimension, which we will discuss later:

**THEOREM 1.5.22.** *If  $f : X \rightarrow C$  is a fiber space as above, then  $\deg f_*\omega_{X/C} \geq 0$ .*<sup>7</sup>

Let's apply this: assume for instance that  $\omega_X \simeq \mathcal{O}_X$ , like in the  $K3$  or abelian case. Since  $f_*\mathcal{O}_X \simeq \mathcal{O}_C$  by the fiber space assumption, we have that

$$\deg f_*\omega_{X/C} = \deg \omega_C^{-1} = 2 - 2g(C) \geq 0,$$

and so  $g(C) \leq 1$ . On the other hand  $\omega_F \simeq \omega_{X|F} \simeq \mathcal{O}_F$ , which means that  $F$  is an elliptic curve.

In the Enriques or bielliptic case  $\omega_X$  is not trivial, but only torsion. The exact same argument shows in any case that  $F$  must be an elliptic curve, since  $\deg \omega_F = 0$ . As for  $C$ , one may argue by taking an étale base change  $C' \rightarrow C$  which makes the canonical of  $X$  trivial, or apply directly as above an important generalization of Theorem 1.5.22 which says that in fact

$$\deg f_*\omega_{X/C}^{\otimes m} \geq 0, \quad \forall m \geq 1.$$

We will come back to this result in a more general context.

**Note.** A good reference for such results (and further refinements) in the context of surfaces, is [BHPV, §17, §18]. The main tool is the period map for families of curves, and the Torelli theorem. A more algebraic proof can be found in [We].

**REMARK 1.5.23 (Strict inequality).** It is clear that one cannot hope to always have equality in Theorem 1.5.21. The most obvious case is when  $C = \mathbf{P}^1$ , but  $\kappa(X) \geq 0$ ; see for instance Example 1.5.15, but really any surface endowed with a basepoint-free pencil. However it may also happen when both sides are finite; see Examples 1.5.19 and 1.5.20. We will see later on that equality does hold though when  $f$  is a smooth morphism.

## 1.6. Vanishing theorems

In this section we deal only with varieties defined over  $\mathbf{C}$ . The results can be shown to hold for all algebraically closed fields in characteristic 0, but are known to fail as stated in positive characteristic. We start with perhaps the best known vanishing theorem.

<sup>7</sup>Note that  $f_*\omega_{X/C}$  is a torsion-free sheaf on a smooth curve, hence it is locally free. Then  $\deg f_*\omega_{X/C}$  is simply its first Chern class, or the degree of its determinant line bundle.

**THEOREM 1.6.1 (Kodaira Vanishing).** *Let  $X$  be a smooth complex projective variety of dimension  $n$ , and let  $L$  be an ample line bundle on  $X$ . Then*

$$H^i(X, \omega_X \otimes L) = 0 \text{ for all } i > 0.$$

*Equivalently,*

$$H^i(X, L^{-1}) = 0 \text{ for all } i < n.$$

Kodaira Vanishing is the special case  $p = n$  of the following result about all bundles of holomorphic forms.

**THEOREM 1.6.2 (Nakano Vanishing).** *Let  $X$  be a smooth complex projective variety of dimension  $n$ , and  $L$  an ample line bundle on  $X$ . Then*

$$H^q(X, \Omega_X^p \otimes L) = 0 \text{ for } p + q > n,$$

*or equivalently*

$$H^q(X, \Omega_X^p \otimes L^{-1}) = 0 \text{ for } p + q < n.$$

Here I will prove these theorems using a method first introduced by Kollár, based on what are called injectivity theorems. The approach to the proof is due to Esnault-Viehweg; a lot about this can be found in their book [EV].

**DEFINITION 1.6.3 (Forms with log-poles).** Let  $X$  be a smooth variety, and  $D$  a smooth effective divisor on  $X$ . The sheaf of 1-forms on  $X$  with log-poles along  $D$  is

$$\Omega_X^1(\log D) = \Omega_X^1 \left\langle \frac{df}{f} \right\rangle, \quad f \text{ local equation for } D.$$

Concretely, if  $z_1, \dots, z_n$  are local coordinates on  $X$ , chosen such that  $D = (z_n = 0)$ , then  $\Omega_X^1(\log D)$  is locally generated by  $dz_1, \dots, dz_{n-1}, \frac{dz_n}{z_n}$ . This is a free system of generators, so  $\Omega_X^1(\log D)$  is locally free of rank  $n$ . For any integer  $p$ , we define

$$\Omega_X^p(\log D) := \bigwedge^p (\Omega_X^1(\log D)).$$

**LEMMA 1.6.4.** *There are short exact sequences:*

$$(i) \quad 0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log D) \longrightarrow \Omega_D^{p-1} \longrightarrow 0.$$

$$(ii) \quad 0 \longrightarrow \Omega_X^p(\log D)(-D) \longrightarrow \Omega_X^p \longrightarrow \Omega_D^p \longrightarrow 0.$$

**PROOF.** I will sketch the proof for  $p = 1$ ; in general it is only notationally more complicated. The comprehensive source for this is [EV] Section 2.

Choose local analytic coordinates  $z_1, \dots, z_n$  so that  $D = (z_n = 0)$ . For (i), the map on the right is the *residue map* along  $D$

$$\text{res}_D : \Omega_X^1(\log D) \longrightarrow \mathcal{O}_D$$

given by

$$f_1 dz_1 + \dots + f_{n-1} dz_{n-1} + f_n \frac{dz_n}{z_n} \mapsto f_n|_D,$$

where  $f_1, \dots, f_n$  are local functions on  $X$ . The right hand side is 0 if one can write  $f = z_n \cdot g$  for an arbitrary regular function  $g$ . Therefore we can see the kernel as being locally generated by  $dz_1, \dots, dz_n$ , hence isomorphic to  $\Omega_X^1$ .

For (ii), the map on the right is given by restriction of forms. Since locally  $D = (z_n = 0)$ , the kernel of the restriction map  $\Omega_X^1 \rightarrow \Omega_D^1$  is locally generated by  $z_n dz_1, \dots, z_n dz_{n-1}, dz_n$ . But these obviously generate the subsheaf  $\Omega_X^1(\log D)(-D) \subset \Omega_X^1(\log D)$ .  $\square$

**Cyclic covers.** I will state here a useful technical result needed in order to “take  $m$ -th roots” of divisors  $B \in |mD|$  with  $m \geq 2$ . For a thorough survey and other useful covering constructions see [La] 4.1.B and [EV] Section 3.

**PROPOSITION 1.6.5.** *Let  $X$  be a variety over an algebraically closed field  $k$ , and let  $L$  be a line bundle on  $X$ . Let  $0 \neq s \in H^0(X, L^{\otimes m})$  for some  $m \geq 1$ , with  $D = Z(s) \in |mL|$ . Then there exists a finite flat morphism  $f : Y \rightarrow X$  of degree  $m$ , where  $Y$  is a scheme over  $k$  such that if  $L' = f^*L$ , there is a section*

$$s' \in H^0(Y, L') \text{ satisfying } (s')^m = f^*s.$$

Moreover:

- if  $X$  and  $D$  are smooth, then so are  $Y$  and  $D' = Z(s')$ .
- the divisor  $D'$  maps isomorphically onto  $D$ .
- there is a canonical isomorphism  $f_*\mathcal{O}_Y \simeq \mathcal{O}_X \oplus L^{-1} \oplus \dots \oplus L^{-(m-1)}$ .
- for every  $p \geq 1$ , one has

$$f_*\Omega_Y^p \simeq \Omega_X^p \oplus \bigoplus_{i=1}^{m-1} \Omega_X^p(\log D) \otimes L^{-i}.$$

**PROOF.** Let's first do this construction locally: assume that  $X = \text{Spec } A$ , and think of  $s$  as a function  $s \in A$ . Then, introducing a new variable  $t$ , one can simply define

$$Y = \text{Spec } \frac{A[t]}{(t^m - s)} \subset X \times \mathbf{A}^1.$$

The natural morphism from  $A$  to this new ring (or the projection onto the first factor of  $X \times \mathbf{A}^1$ ) induces a map  $f : Y \rightarrow X$ , which is clearly finite. If  $X$  and  $D$  are smooth, we can more specifically assume that  $X$  has a coordinate system  $x_1, \dots, x_n$  such that  $s = x_1$ . Then  $Y$  has a coordinate system  $y_1, \dots, y_n$  with  $t = y_1$ , so that the map  $f$  can be described as

$$(1.6.1) \quad (y_1, y_2, \dots, y_n) \mapsto (y_1^m, y_2, \dots, y_n).$$

It follows that  $Y$  is smooth as well, and we also see that  $f$  is ramified exactly over  $D = (x_1 = 0)$ , where it is in fact maximally ramified. Note moreover that  $D' = (y_1 = 0)$  maps isomorphically onto  $D$ .

Note furthermore that the ring  $B = A[t]/(t^m - s)$ , which can be identified with the sheaf  $f_*\mathcal{O}_Y$ , admits a decomposition

$$(1.6.2) \quad B = \bigoplus_{i=0}^{m-1} A \cdot t^i,$$

and we know that  $t^m = s \in A$ . Now the group  $\mu_m$  of  $m$ -th roots of unity acts on  $B$  (so on  $Y$ ) as follows: if  $\mu$  is a primitive  $m$ -th root, then  $\mu$  acts on  $t$  by  $\mu \cdot t$ . It is clear that the eigenspace associated to  $\mu^i$  is precisely the summand  $A \cdot t^i$  in the decomposition above.

This is the local version. Now start with global  $X$  and  $D$ , where  $D$  is a divisor associated to a section  $s$  of  $L^{\otimes m}$ . Choose an affine open cover of  $X$  on which  $L$  can be trivialized, and for each open  $U_i \subset X$  in this cover, think of  $s_{U_i}$  as a function  $s_i$  on  $U_i$ . The construction above can be performed to get

$$f_i : Y_i = \text{Spec} \frac{\mathcal{O}_X(U_i)[t_i]}{(t_i^m - s_i)} \longrightarrow U_i.$$

If  $U_i$  and  $U_j$  are open sets in the cover, the line bundle comes with transition functions  $g_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$ ; the transition functions of  $L^{\otimes m}$  are therefore  $g_{ij}^m$ , so the  $s_i$  satisfy

$$s_i = g_{ij}^m \cdot s_j.$$

We can now glue  $Y_i$  and  $Y_j$  over  $U_i \cap U_j$  by using the rule

$$(1.6.3) \quad t_i = g_{ij} \cdot t_j,$$

which is compatible with the formula above since  $t_i^m = s_i$ . We get a variety  $Y$ , due to the fact that the gluing behaves well on triple overlaps because of the cocycle condition

$$g_{ij} \cdot g_{jk} \cdot g_{ki} = 1$$

satisfied by the transition functions. The  $g_{ij}$  are also the transition functions of the line bundle  $L' = f^*L$ , and so (1.6.3) implies that the  $t_i$  glue to give a global section  $t \in H^0(Y, L')$ . Obviously  $t^m = f^*s$ . Since  $f$  is a finite morphism, if  $X$  is projective then  $Y$  is projective as well. Also, the considerations in the local case apply to say that if  $X$  and  $D$  are smooth, then  $Y$  is also smooth.

Finally, we need to establish the decomposition formulas for push-forwards of bundles of holomorphic forms. Using an open cover as above, recall that over  $U_j$  the sheaf  $f_*\mathcal{O}_Y$  can be described as the  $A_j$ -algebra

$$B_j := \text{Spec} \frac{A_j[t_j]}{(t_j^m - s_j)} = \bigoplus_{i=0}^{m-1} A_j \cdot t_j^i,$$

where the decomposition into a direct sum of free rank 1  $A_j$ -modules on the right hand side corresponds to the eigenspaces of the  $\mu_m$ -action. Each of these glue to a line bundle; note that since the  $t_j^i$  transform according to the formula

$$t_j^i = g_{jk}^i \cdot t_k^i,$$

it follows that this line bundle is that given by the transition functions  $g_{jk}^{-i}$ , i.e.  $L^{-i}$ . This proves the formula for  $f_*\mathcal{O}_Y$ .

Let's conclude by proving the formula for  $f_*\Omega_Y^1$ ; that for arbitrary  $p$  is left as an exercise. On one of the open sets  $U_j$  of our cover, consider a local coordinate system  $x_1, \dots, x_n$  on  $X$  such that  $D = (x_1 = 0)$  as above, so that  $\Omega_X^1(\log D)$  is generated by  $dx_1/x_1, dx_2, \dots, dx_n$ . Consider also a coordinate system  $y_1, \dots, y_n$  on  $Y$  so that the mapping is given by (1.6.1), and  $t_j$  corresponds to  $y_1$ . Over  $U_j$  we have

$$f_*\Omega_Y^1(U_j) \simeq \bigoplus_{k=1}^n B_j \cdot dy_k \simeq \bigoplus_{k=1}^n \bigoplus_{i=0}^{m-1} A_j \cdot y_1^i dy_k.$$

Note now that for  $k \geq 2$  we have  $y_1^i dy_k = y_1^i dx_k$ . For  $k = 1$ , the formula  $x_1 = y_1^m$  implies that  $dx_1 = m \cdot y_1^{m-1} dy_1$ , and in particular  $dx_1/x_1 = m \cdot dy_1/y_1$ . We conclude that

$$y_1^i dy_1 = \frac{1}{m} y_1^{i+1} \frac{dx_1}{x_1}.$$

The eigenspaces of  $f_*\Omega_Y^1(U_j)$  under the action of  $\mu_m$  are obtained by putting together the terms which contain the same power of  $y_1$ , and one easily checks that they correspond to the summands in the statement.  $\square$

We will use Proposition 1.6.5 and some basic information coming from Hodge theory in order to prove the following “injectivity theorem”, originally due to Kollár, which turns out to be stronger than Kodaira vanishing.

**THEOREM 1.6.6.** *Let  $X$  be a smooth projective variety,  $L$  a line bundle on  $X$ , and a nontrivial section  $s \in H^0(X, L^{\otimes m})$  such that  $D = Z(s)$  is a smooth divisor. Then, for each  $j$ , the map*

$$H^j(X, \omega_X \otimes L) \longrightarrow H^j(X, \omega_X \otimes L^{\otimes m+1})$$

*induced by multiplication by  $s$  is injective.*

**PROOF.** We use the construction and notation of Proposition 1.6.5. Since  $f$  is finite, we obtain isomorphisms

$$H^j(Y, \mathcal{O}_Y) \simeq H^j(X, \mathcal{O}_X) \oplus \bigoplus_{i=1}^{m-1} H^j(X, L^{-i})$$

and

$$H^j(Y, \Omega_Y^p) \simeq H^j(X, \Omega_X^p) \oplus \bigoplus_{i=1}^{m-1} H^j(X, \Omega_X^p(\log D) \otimes L^{-i}).$$

We consider now the exterior derivative

$$d : \mathcal{O}_Y \longrightarrow \Omega_Y^1,$$

which is a  $\mathbf{C}$ -linear sheaf homomorphism. This induces for each  $j$  a homomorphism on cohomology

$$d : H^j(Y, \mathcal{O}_Y) \longrightarrow H^j(Y, \Omega_Y^1).$$

But Hodge theory tells us that this homomorphism is always *zero*; this is a special case of the degeneration at  $E_1$  of the Hodge-to-de Rham spectral sequence. In more elementary terms, the reason is that these two spaces are isomorphic to the spaces  $H^{0,j}(Y)$

and  $H^{1,j}(Y)$  of forms of the corresponding types. But each element in  $H^{0,j}(Y)$  can be represented by a harmonic form, and all such forms are  $d$ -closed.

Note that  $d$  is compatible with the decompositions above (exercise; see also the proof of Lemma 1.6.7 below), and so it induces maps

$$d : H^j(X, L^{-1}) \longrightarrow H^j(X, \Omega_X^1(\log D) \otimes L^{-1})$$

which are also identically zero. Now recall from Lemma 1.6.4 that we have a residue mapping  $\Omega_X^1(\log D) \rightarrow \mathcal{O}_D$ . Tensoring it with  $L^{-1}$  and passing to cohomology, we finally get that the induced homomorphism

$$H^j(X, L^{-1}) \longrightarrow H^j(D, L_{|D}^{-1})$$

is zero as well. But Lemma 1.6.7 below tells us that up to scalar this is the same as the homomorphism induced from the short exact sequence

$$0 \longrightarrow L^{-1}(-D) \longrightarrow L^{-1} \longrightarrow L_{|D}^{-1} \longrightarrow 0.$$

Looking at the long exact sequence on cohomology, it follows that the induced homomorphisms

$$H^j(X, L^{-1}(-D)) \longrightarrow H^j(X, L^{-1})$$

are all surjective. Recalling that  $L^{\otimes m} \simeq \mathcal{O}_X(D)$ , Serre duality implies the statement we want.  $\square$

LEMMA 1.6.7. *The homomorphism*

$$H^j(X, L^{-1}) \longrightarrow H^j(D, L_{|D}^{-1})$$

*above, obtained using the residue map, is the same as the natural homomorphism induced by restriction, after multiplication by  $m$ .*

PROOF. We in fact prove the pre-cohomology statement that the mapping

$$L^{-1} \longrightarrow \Omega_X^1(\log D) \otimes L^{-1} \longrightarrow L_{|D}^{-1}$$

obtained by composing  $d$  on the eigensheaves corresponding to  $\mu$  with the residue map along  $D$  is equal to the restriction map up to a factor of  $m$ . We are then allowed to work in local coordinates, and we use the notation in the proof of Proposition 1.6.5.

In local coordinates on an open set  $U$ , the summand  $L^{-1}$  of  $f_*\mathcal{O}_Y$  is generated by elements of the form  $fy_1$ , with  $f \in \mathcal{O}_X(U)$ . Note that

$$d(fy_1) = dfy_1 + fdy_1 = y_1 \left( df + \frac{f}{m} \cdot \frac{dx_1}{x_1} \right),$$

which is a section of  $\Omega_X^1(\log D) \otimes L^{-1}$  over  $U$ . Its residue along  $(x_1 = 0)$  is equal to  $y_1 \cdot \frac{f}{m}$  restricted to  $(x_1 = 0)$ , which after multiplication by  $m$  coincides with the restriction of  $fy_1$ .  $\square$

**Proof of Theorem 1.6.1 using Theorem 1.6.6.** Let  $L$  be an ample line bundle. Then there exists  $m \gg 0$  such that  $L$  is very ample and

$$H^i(X, \omega_X \otimes L^{\otimes m+1}) = 0 \quad \text{for all } i > 0.$$

But the linear system  $|mL|$  contains a smooth divisor, so we can apply Theorem 1.6.6 to deduce that  $H^i(X, \omega_X \otimes L)$  embeds in this space.

**Kollár vanishing.** Theorem 1.6.6 also leads to important generalization of Kodaira Vanishing to higher direct images of canonical bundles; the point is to use the more general case when  $L$  is not necessarily ample.

**THEOREM 1.6.8 (Kollár Vanishing).** *Let  $f : X \rightarrow Y$  be a morphism from a smooth projective variety  $X$  to a projective variety  $Y$ , and let  $L$  be an ample line bundle on  $Y$ . Then*

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0, \quad \text{for all } i \text{ and all } j > 0.$$

**PROOF.** Let  $m$  be a sufficiently large integer such that  $L^{\otimes m}$  is very ample. If  $B \in |mL|$  is a general element and  $D = f^*B$ , then by Bertini's theorem  $D$  is a smooth hypersurface on  $X$ . Let's denote

$$f_D : D \longrightarrow B$$

the restriction of  $f$  to  $D$ . By the adjunction formula  $\omega_D \simeq \omega_X(D)|_D$ , while on the other hand  $\mathcal{O}_X(D) \simeq f^*L^{\otimes m}$ , so

$$\omega_D \simeq \omega_{X|D} \otimes g^*L|_B^{\otimes m}.$$

It follows that we have a short exact sequence

$$0 \longrightarrow \omega_X \otimes f^*L \longrightarrow \omega_X \otimes f^*L^{\otimes m+1} \longrightarrow \omega_D \otimes f_D^*L|_B \longrightarrow 0.$$

Pushing this sequence forward, we obtain a long exact sequence

$$\cdots \longrightarrow R^i f_* \omega_X \otimes L \xrightarrow{\cdot B} R^i f_* \omega_X \otimes L^{\otimes m+1} \longrightarrow R^i f_{D*} \omega_D \otimes L|_B \longrightarrow \cdots$$

We can however choose the divisor  $B$  sufficiently general, such that the mapping

$$R^i f_* \omega_X \otimes L \xrightarrow{\cdot B} R^i f_* \omega_X \otimes L^{\otimes m+1}$$

is in fact injective; this follows from Exercise 1.6.9 below. In this case the long exact sequence above reduces to a collection of short exact sequences

$$0 \longrightarrow R^i f_* \omega_X \otimes L \xrightarrow{\cdot B} R^i f_* \omega_X \otimes L^{\otimes m+1} \longrightarrow R^i f_{D*} \omega_D \otimes L|_B \longrightarrow 0.$$

By induction on dimension, we may assume that

$$H^j(Y, R^i f_{D*} \omega_D \otimes L|_B) = 0, \quad \text{for all } i \text{ and all } j > 0.$$

Using Serre vanishing, we may also choose  $m$  large enough so that the higher cohomology of all  $R^i f_* \omega_X \otimes L^{\otimes m+1}$  vanishes. Combined with the inductive assumption about the right-most sheaf, this implies first of all that

$$H^j(Y, R^i f_* \omega_X \otimes L) = 0 \quad \text{for all } j \geq 2.$$

For the final case  $j = 1$  we need to use the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \omega_X \otimes L) \implies H^{p+q}(X, \omega_X \otimes f^*L).$$

We have already shown that  $E_2^{p,q} = 0$  for  $p \geq 2$  and all  $q$ ; this can be easily seen to imply that

$$E_2^{1,i} = E_\infty^{1,i},$$

since all the terms coming into or out of the  $E_k^{1,i}$  are 0 for each  $k$ . Since  $E_\infty^{1,i}$  is the lowest term of the filtration on  $H^{i+1}$ , this implies that for each  $i$  we have an injection

$$E_2^{1,i} = H^1(Y, R^i f_* \omega_X \otimes L) \hookrightarrow H^{i+1}(X, \omega_X \otimes f^* L).$$

On the other hand, we apply Theorem 1.6.6 to the (semiample) line bundle  $f^* L$  and to the divisor  $D$  on  $X$ , to conclude that the natural maps

$$H^{i+1}(X, \omega_X \otimes f^* L) \xrightarrow{\cdot D} H^{i+1}(X, \omega_X \otimes f^* L^{\otimes m+1})$$

are injective for all  $i$ . But the composition of these two injections also factors through the bottom left term in the commutative diagram

$$\begin{array}{ccc} H^1(Y, R^i f_* \omega_X \otimes L) & \longrightarrow & H^{i+1}(X, \omega_X \otimes f^* L) \\ \downarrow & & \downarrow \\ H^1(Y, R^i f_* \omega_X \otimes L^{\otimes m+1}) & \longrightarrow & H^{i+1}(X, \omega_X \otimes f^* L^{\otimes m+1}) \end{array}$$

and this is 0 for  $m \gg 0$  by Serre Vanishing. We finally conclude that

$$H^1(Y, R^i f_* \omega_X \otimes L) = 0$$

as well. □

**EXERCISE 1.6.9.** Let  $\mathcal{F}$  be a coherent sheaf and  $L$  a very ample line bundle on a projective variety  $X$ . If  $s \in H^0(X, L)$  is a general section, then the induced morphism

$$\mathcal{F} \xrightarrow{\cdot s} \mathcal{F} \otimes L$$

is injective.

## 1.7. Castelnuovo-Mumford regularity

An effective link between vanishing and global generation is provided by the theory of Castelnuovo-Mumford regularity. This is usually defined with respect to  $\mathcal{O}_{\mathbf{P}^n}(1)$  on a projective space, but we can consider a slightly more general class of line bundles. For a complete treatment, see [La, §1.8].

**DEFINITION 1.7.1.** Let  $X$  be a projective variety, and  $L$  an ample and globally generated line bundle on  $X$ . A coherent sheaf  $\mathcal{F}$  on  $X$  is called  *$m$ -regular with respect to  $L$*  if

$$H^i(X, \mathcal{F} \otimes L^{\otimes m-i}) = 0 \quad \forall i > 0.$$

**THEOREM 1.7.2 (Castelnuovo-Mumford Lemma).** *Let  $X$  be a projective variety, and  $L$  an ample and globally generated line bundle on  $X$ . Let  $\mathcal{F}$  be a coherent sheaf on  $X$  which is  $m$ -regular with respect to  $L$ , and let  $k \geq 0$ . Then:*

(i)  $\mathcal{F}$  is  $(m+k)$ -regular with respect to  $L$ .

(ii)  $\mathcal{F} \otimes L^{\otimes m+k}$  is globally generated.

(iii) The multiplication map

$$H^0(X, \mathcal{F} \otimes L^{\otimes m}) \otimes H^0(X, L^{\otimes k}) \longrightarrow H^0(X, \mathcal{F} \otimes L^{\otimes m+k})$$

is surjective.

PROOF. Note first that it is enough to prove only (i) and (iii). Indeed, if we know (iii) for all  $k$ , we can combine it with the fact that by Serre's theorem  $\mathcal{F} \otimes L^{\otimes m+k}$  is globally generated for  $k \gg 0$ . But we have a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{F} \otimes L^{\otimes m}) \otimes H^0(X, L^{\otimes k}) \otimes \mathcal{O}_X & \longrightarrow & H^0(X, \mathcal{F} \otimes L^{\otimes m+k}) \otimes \mathcal{O}_X \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{F} \otimes L^{\otimes m}) \otimes L^{\otimes k} & \longrightarrow & \mathcal{F} \otimes L^{\otimes m+k} \end{array}$$

where the vertical and bottom horizontal maps are obtained from the evaluation of global sections of the sheaves in question. It follows that for  $k \gg 0$  the composition of the top horizontal and left vertical maps is surjective. Therefore the bottom horizontal map is surjective, which means precisely that  $\mathcal{F} \otimes L^{\otimes m}$  is globally generated. Also, because of the inductive nature of (i), it is in fact enough to prove (i) and (iii) for  $k = 1$ .

Denote  $V = H^0(X, L)$ , and say  $\dim V = n$ . Since  $L$  is globally generated, we have a surjective map

$$V \otimes \mathcal{O}_X \xrightarrow{\text{ev}} L.$$

Thinking of this as a nowhere-vanishing section of the vector bundle  $V^\vee \otimes L$ , we can associate to it a Koszul complex (see [La, Appendix B.2]):

$$0 \longrightarrow \bigwedge^n V \otimes L^{\otimes -n} \longrightarrow \dots \longrightarrow \bigwedge^2 V \otimes L^{\otimes -2} \longrightarrow V \otimes L^{\otimes -1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that this is an exact complex, and the kernels (= cokernels) of the maps in the complex are all locally free. Indeed, recall that on projective space  $\mathbf{P} = \mathbf{P}(V)$  we have the Euler sequence

$$0 \longrightarrow \Omega_{\mathbf{P}}^1 \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0,$$

and so the kernel of the map  $V \otimes L^{\otimes -1} \rightarrow \mathcal{O}_X$  is the vector bundle  $f^*\Omega_{\mathbf{P}}^1$ , with  $f : X \rightarrow \mathbf{P}$  the morphism induced by  $L$ . Then it is not hard to see that the other kernels in the Koszul complex are isomorphic to  $f^*\Omega_{\mathbf{P}}^i$ .

Twisting the Koszul complex by  $\mathcal{F} \otimes L^{\otimes m+1}$ , we then get another exact complex

$$0 \longrightarrow \bigwedge^n V \otimes \mathcal{F} \otimes L^{\otimes m+1-n} \longrightarrow \dots \longrightarrow \bigwedge^2 V \otimes \mathcal{F} \otimes L^{\otimes m-1} \longrightarrow V \otimes \mathcal{F} \otimes L^{\otimes m} \longrightarrow \mathcal{F} \otimes L^{\otimes m+1} \longrightarrow 0.$$

Since  $\mathcal{F}$  is  $m$ -regular, we have that

$$(1.7.1) \quad H^i(X, \bigwedge^{i+1} V \otimes \mathcal{F} \otimes L^{\otimes m-i}) = 0, \quad \forall i > 0.$$

Chasing cohomology inductively from left to right in the exact sequence, applying the vanishing in (1.7.1) at each step, we finally obtain surjectivity at the  $H^0$ -level on the right, i.e. that of the map

$$V \otimes H^0(X, \mathcal{F} \otimes L^{\otimes m}) \longrightarrow H^0(X, \mathcal{F} \otimes L^{\otimes m+1})$$

which is exactly (iii) for  $k = 1$ . To prove (i), we instead twist the Koszul complex by  $\mathcal{F} \otimes L^{\otimes m+1-i}$  and apply a similar inductive argument using (1.7.1) in order to conclude the vanishing of  $H^i$  at the rightmost term, for each  $i > 0$ .  $\square$

EXERCISE 1.7.3. A coherent sheaf  $\mathcal{F}$  on  $\mathbf{P} = \mathbf{P}^n$  is  $m$ -regular (with respect to  $\mathcal{O}_{\mathbf{P}}(1)$ ) if and only if it admits a resolution of the form

$$\cdots \longrightarrow \bigoplus \mathcal{O}_{\mathbf{P}}(-m-2) \longrightarrow \bigoplus \mathcal{O}_{\mathbf{P}}(-m-1) \longrightarrow \bigoplus \mathcal{O}_{\mathbf{P}}(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The main statement we will extract from Theorem 1.7.2 is the following:

COROLLARY 1.7.4. *If  $\mathcal{F}$  is 0-regular with respect to  $L$ , then  $\mathcal{F}$  is globally generated.*

In conjunction with the standard vanishing theorems, this has consequences that will be relevant to our main topic of study.

COROLLARY 1.7.5. (i) *Let  $X$  be a smooth projective complex variety of dimension  $n$ , and  $L$  an ample and globally generated line bundle on  $X$ . Then*

$$\omega_X \otimes L^{\otimes m}$$

*is globally generated for all  $m \geq n + 1$ .*

(ii) *More generally, if  $f : X \rightarrow Y$  is a morphism from a smooth projective complex variety  $X$  to a projective variety  $Y$  of dimension  $n$ , and  $L$  is an ample and globally generated line bundle on  $Y$ , then*

$$R^i f_* \omega_X \otimes L^{\otimes m}$$

*is globally generated for all  $i$  and all  $m \geq n + 1$ .*

PROOF. Kodaira vanishing (for (i)) and Kollár Vanishing (for (ii)) imply that  $\omega_X \otimes L^{\otimes m}$ , and  $R^i f_* \omega_X \otimes L^{\otimes m}$  respectively, are 0-regular with respect to  $L$ . We then apply Corollary 1.7.4.  $\square$

REMARK 1.7.6 (**Fujita's Conjecture**). T. Fujita has formulated one of the most appealing conjectures in higher dimensional geometry, saying that if  $L$  is an ample line bundle on a smooth projective variety of dimension  $n$ , then  $\omega_X \otimes L^{\otimes m}$  is globally generated for  $m \geq n + 1$ , and very ample for  $m \geq n + 2$ .<sup>8</sup>

The Corollary above shows that this is true (over  $\mathbf{C}$ ) when  $L$  is *ample and globally generated*. The general case is much more complicated: the global generation statement is known in dimension two (Reider), three (Ein-Lazarsfeld), four (Kawamata) and five (Ye-Zhu), and in general if the bound  $n + 1$  is replaced by  $\binom{n+1}{2}$  (Angehrn-Siu). Almost nothing is known about very ampleness in dimension three or more.

<sup>8</sup>On a smooth projective curve of genus  $g$ , this is equivalent to the well-known statement that a line bundle is globally generated if it has degree at least  $2g$ , and very ample if it has degree at least  $2g + 1$ .

### 1.8. Log-resolutions, birational transformations, Kawamata-Viehweg

We now discuss briefly a few results from Hironaka's package of resolution of singularities, and put them to a first use by proving a useful generalization of Kodaira Vanishing.

**DEFINITION 1.8.1.** Let  $X$  be a smooth variety. An effective divisor  $D = \sum_i D_i$  on  $X$  has *simple normal crossings* if each  $D_i$  is smooth and around each point of  $X$  there is a coordinate system  $x_1, \dots, x_n$  such that locally  $D$  is given by

$$x_1 \cdot \dots \cdot x_k = 0 \text{ for some } k \leq n.$$

More generally, a  $\mathbb{Q}$ -divisor  $\sum_i d_i D_i$  has *simple normal crossing support* if  $\sum_i D_i$  has simple normal crossings.

**DEFINITION 1.8.2 (Log-resolution).** (i) Say  $X$  is a smooth variety and  $D = \sum_i d_i D_i$  an effective  $\mathbb{Q}$ -divisor on  $X$ . A *log-resolution* of  $D$  is a projective birational morphism  $f : Y \rightarrow X$  with  $Y$  smooth, such that if  $E$  is the exceptional divisor of  $f$  (the sum of the divisors contracted by  $f$ ), then

$$f^{-1}(D) \cup E$$

is a divisor with simple normal crossings support.

(ii) More generally, let  $X$  be an arbitrary variety and  $D = \sum_i d_i D_i$  a Weil  $\mathbb{Q}$ -divisor on  $X$ . A *log-resolution* of the pair  $(X, D)$  is a projective birational morphism  $f : Y \rightarrow X$  with  $Y$  smooth, such that if  $E$  is the exceptional divisor of  $f$ , then

$$f^{-1}(D) \cup E$$

is a divisor with simple normal crossings support.

The following is Hironaka's celebrated theorem:

**THEOREM 1.8.3 (Hironaka resolution).** *Over a field of characteristic 0, every pair  $(X, D)$  as in (ii) above has a log-resolution.*

This implies in particular that every variety (over a field of characteristic 0) has a resolution of singularities, i.e. a projective birational morphism from a smooth variety. As for how this is approached, Hironaka in fact showed the following more general statement:

**THEOREM 1.8.4 (Hironaka principalization).** *Let  $X$  be a smooth variety, and  $\mathcal{J}$  an ideal sheaf on  $X$ . Then there exists a birational morphism  $f : Y \rightarrow X$  obtained as a composition of blow-ups along smooth centers contained in  $\text{Supp}(\mathcal{O}_X/\mathcal{J})$ , such that  $f^*\mathcal{J}$  is locally a principal ideal.<sup>9</sup>*

To deduce Theorem 1.8.3 from this, one roughly proceeds as follows: first embed  $X$  into a smooth variety  $Z$ . (In general this may only work locally, but let's assume for simplicity that it can be done; for instance one can always use a projective space for quasi-projective varieties.) We can then consider a birational morphism  $f : W \rightarrow Z$  which

<sup>9</sup>This means that around each point there exists a coordinate system  $x_1, \dots, x_n$  such that  $f^*\mathcal{J}$  is given by a single monomial in the  $x_i$ 's.

principalizes  $\mathcal{J}_X$ , as in Theorem 1.8.4. Since  $f$  is a composition of smooth blow-ups, it follows that at some point in the process  $X$  is contained in a center of one of the blow-ups. But since  $f^{-1}(X)$  is a divisor on  $W$ , it means that when this happens, the center must in fact be  $X$  itself. This in particular means that one can resolve the singularities of  $X$ , after which one can replace  $D$  by its proper transform plus the exceptional locus on the smooth model. We can then assume that  $X$  is smooth, and then again apply the principalization theorem for  $\mathcal{J}_D$  on  $X$ .

In this course we will mostly apply Theorem 1.8.3 when  $X$  is smooth. In this case  $D$  is  $\mathbb{Q}$ -Cartier, and the theorem is simply saying that after a birational modification we can arrange that (the proper transform of)  $D$  has simple normal crossing support, and intersects the exceptional locus of the modification transversely.

EXAMPLE 1.8.5. (i) Let  $D = (y^2 = x^2 + x^3) \subset \mathbb{A}^2$  be an irreducible nodal curve in the plane. Then the blow-up  $f : \text{Bl}_0 \mathbb{A}^2 \rightarrow \mathbb{A}^2$  is a log-resolution of the pair  $(\mathbb{A}^2, D)$ . Note that  $D$  itself is normal crossings in a neighborhood of the node, but it is a singular irreducible divisor, so the *simple* normal crossings condition is not satisfied.

(ii) Let  $D = (y^2 = x^3) \subset \mathbb{A}^2$  be an irreducible cuspidal curve in the plane. This time the branches of  $D$  at the cusp do not intersect transversely, and  $f : \text{Bl}_0 \mathbb{A}^2 \rightarrow \mathbb{A}^2$  is not a log-resolution any more. In fact one needs to blow up two more times in order to achieve simple normal crossings. I will draw the picture on the board, but see also [Ha] V.3.9.1.

(iii) Cones over smooth hypersurfaces in  $\mathbb{P}^n \dots$

**Vanishing for higher direct images.** I will take for granted the following local vanishing statement, which is a fundamental result on birational morphisms:<sup>10</sup>

THEOREM 1.8.6. *Let  $f : Y \rightarrow X$  be a birational morphism between smooth varieties. Then*

$$R^i f_* \mathcal{O}_Y = 0 \quad \text{for } i > 0.$$

This is well known (but not trivial) in characteristic 0, showing it first for a blow-up along a smooth subvariety (using the theorem on formal functions), and then using the fact that  $f$  can be dominated by another birational morphism which is a composition of blow-ups with smooth centers; this last thing of course uses the statement of the principalization theorem. If resolution were known in characteristic  $p > 0$ , the argument would go through; at the moment this is not the case. However, the statement above was recently proved, with different methods, by Chatzistamatiou-Rülling.

COROLLARY 1.8.7. *Let  $f : Y \rightarrow X$  be a birational morphism between smooth varieties. Then*

$$f_* \omega_Y \simeq \omega_X \quad \text{and} \quad R^i f_* \omega_Y = 0 \quad \text{for } i > 0.$$

PROOF. Recall that in addition to Theorem 1.8.6 we also have the basic statement that  $f_* \mathcal{O}_Y \simeq \mathcal{O}_X$ , and therefore  $\mathbf{R}f_* \mathcal{O}_Y \simeq \mathcal{O}_X$ . But in our situation Grothendieck duality says

$$\mathbf{R}f_* \omega_Y \simeq \mathbf{R}\mathcal{H}om(\mathbf{R}f_* \mathcal{O}_Y, \omega_X)$$

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<sup>10</sup>This result could be phrased as saying that smooth varieties have rational singularities.

in the derived category of sheaves on  $X$ , and combining these two facts we immediately get the conclusion. More precisely, the statement of the Corollary is equivalent to that of Theorem 1.8.6.  $\square$

**EXERCISE 1.8.8.** Use Theorem 1.8.6 to deduce that the Hodge numbers  $h^{0,i} = h^i(X, \mathcal{O}_X)$  are birational invariants for all  $i$ . Give examples showing that other Hodge numbers are not birational invariants.

**Vanishing for  $\mathbb{Q}$ -divisors.** We can now establish the following useful generalization of Kodaira Vanishing; since a few of the details will only be sketched, note that I am following the argument in [La] 4.3, which goes along the lines of Kawamata's original approach.

**THEOREM 1.8.9 (Kawamata-Viehweg Vanishing).** *Let  $X$  be a smooth complex projective variety of dimension  $n$ , and let  $L$  be a big and nef line bundle on  $X$ . Then*

$$H^i(X, \omega_X \otimes L) = 0 \text{ for all } i > 0.$$

*More generally, the same conclusion holds if  $L$  is a line bundle on  $X$  such that*

$$L \sim_{\mathbb{Q}} A + D,$$

*with  $A$  a big and nef  $\mathbb{Q}$ -divisor, and  $D = \sum_i a_i D_i$  a  $\mathbb{Q}$ -divisor with simple normal crossings support satisfying  $0 \leq a_i < 1$  for all  $i$ .*

**PROOF.** I will divide the proof into a few steps; in the first three steps we will assume that  $L$  is a big and nef line bundle, and  $D = 0$ , while the last deals with the general case.

*The line bundle case.* Note to begin with that since  $L$  is big, in general there exist an  $m > 0$ , an ample line bundle  $A$ , and an effective divisor  $E$ , such that

$$(1.8.1) \quad L^{\otimes m} \simeq A \otimes \mathcal{O}_X(E).$$

*Step 1.* We first show that if  $A$  is an ample line bundle, and  $E \subset X$  is a reduced simple normal crossings divisor on  $X$ , then

$$H^i(X, \omega_X \otimes A \otimes \mathcal{O}_X(E)) = 0 \text{ for all } i > 0.$$

Let's assume first that  $E$  is a smooth divisor. Twisting the defining sequence for  $E$  by  $\omega_X \otimes A$ , we have a short exact sequence

$$0 \longrightarrow \omega_X \otimes A \longrightarrow \omega_X \otimes A \otimes \mathcal{O}_X(E) \longrightarrow \omega_E \otimes A|_E \longrightarrow 0$$

where for the last term we used the adjunction formula

$$\omega_E \simeq (\omega_X \otimes \mathcal{O}_X(E))|_E.$$

The statement follows then immediately by passing to cohomology and using Kodaira Vanishing for the left and right terms in the short exact sequence.

In general we have  $E = E_1 + \cdots + E_k$ , where  $E_j$  are smooth divisors with transverse intersections. The statement can be easily proved by induction on  $k$ , using exact sequences of the form

$$0 \longrightarrow \omega_X \otimes A \otimes \mathcal{O}_X(E_1 + \cdots + E_{j-1}) \longrightarrow \omega_X \otimes A \otimes \mathcal{O}_X(E_1 + \cdots + E_j) \longrightarrow$$

$$\longrightarrow \omega_{E_j} \otimes A|_{E_j} \otimes \mathcal{O}_{E_j}(E_1 + \cdots + E_{j-1}) \longrightarrow 0$$

*Step 2.* In this step we show that we can reduce the general statement to the case where in (1.8.1) we have that  $E$  has simple normal crossings support. Starting with an arbitrary  $E$ , we consider  $\mu : Y \rightarrow X$  a log-resolution of  $E$ , so that  $\mu^*E + F$  has simple normal crossings support, where  $F$  is the exceptional divisor of  $\mu$ .

Assuming that we proved that

$$(1.8.2) \quad H^i(Y, \omega_Y \otimes \mu^*L) = 0 \quad \text{for all } i > 0,$$

this implies the vanishing we want on  $X$ , as  $\mu_*\omega_Y \simeq \omega_X$  and  $R^i\mu_*\omega_Y = 0$  for  $i > 0$ , by Theorem 1.8.7.

Let's now write

$$\mu^*E = \sum_j a_j E_j,$$

with the convention that  $a_j \geq 0$ , so that we may assume that the sum contains all the exceptional divisors of  $\mu$  among the  $E_j$ . Note that we have

$$\mu^*L^{\otimes m} \simeq \mu^*A \otimes \mathcal{O}_Y\left(\sum_j a_j E_j\right).$$

To conclude, one appeals to a version of the Negativity Lemma, stating that for some  $k \gg 0$ , there exist  $b_j \geq 0$  such that

$$\mu^*A^{\otimes k} \otimes \mathcal{O}_Y\left(-\sum_j b_j E_j\right)$$

is ample, where the sum runs over the exceptional divisors of  $\mu$  (and so with the same convention as above we can assume that it runs over all  $E_j$ ).<sup>11</sup> But now we can write

$$\mu^*L^{\otimes mk} \simeq \left(\mu^*A^{\otimes k} \otimes \mathcal{O}_Y\left(-\sum_j b_j E_j\right)\right) \otimes \mathcal{O}_Y\left(\sum_j (ka_j + b_j)E_j\right),$$

which is of the form required at the beginning of this reduction step.

*Step 3.* In this last step we conclude the proof assuming that  $E$  in (1.8.1) has simple normal crossings support, which is the outcome of Step 2. Write

$$E = \sum_{i=1}^t e_i E_i, \quad e_i > 0,$$

and define  $e = e_1 \cdots e_t$  and  $e'_i = e/e_i$ . Now by Kawamata's covering construction, see Proposition 1.8.10 below, there exists a finite cover  $f : Y \rightarrow X$  with  $Y$  smooth projective, and a simple normal crossings divisor  $E' = \sum_{i=1}^t E'_i$  on  $Y$ , such that

$$f^*E_i = m e'_i E'_i \quad \text{for all } i = 1, \dots, k.$$

---

<sup>11</sup>This can be treated more generally, but note that according to Hironaka's theorem it is enough to choose the log-resolution to be a composition of blow-ups with smooth centers, and then check the property for one such blow-up; I will leave this last statement as an exercise.

Given (1.8.1), we consequently have

$$f^*L^{\otimes m} \simeq f^*A \otimes \mathcal{O}_Y(meE').$$

Using additive notation somewhat abusively, we can rewrite this as

$$mf^*L \sim f^*A + meE'.$$

This implies the equivalence

$$me(f^*L - E') \sim f^*A + m(e' - 1)f^*L,$$

and note that the right hand side is a divisor  $A'$  such that  $A'' = A'/me$  is also Cartier. But  $A'$  is ample: indeed,  $f^*A$  is ample since  $f$  is finite, while  $f^*L$  is nef since  $L$  is so. We finally obtain the isomorphism

$$f^*L \simeq A'' \otimes \mathcal{O}_Y(E'),$$

with  $A''$  ample, and  $E'$  a reduced simple normal crossings divisor.

We are now in a position to apply Step 1, by which we have

$$H^i(Y, \omega_Y \otimes f^*L) = 0 \quad \text{for all } i > 0.$$

As  $\omega_X$  is a direct summand of  $f_*\omega_Y$  via the trace map, we obtained the desired vanishing using the projection formula.

*Step 4.* This step deals with the general  $\mathbb{Q}$ -divisor case: recall that we are assuming that  $D = \sum_{i=1}^k a_i D_i$  is a divisor with simple normal crossings support, with  $0 < a_i < 1$ .

The strategy is to prove the statement by induction on  $k$ . The case  $k = 0$  is the line bundle case proved above. Assume now that  $k > 0$ , and let's write  $a_1 = \frac{p}{q}$ . Note that  $0 < p \leq q - 1$ . Just as in Step 3, one considers a Kawamata cover associated to the divisor  $D_1$ ; concretely, there exists a finite morphism  $f : Y \rightarrow X$ , with  $Y$  smooth projective, such that on  $Y$  the divisor  $D_1$  becomes divisible by  $d$ . In other words, we have

$$L' := f^*L \sim_{\mathbb{Q}} A' + cD'_1 + \sum_{i=2}^k a_i D'_i,$$

where  $A' = f^*A$  and  $D'_i = f^*D_i$ , still satisfying the fact that  $\sum D'_i$  has simple normal crossings.

By induction we can now assume that the line bundle  $L' \otimes \mathcal{O}_Y(-cD'_1)$  satisfies

$$H^i(Y, \omega_Y \otimes L' \otimes \mathcal{O}_Y(-cD'_1)) = 0 \quad \text{for all } i > 0.$$

On the other hand, just as in Proposition 1.6.5, it is standard that in the covering construction above we have that  $f_*(L' \otimes \mathcal{O}_Y(-cD'_1) \otimes \omega_Y)$  contains  $\omega_X \otimes L$  as a direct summand, which gives the vanishing we want.  $\square$

Here is the more refined covering construction that was used in the proof above. I will not include the proof here, but a very good treatment is given in [La, §4.1B].

**PROPOSITION 1.8.10 (Kawamata covers).** *Let  $X$  be a smooth variety, and  $D = \sum_{i=1}^t D_i$  be a simple normal crossings divisor on  $X$ . Given positive integers  $m_1, \dots, m_k$ , there exists a finite flat morphism  $f : Y \rightarrow X$  with  $Y$  smooth, and a simple normal crossings divisor  $D' = \sum_{i=1}^t D'_i$  on  $Y$ , such that*

$$f^* D_i = m_i D'_i \quad \text{for all } i = 1, \dots, k.$$

Finally, similarly to the proof of Theorem 1.6.8 and the proof above, one can also obtain the following more general statement:

**THEOREM 1.8.11 (Kollár vanishing for  $\mathbb{Q}$ -divisors).** *Let  $f : X \rightarrow Y$  be a morphism from a smooth projective variety  $X$  to a projective variety  $Y$ , and let  $L$  be a line bundle on  $X$  such that*

$$L \sim_{\mathbb{Q}} f^* N + D,$$

*with  $N$  a nef and big  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $Y$ , and  $D = \sum_i d_i D_i$  a  $\mathbb{Q}$ -divisor with simple normal crossings support satisfying  $0 \leq d_i < 1$  for all  $i$ . Then*

$$H^j(Y, R^i f_*(\omega_X \otimes L)) = 0, \quad \text{for all } i \text{ and all } j > 0.$$



## CHAPTER 2

### Iitaka's subadditivity conjecture

In this chapter we work with varieties defined over an algebraically closed field of characteristic 0.

#### 2.1. Statement and intuitive approach

Considerations as in the section on surfaces, and further work in higher dimensions, led Iitaka to formulate a famous conjecture that was one of the main reasons for the development of the minimal model program.

**CONJECTURE 2.1.1 (Iitaka's  $C_{n,m}$  conjecture).** *Let  $f: X \rightarrow Y$  be an algebraic fiber space, with  $X$  and  $Y$  smooth projective varieties, and with general fiber  $F$ . Then*

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

**REMARK 2.1.2.** Recall that the Easy Addition formula in Example 1.3.11 says that in the situation of the conjecture we always have

$$\dim Y + \kappa(F) \geq \kappa(X).$$

**REMARK 2.1.3 (Obvious cases).** (1) The conjecture is obviously true when  $\kappa(Y) = -\infty$  or  $\kappa(F) = -\infty$ . In the latter case, by Easy Addition we also have  $\kappa(X) = -\infty$ .

(2) The conjecture is also clear when  $X$  is of general type. In this case, by Easy Addition we also have that  $F$  is of general type, while equality holds if and only if  $Y$  is of general type as well.

**Intuitive approach to the conjecture.** In the set-up above, denote

$$\mathcal{F}_m := f_*\omega_{X/Y}^{\otimes m}, \quad m \geq 1.$$

Since  $f$  is generically flat, at a general point of  $y$  we can apply the Base Change theorem and deduce that  $\mathcal{F}_m$  is a torsion-free coherent sheaf on  $Y$  of generic rank

$$\mathrm{rk}(\mathcal{F}_m) = P_m(F) = h^0(F, \omega_F^{\otimes m}).$$

By the projection formula we have

$$f_*\omega_X^{\otimes m} \simeq \mathcal{F}_m \otimes \omega_Y^{\otimes m},$$

and so

$$P_m(X) = h^0(Y, \mathcal{F}_m \otimes \omega_Y^{\otimes m}), \quad \forall m \geq 1.$$

As we will see later, one of the key properties all fiber spaces is that they come with naturally attached positivity properties, reflected precisely in the canonically defined sheaves

$\mathcal{F}_m$ . Let's imagine for a moment that these sheaves were generically globally generated for  $m$  sufficiently large and divisible. (This is usually not true, but the properties I am alluding to go roughly in this direction.) We can then apply Exercise 2.1.4 below to deduce that there exists a sheaf inclusion

$$\bigoplus_{\text{rk}(\mathcal{F}_m)} \omega_Y^{\otimes m} \hookrightarrow \mathcal{F}_m \otimes \omega_Y^{\otimes m}.$$

Passing to global sections, we conclude that

$$P_m(X) \geq P_m(F) \cdot P_m(Y),$$

which after comparing the rate of growth on the two sides implies the inequality in Conjecture 2.1.1.

**EXERCISE 2.1.4.** Let  $\mathcal{F}$  be a generically globally generated coherent sheaf of generic rank  $r$  on a variety  $X$ . Then there exists an sheaf inclusion

$$\mathcal{O}_X^{\oplus r} \hookrightarrow \mathcal{F}.$$

**EXAMPLE 2.1.5 (Calabi-Yau fibers).** The discussion simplifies further if we assume  $\omega_F \simeq \mathcal{O}_F$  for the general fiber of  $F$ . We have in particular  $\text{rk}(\mathcal{F}_m) = 1$  for all  $m \geq 1$ , and one needs to show that  $\kappa(X) \geq \kappa(Y)$ .

Let's note that in this case it would in fact enough to show that  $H^0(Y, \mathcal{F}_m) \neq 0$  for some  $m$ . Since  $\mathcal{F}_m$  is a line bundle, this is equivalent to being generically globally generated, and so the discussion in the intuitive approach applies. On the other hand, once we know this for one  $m$ , we know it for all its multiples. Indeed, we would then have  $H^0(X, \omega_{X/Y}) \neq 0$ , and taking powers of a non-zero section on  $X$ , we obtain that

$$h^0(X, \omega_{X/Y}^{\otimes mk}) = h^0(Y, \mathcal{F}_{mk}) \neq 0, \quad \forall k \geq 1.$$

The actual proof does indeed go via producing nontrivial sections, however not directly for  $\mathcal{F}_m$  but for more complicated auxiliary constructions.

**REMARK 2.1.6 (Deformation invariance of plurigenera).** Siu (reference!) has shown that the plurigenera of a smooth projective complex variety are invariant under deformation. In other words, if  $f: X \rightarrow Y$  is a smooth fiber space, with  $X$  and  $Y$  quasi-projective, then the plurigenera  $P_m(F)$  of the fibers of  $f$  are constant. The proof of this theorem requires analytic methods; see also [La, §11.5] for an algebraic proof in the case of varieties of general type.

In particular, for fiber spaces as in Conjecture 2.1.1, if  $V \subseteq Y$  is the non-empty Zariski open set over which the fibers of  $f$  are smooth, then  $P_m(F)$  are the same for all fibers over points of  $V$ , and hence the same is true for  $\kappa(F)$ .

As  $f$  is in particular flat, by Grauert's theorem (see e.g. [Ha, III, Corollary 12.9]), this also implies that all the sheaves  $\mathcal{F}_m$  are locally free over  $V$ .

## 2.2. Positivity for vector bundles

**Ampleness and nefness.** The natural extensions of the standard positivity properties for line bundles are the following:

DEFINITION 2.2.1. Let  $X$  be a projective scheme, and  $E$  a vector bundle on  $X$ . Then  $E$  is called *ample*, or *nef*, if  $\mathcal{O}_{\mathbf{P}}(1)$  is an ample, or nef, line bundle on  $\mathbf{P} = \mathbf{P}(E)$ .

Here I will only give a glimpse of some useful properties of ample and nef vector bundles. For a complete treatment, see [La] Ch.6.

EXERCISE 2.2.2. If  $E$  and  $F$  are ample (nef) vector bundles on  $X$ , then  $E \oplus F$  is ample is ample (nef).

LEMMA 2.2.3. *Let  $E$  be an ample (nef) vector bundle on a projective scheme  $X$  over a field of characteristic zero. Then:*

(i) *If  $E \rightarrow G$  is a quotient vector bundle, then  $G$  is ample (nef). In particular, globally generated vector bundles are nef (as they are quotients of trivial bundles).*

(ii)  *$S^k E$  is ample (nef) for all  $k \geq 1$ .*

(iii)  *$E^{\otimes k}$  is ample (nef) for any  $k \geq 1$ . Consequently,  $\wedge^k E$  is ample (nef) for any  $k \geq 1$ , and so  $\det E$  is an ample (nef) line bundle.*

PROOF. Here is a brief sketch:

(i) This holds in arbitrary characteristic. Since our projective bundles parametrize one-dimensional quotients, the surjection  $E \rightarrow G$  corresponds to an inclusion  $\mathbf{P}(G) \subseteq \mathbf{P}(E)$ , such that the restriction of  $\mathcal{O}_{\mathbf{P}(E)}(1)$  is  $\mathcal{O}_{\mathbf{P}(G)}(1)$ . The assertion is then clear.

(ii) We first show that  $S^m E$  is ample for  $m \gg 0$ . Since the ampleness of  $E$  means by definition the ampleness of  $\mathcal{O}_{\mathbf{P}}(1)$  on  $\pi : \mathbf{P} = \mathbf{P}(E) \rightarrow X$ , and since  $\pi_* \mathcal{O}_{\mathbf{P}}(m) \simeq S^m E$ , an argument completely similar to Serre's theorem implies that for any coherent sheaf  $\mathcal{F}$  on  $X$  there exists some positive integer  $m_0 = m_0(\mathcal{F})$  such that

$$S^m E \otimes \mathcal{F} \text{ is globally generated for } m \geq m_0.$$

In particular, we can take  $\mathcal{F} = A^{-1}$ , where  $A$  is an ample line bundle on  $X$ . We deduce that there is a surjection

$$\bigoplus A \longrightarrow S^m E \longrightarrow 0$$

and hence by part (i)  $S^m E$  is ample, for every  $m \geq m_0$ .

Now fix an arbitrary  $k \geq 1$ . According to Exercise 2.2.4 below, for each  $\ell \geq 1$ , there exists a finite map

$$\varphi : \mathbf{P}(S^k E) \longrightarrow \mathbf{P}(S^{k\ell} E) \text{ with } \mathcal{O}_{\mathbf{P}(S^k E)}(1) \simeq \varphi^* \mathcal{O}_{\mathbf{P}(S^{k\ell} E)}(1).$$

We've seen above that the line bundle on the right hand side is ample for  $\ell \gg 0$ , so  $\mathcal{O}_{\mathbf{P}(S^k E)}(1)$  is ample as well, being its pullback by a finite map. With a little care, the analogous nefness result is a simple application of what we just proved about ampleness, combined with the fact that nef divisors are limits of ample  $\mathbb{Q}$ -divisors.

(iii) Since all the other linear algebra constructions are quotients of tensor products, it is enough to show that  $E^{\otimes k}$  is ample (nef) for  $k \geq 1$ . More generally, we show that if  $E$  and  $F$  are ample (nef) vector bundles, then so is  $E \otimes F$ . But note that  $E \otimes F$  is a

direct summand of  $S^2(E \oplus F)$ , and so the result follows combining Exercise 2.2.2 and part (ii).  $\square$

EXERCISE 2.2.4. Let  $E$  be a vector bundle on a projective scheme  $X$  over a field of characteristic zero, and let  $k, \ell \geq 1$  be two integers. Then there exists a finite (onto its image) morphism

$$\varphi : \mathbf{P}(S^k E) \longrightarrow \mathbf{P}(S^{k\ell} E)$$

compatible with  $\mathcal{O}(1)$ , i.e. such that  $\varphi^* \mathcal{O}_{\mathbf{P}(S^{k\ell} E)}(1) \simeq \mathcal{O}_{\mathbf{P}(S^k E)}(1)$ . (Hint: think of a Veronese-type construction.)

LEMMA 2.2.5. *Let  $E$  be a locally free sheaf on a smooth projective variety  $X$ . If there exists a line bundle  $L$  on  $X$  such that  $S^m E \otimes L$  is nef for all  $m \gg 0$ , then  $E$  is nef. In particular, this happens when  $E^{\otimes m} \otimes L$  is globally generated for  $m \gg 0$ .*

PROOF. Recall that, with the usual notation, we have

$$\pi_* \mathcal{O}_{\mathbf{P}}(m) \simeq S^m E, \quad \forall m \geq 0.$$

Using this, and the fact that  $\mathcal{O}_{\mathbf{P}}(m)$  is globally generated when restricted to the fibers, the adjunction mapping  $\pi^* \pi_* \mathcal{O}_{\mathbf{P}}(m) \rightarrow \mathcal{O}_{\mathbf{P}}(m)$  gives a surjective homomorphism

$$\pi^* S^m E \otimes \pi^* L \longrightarrow \mathcal{O}_{\mathbf{P}}(m) \otimes \pi^* L.$$

It follows that  $\mathcal{O}_{\mathbf{P}}(m) \otimes \pi^* L$  is a nef line bundle for all  $m \gg 0$ . This in turn implies that  $\mathcal{O}_{\mathbf{P}}(1)$  is nef by Exercise 1.4.5. The last assertion follows from the fact that  $S^m E$  is a quotient of  $E^{\otimes m}$ , since globally generated vector bundles are nef.  $\square$

EXERCISE 2.2.6. A vector bundle  $E$  on  $X$  is nef if and only if for every ample line bundle  $H$  on  $X$  and every integer  $\alpha > 0$ , there exists an integer  $\beta > 0$  such that  $S^{\alpha\beta} E \otimes H^{\otimes\beta}$  is globally generated.

**Motivation: positivity for families of curves.** We will be interested in the positivity properties of direct images of relative pluricanonical bundles. A first hint towards understanding this comes from looking at morphisms where the fibers have dimension 1, so in particular the general one is a smooth projective curve. In this case, the problem is intimately related to the existence and projectivity of the moduli space of stable curves. The discussion here is just in order to explain the picture; I will not define the terminology or give details.

Fix a genus  $g \geq 1$ , and recall that there exists a quasi-projective variety  $M_g$  which is a coarse moduli space for isomorphism classes of smooth projective curves of genus  $g$ . It admits a projective compactification  $\overline{M}_g$ , which parametrizes isomorphism classes of *stable* curves. If  $g = 1$ , the dimension of  $M_g$  is 1, otherwise it is equal to  $3g - 3$ . One important (and unfortunate) feature of  $M_g$  and  $\overline{M}_g$  is that they are only coarse moduli spaces; what captures the properties of families of curves somewhat more accurately, but at the same time are more technical objects, are the Deligne-Mumford stacks  $\mathcal{M}_g$  and  $\overline{\mathcal{M}}_g$ , whose associated coarse moduli spaces are  $M_g$  and  $\overline{M}_g$ .

Over  $\overline{M}_g$  sits the *universal curve*

$$\pi : \overline{C}_g \longrightarrow \overline{M}_g,$$

whose fiber over a point in  $\overline{M}_g$  is precisely the curve parametrized by that point. These spaces are not smooth, but their singularities are mild enough that we can still talk about  $\omega_{\overline{C}_g/\overline{M}_g}$ , whose restriction to each fiber is the dualizing sheaf  $\omega_C$  of the stable curve  $C$ . We can then consider

$$\mathcal{H}_m := \pi_* \omega_{\overline{C}_g/\overline{M}_g}^{\otimes m}, \quad \forall m \geq 1.$$

These are vector bundles on  $\overline{M}_g$ , since  $\pi$  is flat and the dimension of the space of sections  $H^0(C, \omega_C^{\otimes m})$  depends only on  $g$  and not on  $C$ . For  $m = 1$ , this is the celebrated *Hodge bundle*, of rank  $g$ . Here are some important theorems about these bundles:

- $\mathcal{H}_m$  is nef for all  $m \geq 1$ ; in particular so is  $\det \mathcal{H}_m$ .
- $\det \mathcal{H}_m$  is ample for  $m \gg 0$ .

Finally, let  $f : X \rightarrow Y$  be a surjective morphism of, say, smooth projective varieties such that its general fiber is a smooth projective curve of genus  $g$ . Let's assume for simplicity that  $f$  is flat, so that all fibers are 1-dimensional; by a process called stable reduction, after a finite base change we can even assume that all fibers are stable curves, so let's say that this is the case.

In other words, we consider a flat family  $f : X \rightarrow Y$  of stable curves of genus  $g$ . Let's even assume that it comes by base change from the moduli space, i.e. that there exists a morphism  $\varphi : Y \rightarrow \overline{M}_g$  and a fiber diagram

$$\begin{array}{ccc} X & \longrightarrow & \overline{C}_g \\ f \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\varphi} & \overline{M}_g \end{array}$$

(This is strictly the case only after a finite cover, which is a reflection of the fact that  $\overline{M}_g$  is not a fine moduli space; note however that our end goal, which is positivity in a loose sense, behaves well with respect to finite maps.)

At least if  $\varphi$  is flat, since  $\pi$  is flat with reduced fibers, by general duality theory that will be discussed later, the relative dualizing sheaf is compatible with base change. We conclude that

$$\mathcal{F}_m = f_* \omega_{X/Y}^{\otimes m} \simeq \varphi^* \mathcal{H}_m \quad \text{and} \quad \det \mathcal{F}_m \simeq \varphi^* \det \mathcal{H}_m.$$

Thus the basic properties of  $\mathcal{H}_m$  imply positivity for families; it follows that for stable families of curves  $f$  as above we have:

- $\mathcal{F}_m$  is a nef vector bundle, and  $\det \mathcal{F}_m$  is a nef line bundle.
- if  $f$  is a finite morphism, then  $\det \mathcal{F}_m$  is ample for  $m \gg 0$ .
- more generally, for arbitrary  $f$ , we have that  $\kappa(Y, \det \mathcal{F}_m) \geq \dim \varphi(Y)$  (the “variation” of the family in moduli).

It is worth noting however that beyond the intuitive picture, in general things go mostly the other way: one tries to prove positivity results for  $\mathcal{F}_m$  for every (stable) family

by other means precisely in order to deduce the projectivity of moduli spaces, according to a strategy introduced by Kollár and Viehweg.

### 2.3. Vanishing and global generation for direct images of pluricanonical bundles

I will now explain a vanishing theorem that will allow us to give algebraic proofs of the positivity results for direct images of pluricanonical bundles that we are after. It is inspired by the following observation, which shows that Kodaira vanishing can be extended to powers of the canonical bundle. All the varieties considered in this section are over the complex numbers.

**PROPOSITION 2.3.1.** *Let  $X$  be a smooth projective variety,  $L$  an ample line bundle on  $X$ , and  $k \geq 1$  an integer. Then*

$$H^i(X, \omega_X^{\otimes k} \otimes L^{k(n+1)-n}) = 0, \quad \forall i > 0.$$

**PROOF.** For clarity, I'll use additive notation. We write

$$kK_X + (k(n+1) - n)L = K_X + (k-1)(K_X + (n+1)L) + L.$$

Recall that Fujita's conjecture predicts that  $K_X + (n+1)L$  is globally generated; the weaker statement that it is nef is however already known as part of Mori's proof of the Cone and Rationality theorem. It follows that

$$(k-1)(K_X + (n+1)L) + L$$

is an ample line bundle, and therefore Kodaira Vanishing applies.  $\square$

Recall also Fujita's conjecture, predicting that if  $L$  is an ample line bundle on a smooth projective variety  $X$  of dimension  $n$ , then  $\omega_X \otimes L^{\otimes k}$  is globally generated for  $k \geq n+1$ ; therefore the same should hold for  $\omega_X^{\otimes m} \otimes L^{\otimes mk}$  for all  $m \geq 1$ . By analogy with this, and with the Proposition above, Schnell and I proposed in [PS2] the following relative version:

**CONJECTURE 2.3.2.** *Let  $f: X \rightarrow Y$  be a morphism of smooth projective varieties, with  $Y$  of dimension  $n$ , and let  $L$  be an ample line bundle on  $Y$ . Then, for every  $m \geq 1$ :*

(1) *The sheaf*

$$f_* \omega_X^{\otimes m} \otimes L^{\otimes \ell}$$

*is globally generated for  $\ell \geq m(n+1)$ .*

(2) *We have*

$$H^i(Y, f_* \omega_X^{\otimes m} \otimes L^{\otimes \ell}) = 0$$

*for all  $i > 0$  and  $\ell \geq m(n+1) - n$ .*

**REMARK 2.3.3.** Part (2) is not explicitly stated as a conjecture in [PS2], but it is implicit in the text, where it is also shown (see Proposition 2.13 in *loc. cit.*) that it follows from the  $m = 1$  case of a slightly more general klt version of part (1).

**REMARK 2.3.4 (Kawamata's work).**

We currently have a proof of this conjecture when  $L$  is assumed in addition to be globally generated; see [PS2, Theorem 1.4 and Corollary 2.9]. Unlike in the case  $m = 1$  (see Corollary 1.7.5) this does not follow any more as an immediate consequence of Kollár vanishing.

**THEOREM 2.3.5.** *Conjecture 2.3.2 holds if  $L$  is ample and globally generated.*

**PROOF.** It suffices to prove (2), since this in turn implies that the sheaf

$$f_*\omega_X^{\otimes m} \otimes L^{\otimes k}$$

is 0-regular for  $k \geq m(n + 1)$ , which in turn implies (1) by the Castelnuovo-Mumford Lemma.

Focusing on (2), we will first show that we can reduce to the case when the image of the adjunction morphism

$$(2.3.1) \quad f^*f_*\omega_X^{\otimes m} \rightarrow \omega_X^{\otimes m}.$$

is a line bundle. A priori the image is  $\mathfrak{b} \otimes \omega_X^{\otimes m}$ , where  $\mathfrak{b}$  is the relative base ideal of  $\omega_X^{\otimes m}$ . (Note that on the general fiber  $F$  the adjunction morphism is simply the evaluation map  $H^0(F, \omega_F^{\otimes m}) \otimes \mathcal{O}_F \rightarrow \omega_F^{\otimes m}$ .) We consider a log-resolution

$$\mu : \tilde{X} \longrightarrow X$$

of the ideal sheaf  $\mathfrak{b}$ . Since  $\tilde{X}$  and  $X$  are smooth, we have that

$$\mu_*\omega_{\tilde{X}}^{\otimes m} \simeq \omega_X^{\otimes m},$$

and so we can replace  $X$  by  $\tilde{X}$  and  $f$  by  $f \circ \mu$  without changing the conclusion. Going back to the original notation, we can thus assume that the image sheaf of the adjunction morphism (2.3.1) is of the form  $\omega_X^{\otimes m} \otimes \mathcal{O}_X(-E)$  for a divisor  $E$  with simple normal crossings support. We will do so from now on.

Since  $L$  is ample, there is a smallest integer  $k \geq 0$  such that  $f_*\omega_X^{\otimes m} \otimes L^{\otimes k}$  is globally generated. Then  $f^*f_*\omega_X^{\otimes m} \otimes f^*L^{\otimes k}$  is globally generated as well, and so using the adjunction morphism (2.3.1) and Bertini's theorem, we can write

$$\omega_X^{\otimes m} \otimes f^*L^{\otimes k} \simeq \mathcal{O}_X(D + E),$$

with  $D$  smooth and  $D + E$  a divisor with simple normal crossings support. In divisor notation, we obtain

$$(2.3.2) \quad K_X \sim_{\mathbb{Q}} \frac{1}{m}D + \frac{1}{m}E - \frac{k}{m}f^*L.$$

For any integer  $\ell \geq 0$ , using (2.3.2) we can then write the following equivalence:

$$(2.3.3) \quad \begin{aligned} mK_X - \left\lfloor \frac{m-1}{m}E \right\rfloor + \ell f^*L &= K_X + (m-1)K_X - \left\lfloor \frac{m-1}{m}E \right\rfloor + \ell f^*L \\ &\sim_{\mathbb{Q}} K_X + \Delta + \left( \ell - \frac{m-1}{m} \cdot k \right) f^*L, \end{aligned}$$

where

$$\Delta = \frac{m-1}{m}D + \frac{m-1}{m}E - \left\lfloor \frac{m-1}{m}E \right\rfloor$$

is a boundary divisor (meaning  $\Delta = \sum_i d_i \Delta_i$  with  $0 < d_i < 1$ ) with simple normal crossings support.

Observe now that for every effective Cartier divisor  $E' \preceq E$  we have

$$(2.3.4) \quad f_* (\omega_X^{\otimes m} \otimes \mathcal{O}_X(-E')) \simeq f_* \omega_X^{\otimes m}.$$

Indeed, it is enough to have this for  $E$  itself; but this is the base locus of  $\omega_X^{\otimes m}$  relative to  $f$ , so by construction we have that the adjunction morphism factors as

$$f^* f_* \omega_X^{\otimes m} \rightarrow \omega_X^{\otimes m} \otimes \mathcal{O}_X(-E) \hookrightarrow \omega_X^{\otimes m}.$$

The claimed isomorphism follows by noting that the composition

$$f_* \omega_X^{\otimes m} \longrightarrow f_* (\omega_X^{\otimes m} \otimes \mathcal{O}_X(-E)) \longrightarrow f_* \omega_X^{\otimes m}$$

of the push-forward maps is the identity. Using (2.3.4) and the projection formula, we obtain that

$$f_* \left( \omega_X^{\otimes m} \left( - \left\lfloor \frac{m-1}{m}E \right\rfloor \right) \otimes f^* L^{\otimes \ell} \right) \simeq f_* \omega_X^{\otimes m} \otimes L^{\otimes \ell}.$$

On the other hand, because of (2.3.3), the left hand side can also be written as

$$f_* \mathcal{O}_X \left( K_X + \Delta + \left( \ell - \frac{m-1}{m} \cdot k \right) f^* L \right),$$

to which one can apply Kollár vanishing in the form of Theorem 1.8.11 if the number in the parenthesis is positive. In other words,

$$H^i(Y, f_* \omega_X^{\otimes m} \otimes L^{\otimes \ell}) = 0 \quad \text{for all } i > 0 \quad \text{and } \ell > \frac{m-1}{m} \cdot k.$$

Using the Castelnuovo-Mumford Lemma (see Corollary 1.7.4), we conclude that  $f_* \omega_X^{\otimes m} \otimes L^{\otimes \ell}$  is globally generated for  $\ell > \frac{m-1}{m} \cdot k + n$ . But  $k$  was chosen minimal with this property, which means that we must have

$$\frac{m-1}{m} \cdot k + n + 1 \geq k,$$

which translates into  $k \leq m(n+1)$ . Consequently, vanishing holds for all  $\ell \geq m(n+1) - n$ .  $\square$

We rephrase a version of part (2) of the theorem in a form that we will find convenient later on:

**COROLLARY 2.3.6.** *Let  $f: X \rightarrow Y$  be a morphism of smooth projective varieties, with  $Y$  of dimension  $n$ , and let  $L$  be an ample and globally generated line bundle on  $Y$ . Then, for every  $m \geq 1$  the sheaf*

$$f_* \omega_{X/Y}^{\otimes m} \otimes A^{\otimes m}$$

*is globally generated, where  $A := \omega_Y \otimes L^{\otimes n+1}$ .*

**Example: the main conjecture over curves.** There is one case when the main relative Fujita-type Conjecture 2.3.2 can be shown to hold in full, namely when the base of the morphism is a curve. This is not hard to check, but it uses important special facts about vector bundles on curves.

PROPOSITION 2.3.7. *Let  $f: X \rightarrow C$  be a morphism of smooth projective varieties, with  $C$  a curve, and let  $L$  be an ample line bundle on  $C$ . Then, for every  $m \geq 1$ , the vector bundle*

$$f_*\omega_X^{\otimes m} \otimes L^{\otimes \ell}$$

*is globally generated for  $\ell \geq 2m$ .*

PROOF. First, note that since  $C$  is a curve, the sheaf in question is locally free. We can rewrite it as

$$f_*\omega_X^{\otimes m} \otimes L^{\otimes \ell} \simeq f_*\omega_{X/C}^{\otimes m} \otimes \omega_C^{\otimes m} \otimes L^{\otimes \ell}.$$

Now a result of Kawamata [Ka2, Theorem 1] (that we are aiming to generalize extensively, but has been known for a while for curves) says that  $f_*\omega_{X/C}^{\otimes m}$  is a semipositive vector bundle on  $C$ , in the sense that it cannot have quotient line bundles of negative degree. On the other hand

$$\deg(\omega_C^{\otimes m} \otimes L^{\otimes \ell}) \geq m(2g - 2) + \ell \deg L \geq 2g,$$

with  $g$  the genus of  $C$ , as  $\deg L > 0$ . The statement then follows from the following general result.  $\square$

LEMMA 2.3.8. *Let  $E$  be a semipositive vector bundle and  $L$  a line bundle of degree at least  $2g$  on a smooth projective curve  $C$  of genus  $g$ . Then  $E \otimes L$  is globally generated.*

PROOF. It is enough to show that for every  $p \in C$ , one has

$$H^1(C, E \otimes L \otimes \mathcal{O}_C(-p)) = 0,$$

or equivalently, by Serre Duality, that there are no nontrivial homomorphisms

$$E \longrightarrow \omega_C \otimes \mathcal{O}_C(p) \otimes L^{-1}.$$

But the semipositivity of  $E$  means precisely that it does not admit quotient line bundles of negative degree.  $\square$

REMARK 2.3.9. Note that for curves of genus at least 1, the argument in Proposition 2.3.7 shows in fact that  $f_*\omega_X^{\otimes m} \otimes L^{\otimes 2}$  is always globally generated.

**Note on further results.**

## 2.4. Positivity for direct images of pluricanonical bundles

**Viehweg's fiber product trick, I.** Let  $f: X \rightarrow Y$  be a projective surjective morphism of smooth quasi-projective varieties. Consider the the  $s$ -fold fiber product

$$X \times_Y X \times_Y \cdots \times_Y X$$

induced by  $f$ ; it admits a natural map to  $Y$ .

EXERCISE 2.4.1. (i) Show that there is a unique irreducible component of  $X^s$  of  $X \times_Y \cdots \times_Y X$  which dominates  $Y$ .

(ii) If  $f : X \rightarrow Y$  is a smooth algebraic fiber space, then  $X \times_Y \cdots \times_Y X$  is irreducible (hence equal to  $X^s$ ) and smooth, and the morphism  $f^s : X^s \rightarrow Y$  is smooth as well. If  $X_y$  is the fiber of  $f$  over  $y \in Y$ , then one has

$$(f^s)^{-1}(X_y) \simeq X_y \times \cdots \times X_y,$$

the usual  $s$ -fold product.

(iii) Let  $\pi : X = \text{Bl}_y(Y) \rightarrow Y$  be the blow-up of  $Y$  at a point. Show that  $X \times_Y X$  is reducible (so that  $X^2 \neq X \times_Y X$ ).

In this subsection we review Viehweg's trick in its simplest form, namely for smooth fiber spaces. This will be generalized in Proposition 2.4.18.

PROPOSITION 2.4.2. *If  $f$  is a smooth algebraic fiber space, then there is a natural isomorphism of locally free sheaves*

$$f_* \omega_{X^s/Y}^{\otimes m} \simeq (f_* \omega_{X/Y}^{\otimes m})^{\otimes s}.$$

(Note that  $X^s$  is already smooth by Exercise 2.4.1(iii).)

In order to address this, we first discuss some generalities. Consider a fiber product diagram

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{g'} & X \\ f' \downarrow & \searrow \varphi & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array}$$

where  $f$  and  $g$  are smooth morphisms. Since smoothness is preserved by base change and composition, so are  $f'$ ,  $g'$  and  $\varphi$ . In particular  $X \times_Y Z$  is smooth, and irreducible if the fibers of  $f$  are so. Relative dualizing sheaves commute with flat base change, and therefore we have

$$\omega_{X \times_Y Z/Z} \simeq g'^* \omega_{X/Y}.^1$$

By factoring  $\varphi$  as  $g \circ f'$ , this immediately shows that

$$(2.4.1) \quad \omega_{X \times_Y Z/Y} \simeq f'^* \omega_{Z/Y} \otimes g'^* \omega_{X/Y}.$$

PROOF OF PROPOSITION 2.4.2. We prove this by induction on  $s$ , noting that we have fiber product diagrams

$$\begin{array}{ccc} X^s & \xrightarrow{g'} & X \\ f' \downarrow & \searrow f^s & \downarrow f \\ X^{s-1} & \xrightarrow{f^{s-1}} & Y \end{array}$$

<sup>1</sup>In the case of smooth morphisms this is actually clear, since the sheaf of relative Kähler differentials  $\Omega_{X/Y}^1$  is locally free, with  $\det \Omega_{X/Y}^1 \simeq \omega_{X/Y}$ ; on the other hand, Kähler differentials commute with arbitrary base change.

where  $g' = (f^{s-1})'$ . Using (2.4.1), we obtain

$$\omega_{X^s/Y}^{\otimes m} \simeq f'^* \omega_{X^{s-1}/Y}^{\otimes m} \otimes g'^* \omega_{X/Y}^{\otimes m}.$$

Therefore

$$\begin{aligned} f_* \omega_{X^s/Y}^{\otimes m} &= f_*^{s-1} f'_* (f'^* \omega_{X^{s-1}/Y}^{\otimes m} \otimes g'^* \omega_{X/Y}^{\otimes m}) \simeq \\ &\simeq f_*^{s-1} (\omega_{X^{s-1}/Y}^{\otimes m} \otimes f'_* g'^* \omega_{X/Y}^{\otimes m}) \simeq f_*^{s-1} (\omega_{X^{s-1}/Y}^{\otimes m} \otimes f^{s-1*} f_* \omega_{X/Y}^{\otimes m}) \simeq \\ &\simeq f_*^{s-1} \omega_{X^{s-1}/Y}^{\otimes m} \otimes f_* \omega_{X/Y}^{\otimes m} \simeq (f_* \omega_{X/Y}^{\otimes m})^{\otimes s}, \end{aligned}$$

where the first isomorphism follows from the projection formula, the second from the push-pull formula  $f^{s-1*} f_* = f'_* g'^*$  for base change along flat morphisms (see [Ha, Ch.III, Prop.9.3]), the third from the projection formula, and the fourth from the inductive assumption for  $f^{s-1}$ .  $\square$

REMARK 2.4.3. Note that fiber by fiber the isomorphism in Proposition 2.4.2 has a very simple interpretation. If  $F$  is a fiber of  $f$ , then it corresponds to the natural isomorphism

$$H^0(F \times \cdots \times F, \omega_{F \times \cdots \times F}) \simeq (H^0(F, \omega_F^{\otimes m}))^{\otimes s}$$

given by the Künneth formula.

**Positivity for direct images of relative pluricanonical bundles.** I will first address the case of smooth morphisms, which is easiest to explain, and where the (semi)positivity of direct images holds in a strong form. We will later prove a generalization of this statement due to Viehweg, which holds for arbitrary morphisms, but where the conclusion is necessarily weaker; the proofs are similar, and we will obtain them here as relatively quick applications of Corollary 2.3.6.

THEOREM 2.4.4. *Let  $f : X \rightarrow Y$  be a smooth morphism of smooth projective varieties. Then*

$$\mathcal{F}_m = f_* \omega_{X/Y}^{\otimes m}$$

*is a nef vector bundle for all  $m \geq 0$ .*

PROOF. We have seen before that by Siu's deformation invariance of plurigenera the sheaves  $\mathcal{F}_m$  are locally free, of rank  $P_m(F)$ , where  $F$  is any fiber of  $f$ .

To prove nefness, consider the line bundle

$$A := \omega_Y \otimes L^{\otimes n+1},$$

where  $n = \dim Y$  and  $L$  is an ample and globally generated line bundle on  $Y$ . According to Lemma 2.2.5, it suffices to show that  $\mathcal{F}_m^{\otimes s} \otimes A^{\otimes m}$  is globally generated for all  $s \geq 1$ . Note first that we know the result for  $\mathcal{F}_m$  itself, by Corollary 2.3.6.

To prove the statement for arbitrary  $s$ , one uses Viehweg's trick based on the construction explained in the previous subsection; we will make  $\mathcal{F}_m^{\otimes s}$  look like  $\mathcal{F}_m$  itself, so that we can again apply the argument above, but after changing the domain  $X$ . To this end consider the  $s$ -fold fiber product induced by  $f$ ,

$$f^s : X^s = X \times_Y \cdots \times_Y X \longrightarrow Y,$$

see Exercise 2.4.1(ii). Since  $f$  is smooth, by Proposition 2.4.2 we have an isomorphism

$$f_* \omega_{X^s/Y}^{\otimes m} \simeq (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} = \mathcal{F}_m^{\otimes s}.$$

But the left hand side of the isomorphism is again a direct image of a relative pluricanonical bundle, and so we can apply Corollary 2.3.6 to  $f^s$  in order to conclude that  $\mathcal{F}_m^{\otimes s} \otimes A^{\otimes m}$  is globally generated.  $\square$

**REMARK 2.4.5 (Base of dimension one).** Recall that in a previous section we saw that Iitaka's conjecture for surfaces would follow if we knew that for a morphism  $f : S \rightarrow C$  from a surface to a curve one had

$$\deg f_* \omega_{S/C}^{\otimes m} \geq 0, \quad \forall m \geq 1.$$

More generally, let  $f : X \rightarrow C$  be a fiber space with  $X$  a smooth projective variety of arbitrary dimension, and  $C$  a smooth projective curve. If  $f$  is smooth, Theorem 2.4.4 says that  $\mathcal{F}_m = f_* \omega_{X/C}^{\otimes m}$  is a nef vector bundle on  $C$ , which in particular implies that  $\det \mathcal{F}_m$  is nef as well by Lemma 2.2.3. This last assertion is equivalent to

$$\deg \mathcal{F}_m \geq 0, \quad \forall m \geq 1,$$

as the degree of  $\mathcal{F}_m$  is equal to that of its determinant. We will see that the same statement holds even if  $f$  is not necessarily assumed to be smooth, but this requires more work. (Note that since  $C$  is a curve,  $\mathcal{F}_m$  is automatically locally free for any morphism  $f$ , since it is torsion-free.)

**Torsion-free, reflexive, and weakly positive sheaves.** Let  $X$  be an integral scheme of finite type. For an  $\mathcal{O}_X$ -module  $\mathcal{F}$ , we denote by  $\mathcal{F}^\vee$  the sheaf dual of  $\mathcal{F}$ , i.e.

$$\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_X).$$

**DEFINITION 2.4.6.** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is *torsion-free* if  $\mathcal{F}_x$  is a torsion-free  $\mathcal{O}_{X,x}$ -module for all  $x \in X$ . Equivalently, the natural mapping

$$\varphi : \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee}$$

is injective. Moreover,  $\mathcal{F}$  is called *reflexive* if  $\varphi$  is an isomorphism, so that  $\mathcal{F} \simeq \mathcal{F}^{\vee\vee}$ . In general,  $\mathcal{F}^{\vee\vee}$  is called the *reflexive hull* of  $\mathcal{F}$ .

**EXERCISE 2.4.7.** If  $f : X \rightarrow Y$  is a surjective morphism of varieties, and  $\mathcal{F}$  is a torsion-free sheaf on  $X$ , then  $f_* \mathcal{F}$  is torsion-free on  $Y$ .

**EXERCISE 2.4.8.** A coherent sheaf  $\mathcal{F}$  on  $X$  is called a *k-th syzygy sheaf* if locally around each point there exists an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_k \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{G} \longrightarrow 0$$

with  $\mathcal{E}_j$  free for all  $j$ . Show that 1-st syzygy sheaf is equivalent to torsion-free, and 2-nd syzygy sheaf is equivalent to reflexive.

**LEMMA 2.4.9.** *If  $\mathcal{F}$  is a coherent sheaf, then  $\mathcal{F}^\vee$  is reflexive.*

PROOF. First note that  $\mathcal{F}^\vee$  is torsion-free. Indeed,  $\mathcal{F}$  is locally a quotient

$$\mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{F}$$

and dualizing this we obtain a local inclusion of  $\mathcal{F}^\vee$  in a free sheaf. Now dualizing the natural map  $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$  and then composing it with the similar map for  $\mathcal{F}^\vee$  leads to a composition

$$\mathcal{F}^{\vee\vee\vee} \rightarrow \mathcal{F}^\vee \rightarrow \mathcal{F}^{\vee\vee\vee}$$

which can be easily seen to be the identity. It follows that the last map is surjective; it is however also injective, since  $\mathcal{F}^\vee$  is torsion-free.

Alternatively one can use the previous exercise: locally we always have a presentation

$$\mathcal{O}_X^{\oplus r_2} \longrightarrow \mathcal{O}_X^{\oplus r_1} \longrightarrow \mathcal{F} \longrightarrow 0,$$

and dualizing it we realize  $\mathcal{F}$  as a 2-nd syzygy sheaf.  $\square$

PROPOSITION 2.4.10. *If  $\mathcal{F}$  is a coherent sheaf on a smooth variety  $X$ , denote by  $S(\mathcal{F})$  the closed subset of  $X$  where  $\mathcal{F}$  is not locally free. Then, if  $\mathcal{F}$  is a  $k$ -th syzygy sheaf, then*

$$\text{codim}_X S(\mathcal{F}) > k.$$

*In particular:*

(i) *If  $\mathcal{F}$  is torsion-free, then  $\text{codim}_X S(\mathcal{F}) \geq 2$ . In particular, if  $X$  is a smooth curve, then torsion-free is equivalent to locally free.*

(ii) *If  $\mathcal{F}$  is reflexive, then  $\text{codim}_X S(\mathcal{F}) \geq 3$ . In particular, if  $X$  is a smooth surface, then reflexive is equivalent to locally free.*

PROOF. Note that a module over a local ring is free if and only if its projective dimension is 0, and so by definition we have

$$S(\mathcal{F}) = \{x \in X \mid \text{pd } \mathcal{F}_x \geq 1\}.$$

Fix a point  $x \in X$ , and denote  $A = \mathcal{O}_{X,x}$  and  $M = \mathcal{F}_x$ . Our hypothesis says that  $M$  is a finitely generated  $A$ -module that sits in an exact sequence

$$0 \longrightarrow M \longrightarrow A^{\oplus r_k} \longrightarrow \dots \longrightarrow A^{\oplus r_1} \longrightarrow N \longrightarrow 0,$$

with  $N$  another finitely generated  $A$ -module. Using the standard interpretation of projective dimension in terms of Ext groups, we see that  $x \in S(\mathcal{F})$  is equivalent to

$$0 \neq \text{Ext}_A^i(M, A) \simeq \text{Ext}_A^{i+k}(N, A)$$

for some  $1 \leq i \leq n = \dim X$ .

Thus locally there exists a coherent sheaf  $\mathcal{G}$  such that

$$S(\mathcal{F}) = \bigcup_{i=1}^n \text{Supp } \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = \bigcup_{j=k+1}^n \text{Supp } \mathcal{E}xt^j(\mathcal{G}, \mathcal{O}_X).$$

But a well-known application of the Auslander-Buchsbaum theorem says that for any coherent sheaf  $\mathcal{G}$  on a smooth variety, one has

$$\mathrm{codim}_X \mathrm{Supp} \mathcal{E}xt^j(\mathcal{G}, \mathcal{O}_X) \geq j \quad \text{for all } j \geq 0.$$

□

EXERCISE 2.4.11. Check the last assertion in the proof above.

LEMMA 2.4.12. *A coherent sheaf  $\mathcal{F}$  on a smooth variety  $X$  is reflexive if and only if it is torsion-free and the following property holds: for every open set  $U \subseteq X$  and every closed subset  $Z \subseteq U$  of codimension at least 2, the restriction map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus Z)$$

*is an isomorphism. (We sometimes say that  $\mathcal{F}$  is normal.)*

PROOF. Assume first that  $\mathcal{F}$  is reflexive, so clearly also torsion-free. Moreover, locally there exist exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus r_2} \longrightarrow \mathcal{O}_X^{\oplus r_1}$$

and the restriction map  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U \setminus Z)$  is an isomorphism since regular functions extend over codimension two subsets on smooth (or just normal) varieties. This implies the same assertion for  $\mathcal{F}$ .

To prove the opposite implication, note that  $\mathcal{F}$  and  $\mathcal{F}^{\vee\vee}$  are isomorphic outside of the singularity set  $S(\mathcal{F})$ . Since  $\mathcal{F}$  is torsion-free, Proposition 2.4.10 implies that  $S(\mathcal{F})$  has codimension at least 2 (in every open set in  $X$ ). The second hypothesis then implies that  $\varphi: \mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$  is an isomorphism on any open set  $U$ , hence an isomorphism of sheaves. □

**Weak positivity for torsion-free sheaves.** In this section we work over the complex numbers. We will prove a fundamental result of Viehweg on the weak positivity of direct images of relative pluricanonical bundles.

**Notation:** Since this will appear repeatedly, it is convenient to introduce the following notation: if  $\mathcal{F}$  is a coherent sheaf on  $X$  and  $k$  is an integer, then

$$\widehat{S}^k \mathcal{F} := (S^k \mathcal{F})^{\vee\vee}.$$

Note that if  $\mathcal{F}$  is torsion-free  $S^k \mathcal{F}$  injects into  $\widehat{S}^k \mathcal{F}$ , while  $(\mathcal{F}^{\otimes k})^{\vee\vee}$  surjects onto  $\widehat{S}^k \mathcal{F}$ .

DEFINITION 2.4.13. Let  $X$  be a smooth quasi-projective variety. A torsion-free coherent sheaf  $\mathcal{F}$  on  $X$  is *weakly positive* if for every integer  $\alpha > 0$  and every ample line bundle  $H$  on  $X$ , there exists an integer  $\beta > 0$  such that

$$\widehat{S}^{\alpha\beta} \mathcal{F} \otimes H^{\otimes\beta}$$

is generically globally generated.

REMARK 2.4.14. Viehweg's original definition, see e.g. [Vi2], is slightly different, in that it requires the existence of a fixed nonempty open set  $U \subseteq X$  over which global generation holds for every  $\alpha > 0$  and  $H$ . However it is also customary to consider the definition given here, which suffices for our purposes.

**EXAMPLE 2.4.15 (Line bundles).** Let's see what weak positivity means in the case of line bundles on projective varieties. Note that a line bundle  $L$  is generically globally generated iff  $H^0(X, L) \neq 0$ . Using additive notation, we obtain that if a line bundle  $M$  is weakly positive if and only if, in the notation of the definition above, we have that

$$\alpha\beta M + \beta H \text{ is effective.}$$

Dividing by  $\beta$  and letting  $\alpha \rightarrow \infty$ , we see that this is equivalent to  $M$  being in the closure of the cone of effective divisors. i.e. with  $M$  being *pseudoeffective*.

**EXAMPLE 2.4.16 (Nef vector bundles).** If  $E$  is a nef vector bundle on a smooth projective variety  $X$ , then  $E$  is weakly positive. In fact Exercise 2.2.6 give something stronger:  $E$  is nef if and only if for every ample line bundle  $H$  on  $X$  and every integer  $\alpha > 0$ , there exists an integer  $\beta > 0$  such that  $S^{\alpha\beta} E \otimes H^{\otimes\beta}$  is globally generated.

Here is a basic fact we will need; the proof is left as an exercise:

**LEMMA 2.4.17.** *Let  $\mathcal{F}$  and  $\mathcal{G}$  be torsion-free coherent sheaves on  $X$ , such that there exists a generically surjective homomorphism  $\mathcal{F} \rightarrow \mathcal{G}$ . If  $\mathcal{F}$  is weakly positive, then  $\mathcal{G}$  is weakly positive as well. In particular, if  $\mathcal{F}$  is weakly positive, then  $\det \mathcal{F}$  is a pseudoeffective line bundle.*

**Viehweg's fiber product trick, II.**<sup>2</sup> Let now  $f: X \rightarrow Y$  be an arbitrary projective algebraic fiber space, with  $X$  and  $Y$  smooth quasi-projective varieties. For any  $s \geq 1$ , we consider the main component  $X^s$  of the  $s$ -fold fiber product of  $X$  over  $Y$ , as in Exercise 2.4.1, with the induced dominant morphism  $f^s: X^s \rightarrow Y$ . We also consider a resolution of singularities  $\mu: X^{(s)} \rightarrow X^s$ , so that we have a composition

$$f^{(s)}: X^{(s)} \rightarrow Y, \quad f^{(s)} = f^s \circ \mu.$$

Since any two resolutions are dominated by a third, and for a birational morphism  $g: W \rightarrow Z$  between smooth varieties we have  $g_* \omega_W^{\otimes m} \simeq \omega_Z^{\otimes m}$ , we have that the sheaf

$$f_*^{(s)} \omega_{X^{(s)}/Y}^{\otimes m}$$

is independent of the choice of resolution for every  $m \geq 1$ . The general statement extending Proposition 2.4.2 is:

**PROPOSITION 2.4.18.** *For every  $m \geq 1$  and  $s \geq 1$ , there is an inclusion*

$$\left( f_*^{(s)} \omega_{X^{(s)}/Y}^{\otimes m} \right)^{\vee\vee} \hookrightarrow \left( (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} \right)^{\vee\vee}$$

*which is generically an isomorphism. More precisely, it is an isomorphism over the locus where  $f$  is smooth, and more generally where  $f$  is semistable.*

This result is a consequence of the following more general statement, in the same way as Proposition 2.4.2 is a consequence of (2.4.1).

<sup>2</sup>Thanks to Sung Gi Park for lecturing on the material in this section!

**THEOREM 2.4.19 (Viehweg's base change theorem).** *Let  $X$ ,  $Y$  and  $Y'$  be smooth quasi-projective varieties, and consider the commutative diagram*

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\mu} & X' & \xrightarrow{g'} & X \\ & \searrow \tilde{f} & \downarrow f' & & \downarrow f \\ & & Y' & \xrightarrow{g} & Y \end{array}$$

where  $f$  and  $g$  are surjective projective morphisms,  $X'$  is the main component of the fiber product  $X \times_Y Y'$ , and  $\mu$  is a resolution of singularities. Then there exists a canonical inclusion

$$(g_* \tilde{f}_* \omega_{\tilde{X}/Y}^{\otimes m})^{\vee\vee} \hookrightarrow (f_* \omega_{X'/Y}^{\otimes m} \otimes g_* \omega_{Y'/Y}^{\otimes m})^{\vee\vee}.$$

Moreover, this inclusion is an isomorphism at a point  $y \in Y$  if either  $f$  or  $g$  is semistable in a neighborhood of  $y$ .

A key ingredient for the proof is the following:

**LEMMA 2.4.20.** *In the notation of Theorem 2.4.19, assume the  $g$  is flat. Then  $X'$  is Gorenstein, and for each  $m \geq 1$  there is a natural inclusion*

$$\mu_* \omega_{\tilde{X}}^{\otimes m} \hookrightarrow \omega_{X'}^{\otimes m}.$$

Moreover, this inclusion is an isomorphism if  $X'$  has rational singularities.

**PROOF.** It is a general fact that relative dualizing sheaves commute with flat base change; see e.g. [?]. This gives an isomorphism

$$g'^* \omega_{X'/Y}^{\otimes m} \simeq \omega_{X'/Y'}^{\otimes m}$$

and since  $Y'$  is smooth, it follows that  $X'$  is Gorenstein.

The rest of the statement is a general fact for a projective birational morphism  $\mu: \tilde{X} \rightarrow X'$  from a smooth variety to a Gorenstein one. We first factor it through the normalization  $X''$  of  $X'$ , as follows:

$$\mu: \tilde{X} \xrightarrow{\eta} X'' \xrightarrow{\nu} X'.$$

Since  $X''$  is normal,  $\eta$  is an isomorphism over an open set  $V \subseteq X''$  whose complement has codimension at least 2, and since  $\eta_* \omega_{\tilde{X}}^{\otimes m}$  is torsion-free, we obtain an inclusion

$$\eta_* \omega_{\tilde{X}}^{\otimes m} \hookrightarrow (\omega_{X''}^{\otimes m})^{\vee\vee}$$

extending the identity over  $V$ . On the other hand, duality gives us a trace map

$$\nu_* \omega_{X''} \rightarrow \omega_{X'},$$

which by adjunction corresponds to a morphism

$$\omega_{X''} \rightarrow \nu^* \omega_{X'}.$$

Combined with the inclusion above, we obtain a morphism

$$\eta_* \omega_{\tilde{X}}^{\otimes m} \hookrightarrow \omega_{X''} \otimes \nu^* \omega_{X'}^{\otimes m-1}.$$

Finally, applying  $\nu_*$  we get inclusions

$$\mu_*\omega_{\tilde{X}}^{\otimes m} \hookrightarrow \nu_*\omega_{X''} \otimes \omega_{X'}^{\otimes m-1} \hookrightarrow \omega_{X'}^{\otimes m},$$

as stated. For the last statement, note that since  $X'$  is Gorenstein, it has rational singularities if and only if  $\mu_*\omega_{\tilde{X}} \simeq \omega_{X'}$ . □

**PROOF OF THEOREM 2.4.19.** Since the two sheaves in the statement are reflexive, by Lemma 2.4.12 it is enough to define this inclusion away from a closed subset of codimension at least 2. We may therefore assume that  $g$  is flat.<sup>3</sup>

We first claim that there is a natural inclusion

$$(2.4.2) \quad \tilde{f}_*\omega_{\tilde{X}/Y'}^{\otimes m} \hookrightarrow g^*f_*\omega_{X'/Y'}^{\otimes m}.$$

Indeed, applying  $f'_*$  to the inclusion in Lemma 2.4.20, we obtain an inclusion

$$\tilde{f}_*\omega_{\tilde{X}/Y'}^{\otimes m} \hookrightarrow f'_*\omega_{X'/Y'}^{\otimes m}$$

On the other hand, there is an isomorphism

$$f'_*\omega_{X'/Y'}^{\otimes m} \simeq g^*f_*\omega_{X'/Y'}^{\otimes m},$$

which is simply the push-pull formula for flat morphisms, combined with the isomorphism  $g'^*\omega_{X'/Y'}^{\otimes m} \simeq \omega_{X'/Y'}^{\otimes m}$ , which is given by the commutation of relative dualizing sheaves with flat base change.

Now tensoring (2.4.2) with  $\omega_{Y'/Y}^{\otimes m}$  and applying  $g_*$ , we obtain the inclusion

$$g_*\tilde{f}_*\omega_{\tilde{X}/Y'}^{\otimes m} \hookrightarrow f_*\omega_{X'/Y'}^{\otimes m} \otimes g_*\omega_{Y'/Y}^{\otimes m}$$

over the locus where  $f_*\omega_{X'/Y'}^{\otimes m}$  is locally free, using the projection formula. But since this sheaf is torsion-free, it is locally free in codimension one, and therefore the inclusion extends to the entire  $Y$  by taking double duals.

To see the last statement, note that Lemma 2.4.20 also tells us that our inclusion is an isomorphism whenever  $X'$  has rational singularities. We will show that this is the case over every point  $y \in Y$  over which either  $f$  or  $g$  are semistable. TO FILL IN. □

**Viehweg's weak positivity theorem.** The following is the key ingredient introduced by Viehweg for the study of Iitaka's conjecture. We present the simplified proof given in [PS2].

**THEOREM 2.4.21 (Viehweg).** *Let  $f : X \rightarrow Y$  be a surjective morphism of projective varieties. Then, for every  $m \geq 0$ , the sheaf  $f_*\omega_{X/Y}^{\otimes m}$  is weakly positive.*

---

<sup>3</sup>Any morphism with equidimensional fibers between an irreducible Cohen-Macaulay variety and a smooth variety is flat, by the so-called "miracle flatness"; see ??

PROOF. Recall the notation in Viehweg's fiber product trick. If we denote by  $X^{(s)}$  a resolution of singularities of the irreducible component of the  $s$ -fold fiber product of  $X$  over  $Y$ , there is an induced morphism

$$f^{(s)} : X^{(s)} \longrightarrow Y.$$

A torsion-free sheaf injects into its double-dual, so by Proposition 2.4.18 for every  $s \geq 1$  there is an inclusion

$$\varphi : f_* \omega_{X^{(s)}/Y}^{\otimes m} \hookrightarrow \left( (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} \right)^{\vee\vee}$$

which is generically an isomorphism. (Proposition 2.4.2 says that it is for instance an isomorphism on the locus in  $Y$  over which  $f$  is smooth.)

Let  $H$  be an ample line bundle on  $Y$ , and  $k > 0$  an integer such that  $H^{\otimes k}$  is very ample. Corollary 2.3.6 implies that

$$f_* \omega_{X^{(s)}/Y}^{\otimes m} \otimes A^{\otimes m}$$

is globally generated, where  $n = \dim Y$  and  $A = \omega_Y \otimes H^{\otimes k(n+1)}$ . But the generic isomorphism  $\varphi$  above implies then that

$$\left( (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} \right)^{\vee\vee} \otimes A^{\otimes m}$$

is generated by global sections over the locus where  $\varphi$  is an isomorphism.

Since  $H$  is ample, there is an integer  $a$  such that  $H^{\otimes b} \otimes \omega_Y^{\otimes -m}$  is globally generated for all  $b \geq a$ . Taking tensor product, we conclude that

$$\left( (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} \right)^{\vee\vee} \otimes H^{\otimes \beta}$$

is generically globally generated for  $\beta \geq a + mk(n+1)$ , which is independent of  $s$ .

Let now  $\alpha > 0$  be an integer. Since  $\left( (f_* \omega_{X/Y}^{\otimes m})^{\otimes s} \right)^{\vee\vee}$  surjects onto  $\widehat{S}^s f_* \omega_{X/Y}^{\otimes m}$ , taking  $s = \alpha\beta$  with  $\beta$  satisfying the bound above we obtain that

$$\widehat{S}^{\alpha\beta} f_* \omega_{X/Y}^{\otimes m} \otimes H^{\otimes \beta}$$

is generically globally generated. □

REMARK 2.4.22. The proof above gives something a bit stronger than the statement, namely an “effective” version of weak positivity. Indeed, once we fix the very ample line bundle  $H^{\otimes k}$ , then we have the effectively constructed  $A^{\otimes m}$  that can be taken to verify the definition of weak positivity.

## 2.5. Further background

**Multiplication maps.** Let  $X$  be a projective scheme, and  $L$  a line bundle on  $X$ . For each  $m, n \geq 0$  we have multiplication maps on global sections

$$(2.5.1) \quad H^0(X, L^{\otimes m}) \otimes H^0(X, L^{\otimes n}) \longrightarrow H^0(X, L^{\otimes m+n}).$$

In particular, for each  $m \geq 0$  there is a natural map

$$H^0(X, L)^{\otimes m} \longrightarrow H^0(X, L^{\otimes m}).$$

Since the product of sections does not depend on the order of multiplication, this map factors through the symmetric algebra, meaning that the key map to consider is

$$(2.5.2) \quad S^m H^0(X, L) \longrightarrow H^0(X, L^{\otimes m}).$$

**EXAMPLE 2.5.1.** If  $X = \mathbf{P}^n$  and  $L = \mathcal{O}_{\mathbf{P}^n}(d)$ , then the map in (2.5.2) is given by multiplication of polynomials, and is in fact an isomorphism: both sides coincide with the space of homogeneous polynomials of degree  $md$  in  $n + 1$  variables.

**EXERCISE 2.5.2.** If  $\mathfrak{b}_k$  denotes the base ideal of the linear system  $|kL|$ , then show that

$$\mathfrak{b}_m \cdot \mathfrak{b}_n \subseteq \mathfrak{b}_{m+n}.$$

**EXERCISE 2.5.3.** Let  $L$  be an ample line bundle. Then there exists  $m_0 \in \mathbb{N}$  such that the multiplication maps in (2.5.1) are surjective for all  $m, n \geq m_0$ .

Consider now a projective morphism  $f : X \rightarrow Y$  of quasi-projective varieties, and let  $L$  be a line bundle on  $X$ . On any fiber  $F$ , the restriction  $L_F$  induces multiplication maps as in (2.5.2), namely

$$S^m H^0(F, L_F) \longrightarrow H^0(F, L_F^{\otimes m}).$$

It is not hard to check there are also induced morphisms of sheaves

$$\varphi_m : S^m f_* L \longrightarrow f_* L^{\otimes m},$$

which factor the natural morphisms  $(f_* L)^{\otimes m} \rightarrow f_* L^{\otimes m}$ . Note that if  $U \subseteq Y$  is the open set over which  $f$  is flat, and  $V_k \subseteq Y$  is the open set over which  $h^0(F, L_F^{\otimes k})$  is constant, by Grauert's theorem it follows that at a point  $y \in U \cap V_1 \cap V_m$  the morphism  $\varphi_m$  is precisely the multiplication map on the fiber  $F$  over  $y$  described above.

Now the domain and target of  $\varphi_m$  are torsion-free sheaves, and therefore if  $Y$  is smooth they are locally free in codimension one. By Lemma 2.4.12 it follows that  $\varphi_m$  extends uniquely to a morphism

$$(2.5.3) \quad \widehat{\varphi}_m : \widehat{S^m f_* L} \longrightarrow (f_* L^{\otimes m})^{\vee\vee}.$$

Such maps will be useful in what follows.

**Exceptional divisors.** Viehweg devised a clever way of getting rid of divisors that affect our calculations but are exceptional for the fiber space under consideration. The technical lemmas are explained in this subsection.

**LEMMA 2.5.4.** *Let  $f : X \rightarrow Y$  be an algebraic fiber space, with  $X$  and  $Y$  smooth, and let  $L$  be a line bundle on  $X$ . Then for any  $k \geq 1$  there exists an effective divisor  $B$  on  $X$  such that  $\text{codim}_Y f(B) \geq 2$  and*

$$(f_* L^{\otimes m})^{\vee\vee} \simeq f_*(L^{\otimes m}(mB)) \quad \text{for all } m \leq k.$$

**PROOF.** Let  $U$  be the maximal open set on which  $f_* L^{\otimes m}$  is locally free for all  $m \leq k$ , and denote  $V = f^{-1}(U)$ . Since all of these pushforward sheaves are torsion-free, we know that  $Y \setminus U$  has codimension at least 2 in  $Y$ , and therefore

$$(f_* L^{\otimes m})^{\vee\vee} \simeq i_*((f_* L^{\otimes m})|_U) \simeq i_* f_{V*}(L|_V^{\otimes m}) \simeq f_*(j_* L|_V^{\otimes m}),$$

where the maps are summarized in the following diagram, the horizontal maps being the natural inclusions:

$$\begin{array}{ccc} V & \xrightarrow{j} & X \\ f_V \downarrow & & \downarrow f \\ U & \xrightarrow{i} & Y. \end{array}$$

Denote  $D = X \setminus V$ . If the codimension of  $D$  is again at least 2, since  $L^{\otimes m}$  is locally free we have that  $j_*L_{|V}^{\otimes m} \simeq L^{\otimes m}$ , so we can take  $B = 0$ . If  $D$  is a divisor, then

$$j_*L_{|V}^{\otimes m} \simeq L^{\otimes m}(*D) := \bigcup_{p \geq 0} L^{\otimes m}(pD),$$

i.e. the quasi-coherent sheaf of sections of  $L^{\otimes m}$  with poles of arbitrary order along  $D$ . (Locally over some  $\text{Spec } A$ , this is isomorphic to the localization  $A_f$ , where  $f$  is a local equation of  $D$ .) Note then that for each  $m$  we have an ascending chain of coherent sheaves

$$\cdots \subseteq f_*(L^{\otimes m}(pD)) \subseteq f_*(L^{\otimes m}((p+1)D)) \subseteq \cdots \subseteq (f_*L^{\otimes m})^{\vee\vee}.$$

It follows that each such chain must stabilize, and at some  $p_0$  where it does so, we have

$$f_*(L^{\otimes m}(pD)) \simeq (f_*L^{\otimes m})^{\vee\vee} \quad \text{for all } p \geq p_0.$$

Finally, as we are only looking at finitely many  $m$ , choosing  $B$  to be a sufficiently large multiple of  $D$  implies the slightly more precise version in the statement.  $\square$

**LEMMA 2.5.5.** *Let  $f : X \rightarrow Y$  be a morphism of smooth varieties. Then there exists a proper birational morphism  $\tau : Y' \rightarrow Y$  with  $Y'$  smooth, and a resolution of singularities  $X'$  of the main component of  $X \times_Y Y'$ , such that the induced morphism  $f' : X' \rightarrow Y'$  has the property that every divisor  $B'$  in  $X'$  with  $\text{codim}_{Y'} f'(B') \geq 2$  is contained in the exceptional locus of  $\tau' : X' \rightarrow X$ .*

**PROOF.** The main point is the ‘‘flattening’’ theorem due to Hironaka and Gruson-Raynaud, see e.g. [?]: one can find a proper birational morphism  $\tau : Y' \rightarrow Y$  with  $Y'$  smooth such that the induced morphism

$$\tilde{f} : \tilde{X} := (X \times_Y Y')_{\text{main}} \rightarrow Y'$$

is flat. Denoting by  $\mu : X' \rightarrow \tilde{X}$  a resolution of singularities, since  $\tilde{f}$  is flat (hence with all fibers of the same dimension) it follows that if  $B'$  is contracted by  $f'$ , then it must already be exceptional for  $\mu$ , so also for  $\tau'$ .  $\square$

## 2.6. Iitaka's conjecture over a base of general type

In this section we will show that Theorem 2.4.21 can be used to prove Iitaka's conjecture on the subadditivity of the Kodaira dimension when the base is of general type. The more general result below, and the proof presented here, are both due to Viehweg [Vi2].

**THEOREM 2.6.1.** *Let  $f : X \rightarrow Y$  be an algebraic fiber space with  $X$  and  $Y$  smooth projective varieties, and denote by  $F$  the general fiber of  $f$ . Then:*

(i) *If  $L$  is an ample line bundle on  $Y$ , and  $m \geq 1$ , then*

$$\kappa(X, \omega_{X/Y}^{\otimes m} \otimes f^*L) = \kappa(F) + \dim Y.$$

(ii) *If  $Y$  is of general type, then*

$$\kappa(X) = \kappa(F) + \dim Y,$$

*i.e. Iitaka's conjecture holds.*<sup>4</sup>

**PROOF.** We first consider the following technical point: according to Lemma 2.5.5, there exists a smooth birational modification  $\tau : Y' \rightarrow Y$ , and a resolution  $X'$  of the main component of  $X \times_Y Y'$ , giving a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

with the property that every effective divisor  $B$  on  $X'$  such that  $\text{codim}_{Y'} f'(B) \geq 2$  lies in the exceptional locus of  $\tau'$ . Note that for such a divisor  $B$  we have

$$\tau'_* \omega_{X'}^{\otimes m}(mB) \simeq \omega_X^{\otimes m} \quad \text{for all } m \geq 0.$$

Fix now an ample line bundle  $L$  on  $Y$ , and consider the big line bundle  $L' := \tau^*L$  on  $Y'$ . We may assume that the integer  $m$  in the statement is chosen such that  $f'_* \omega_{X'/Y'}^{\otimes m} \neq 0$ ; indeed, if no such  $m$  exists, we clearly have  $\kappa(F) = -\infty$ . By Theorem 2.4.21 there exists an integer  $b > 0$  such that

$$\widehat{S}^{2b} f'_* \omega_{X'/Y'}^{\otimes m} \otimes L'^{\otimes b}$$

is generically globally generated. Moreover, we have seen in (2.5.3) that there exists a morphism

$$\widehat{S}^{2b} f'_* \omega_{X'/Y'}^{\otimes m} \longrightarrow (f'_* \omega_{X'/Y'}^{\otimes 2mb})^{\vee\vee}$$

induced by the relative multiplication map, which is non-trivial at the general point of  $Y$ . On the other hand, by Lemma 2.5.4 there exists an effective divisor  $B$  on  $X'$ , exceptional for  $f'$ , such that

$$(f'_* \omega_{X'/Y'}^{\otimes p})^{\vee\vee} \simeq f'_*(\omega_{X'/Y'}^{\otimes p}(pB)) \quad \text{for all } p \leq 2mb.$$

Putting everything together, it follows that

$$f'_*(\omega_{X'/Y'}^{\otimes 2mb}(2mbB)) \otimes L'^{\otimes b}$$

has a non-zero section. Using the projection formula, this is equivalent to the existence of an inclusion

$$f'^* L'^{\otimes b} \hookrightarrow (\omega_{X'/Y'}(B))^{\otimes 2mb} \otimes f'^* L'^{\otimes 2b}.$$

<sup>4</sup>Note that by Easy Addition, Example 1.3.11, we always have the inequality  $\kappa(F) + \dim Y \geq \kappa(X)$ , hence the main point is the subadditivity predicted by the conjecture.

According to Lemma 1.4.15, we obtain

$$\kappa \left( (\omega_{X'/Y'}(B))^{\otimes m} \otimes f'^* L' \right) = \kappa(F') + \dim Y' = \kappa(F) + \dim Y,$$

where  $F'$  is the general fiber of  $f'$ . For the second equality, note on one hand that  $F'$  and  $F$  are birational, while on the other hand since  $B$  is contracted by  $f'$ , it does not meet  $F'$ , and therefore the restriction of  $\omega_{X'/Y'}(B)$  to  $F'$  is  $\omega_{F'}$ .

To deduce (i), note that since by Easy Addition we have

$$\kappa(F) + \dim Y \geq \kappa(X, \omega_{X/Y}^{\otimes m} \otimes f^* L),$$

it suffices to show that

$$\kappa(X, \omega_{X/Y}^{\otimes m} \otimes f^* L) \geq \kappa \left( (\omega_{X'/Y'}(B))^{\otimes m} \otimes f'^* L' \right),$$

for which in turn, it suffices to have an inclusion

$$\tau'_* (\omega_{X'}^{\otimes m}(mB) \otimes f'^* \omega_{Y'}^{\otimes -m} \otimes f'^* L') \hookrightarrow \omega_{X/Y}^{\otimes m} \otimes f^* L.$$

Now since  $f'^* L' \simeq \tau'^* f^* L$ , and as we have observed at the beginning we have  $\tau'_* \omega_{X'}^{\otimes m}(mB) \simeq \omega_X^{\otimes m}$ , by the projection formula we finally see that it suffices to have an inclusion

$$f'^* \omega_{Y'}^{\otimes -m} \hookrightarrow \tau'^* f^* \omega_Y^{\otimes -m} \simeq f'^* \tau^* \omega_Y^{\otimes -m}.$$

But this follows from the obvious inclusion  $\tau^* \omega_Y \hookrightarrow \omega_{Y'}$ .

To deduce (ii), since  $Y'$  is of general type recall that by Lemma 1.4.11 there exists an inclusion  $L' \hookrightarrow \omega_{Y'}^{\otimes r}$  for some  $r > 0$ . This implies that

$$\kappa(X) = \kappa(X', \omega_{X'}(B)) \geq \kappa \left( (\omega_{X'/Y'}(B))^{\otimes r} \otimes f'^* L' \right),$$

which is equal to  $\kappa(F) + \dim Y$  by the same reasoning as above.  $\square$

## 2.7. A list of known cases of Iitaka's conjecture

A number of significant cases of Iitaka's conjecture are currently known. There are two stages in this story: some important cases were proved in the 1980's, based on developments in birational geometry and Hodge theory. After an almost thirty year gap, a few more significant cases were conquered, due to new techniques of an analytic nature.

In what follows  $f: X \rightarrow Y$  is an algebraic fiber space with  $X$  and  $Y$  smooth projective complex varieties, and with general fiber  $F$ .

Here is a (perhaps incomplete) list of cases in which Iitaka's conjecture holds:

- (1)  $\kappa(Y) = -\infty$ ; this is obvious.
- (2)  $\kappa(F) = -\infty$ ; in this case  $\kappa(X) = -\infty$  by Easy Addition.
- (3)  $\kappa(F) \geq 0$  and  $F$  has a good minimal model [Ka3] (so in particular when  $\dim F \leq 3$ ); this includes the case when  $F$  has semiample (e.g. trivial) canonical bundle and, due to more recent developments in the MMP, the independently proved case when  $F$  is of general type [Ko] (cf. also [Vi3]).

- (4)  $Y$  is of general type [Vi2].
- (5)  $Y$  is an abelian variety [CP]; see also [HPS] for a simplified proof and an extension to the case where  $Y$  is of maximal Albanese dimension.
- (6)  $Y$  is a curve. Note that this is implied by (1), (4) and (5); however it was first proved independently in [Ka2].
- (7)  $Y$  is a surface. Many cases follow from the items above; the essential remaining case of  $K3$  (and Enriques) surfaces was obtained recently in [Ca].
- (8)  $\dim X \leq 6$ . Note that this is now implied by (4), (5) and (7); however it first appeared in [Bi].

**What we do not know.** Despite the important results above, the (unconditional) known cases of Iitaka's conjecture are still quite limited. It is already not known when  $\dim Y = 3$ , for instance when  $Y$  is a Calabi-Yau threefold. It is not known in general when  $\kappa(X) = 0$ , or when  $\kappa(Y) = 0$ , or when  $\kappa(F) = 0$ .

Note however that Kawamata's result in (3) above implies that subadditivity follows from the existence of good minimal models for varieties of non-negative Kodaira dimension; this is one of the main conjectures of the Minimal Model Program, widely believed to be true.



## CHAPTER 3

### Morphisms to abelian varieties

In this chapter we will study the case of morphisms  $f: X \rightarrow A$ , with  $A$  an abelian variety, which has received a lot of attention lately. In particular, we will verify Iitaka's conjecture in this setting, making use of both algebraic and analytic methods.

#### 3.1. Generic vanishing and Fourier-Mukai transform

The main algebraic tool we use is the theory of generic vanishing. This study was initiated by Green and Lazarsfeld [GL1, GL2], in part as an attempt to provide a useful weaker version of Kodaira Vanishing for the canonical bundle, in the absence of twists by positive line bundles. An important addition was provided in work of Simpson [Sim]. The results of Green-Lazarsfeld were extended to include higher direct images of canonical bundles in [Hac]. Further generalizations were provided in [PP1] and [PS1].

The original statements to keep in mind for our purposes are summarized in the following theorem. For any coherent sheaf  $\mathcal{F}$  on an abelian variety  $A$ , we consider for all  $k \geq 0$  the *cohomological support loci*

$$V^k(\mathcal{F}) = \{ P \in \text{Pic}^0(A) \mid H^k(X, \mathcal{F} \otimes P) \neq 0 \}$$

They are closed subsets of  $\text{Pic}^0(A)$ , by the semi-continuity theorem for cohomology.

**THEOREM 3.1.1.** *If  $f: X \rightarrow A$  is a morphism from a smooth projective variety to an abelian variety, then for any  $j, k \geq 0$  we have*

- (1) [Hac]  $\text{codim}_{\text{Pic}^0(A)} V^k(R^j f_* \omega_X) \geq k$ .
- (2) [GL2, Sim] *Every irreducible component of  $V^k(R^j f_* \omega_X)$  is a translate of an abelian subvariety of  $A$  by a torsion point.*

What we will use here are (partial) extensions of these results to pushforwards of pluricanonical bundles  $f_* \omega_X^{\otimes m}$ , for  $m \geq 2$ . We start by providing a general study of the generic vanishing property.

**The GV property and unipotency** Let  $A$  be an abelian variety of dimension  $g$ . The generic vanishing property (1) in the theorem above can be formalized into the following:

**DEFINITION 3.1.2.** A coherent sheaf  $\mathcal{F}$  on  $A$  is called a *GV-sheaf* if

$$\text{codim}_{\text{Pic}^0(A)} V^k(\mathcal{F}) \geq k \quad \text{for all } k \geq 0.$$

We will identify  $\text{Pic}^0(A)$  with the dual abelian variety  $\widehat{A}$ , and denote by  $P$  the normalized Poincaré bundle on  $A \times \widehat{A}$ . It induces the integral transforms

$$\mathbf{R}\Phi_P: \mathbf{D}(A) \longrightarrow \mathbf{D}(\widehat{A}), \quad \mathbf{R}\Phi_P \mathcal{F} := \mathbf{R}p_{2*}(p_1^* \mathcal{F} \otimes P).$$

and

$$\mathbf{R}\Psi_P: \mathbf{D}(\widehat{A}) \longrightarrow \mathbf{D}(A), \quad \mathbf{R}\Psi_P \mathcal{G} := \mathbf{R}p_{1*}(p_2^* \mathcal{G} \otimes P),$$

where for smooth projective variety  $Y$ , the notation  $\mathbf{D}(Y)$  stands for the bounded derived category of coherent sheaves on  $Y$ . These functors are known from [Mu, Theorem 2.2] to be equivalences of derived categories, usually called the Fourier-Mukai transforms; moreover,

$$(3.1.1) \quad \mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^*[-g] \quad \text{and} \quad \mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P = (-1_{\widehat{A}})^*[-g],$$

where  $[-g]$  denotes shifting  $g$  places to the right.

Standard applications of base change (see e.g. [PP2, Lemma 2.1] and [PP1, Proposition 3.14]) lead to the following basic properties of  $GV$ -sheaves:

LEMMA 3.1.3. *Let  $\mathcal{F}$  be a coherent sheaf on  $A$ . Then:*

(1)  *$\mathcal{F}$  is a  $GV$ -sheaf if and only if*

$$\text{codim}_{\widehat{A}} \text{Supp } R^k \Phi_P \mathcal{F} \geq k \quad \text{for all } k \geq 0.$$

(2) *If  $\mathcal{F}$  is a  $GV$ -sheaf, then*

$$V^g(\mathcal{F}) \subseteq \dots \subseteq V^1(\mathcal{F}) \subseteq V^0(\mathcal{F}).$$

Here is a brief sketch of the proof of part (1): note that the restriction of  $p_1^* \mathcal{F} \otimes P$  to a fiber  $A \times \{\alpha\}$  of  $p_2$  is isomorphic to the sheaf  $\mathcal{F} \otimes \alpha$  on  $A$ , and so fiberwise we are looking at the cohomology groups  $H^k(A, \mathcal{F} \otimes \alpha)$ . A simple application of the theorem on cohomology and base change then shows for every  $m \geq 0$  that

$$\bigcup_{k \geq m} \text{Supp } R^k \Phi_P \mathcal{F} = \bigcup_{k \geq m} V^k(\mathcal{F}).$$

This implies the result by descending induction on  $k$ . Part (2) relies on the cohomology and base change theorem as well.

LEMMA 3.1.4. *If  $\mathcal{F}$  is a  $GV$ -sheaf on  $A$ , then  $\mathcal{F} = 0$  if and only if  $V^0(\mathcal{F}) = \emptyset$ .*

PROOF. By Lemma 3.1.3, we see that  $V^0(\mathcal{F}) = \emptyset$  is equivalent to  $V^k(\mathcal{F}) = \emptyset$  for all  $k \geq 0$ , which by base change is in turn equivalent to  $\mathbf{R}\Phi_P \mathcal{F} = 0$ . By Mukai's derived equivalence, this is equivalent to  $\mathcal{F} = 0$ .  $\square$

We next recall a useful concept introduced in [Mu].

DEFINITION 3.1.5. A vector bundle  $U$  on  $A$  is called *unipotent* if it has a filtration

$$0 = U_0 \subset U_1 \subset \dots \subset U_n = U$$

such that  $U_i/U_{i-1} \simeq \mathcal{O}_A$  for all  $i = 1, \dots, n$ . Note in particular that  $\det U \simeq \mathcal{O}_A$ . More generally,  $U$  is called *homogeneous* if it has a filtration

$$0 = U_0 \subset U_1 \subset \dots \subset U_n = U$$

such that  $U_i/U_{i-1}$  is isomorphic to a line bundle in  $\text{Pic}^0(A)$  for all  $i = 1, \dots, n$ .

The key way in which generic vanishing will be used below is by means of the following result.

**PROPOSITION 3.1.6.** *Let  $\mathcal{F}$  be a GV-sheaf on an abelian variety  $A$ . If  $V^0(\mathcal{F}) = \{0\}$ , then  $\mathcal{F}$  is a unipotent vector bundle.*

**PROOF.** By [Mu, Example 2.9], if  $g = \dim A$ , then  $\mathcal{F}$  is a unipotent vector bundle if and only if

$$(3.1.2) \quad R^i \Phi_P \mathcal{F} = 0 \quad \text{for all } i \neq g \quad \text{and} \quad R^g \Phi_P \mathcal{F} = \mathcal{G},$$

where  $\mathcal{G}$  is a coherent sheaf supported at the origin  $0 \in \widehat{A}$ . To review the argument, notice that if this is the case, then if  $l = \text{length}(\mathcal{G}) > 0$ , we have  $h^0(\widehat{A}, \mathcal{G}) \neq 0$  and so there is a short exact sequence

$$0 \longrightarrow k(0) \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}' \longrightarrow 0$$

where  $\mathcal{G}'$  is a coherent sheaf supported at the origin  $0 \in \widehat{A}$ , with  $\text{length}(\mathcal{G}') = l - 1$ . Applying  $\mathbf{R}\Psi_P$  we obtain a short exact sequence of vector bundles on  $A$

$$0 \longrightarrow \mathcal{O}_A \longrightarrow R^0 \Psi_P \mathcal{G} \longrightarrow R^0 \Psi_P \mathcal{G}' \longrightarrow 0,$$

and by (3.1.1) we have  $R^0 \Psi_P \mathcal{G} = (-1_A)^* \mathcal{F}$ . It is then not hard to see that  $\mathcal{F}' = R^0 \Psi_P \mathcal{G}'$  also satisfies the hypotheses in (3.1.2) and so, proceeding by induction on  $l$ , we may assume that  $\mathcal{F}'$  is a unipotent vector bundle. It follows that  $\mathcal{F}$  is also a unipotent vector bundle as well (since it is an extension of a unipotent vector bundle by  $\mathcal{O}_A$ ).

We now check that the two conditions in (3.1.2) are satisfied. By Lemma 3.1.3(2) the hypothesis implies that

$$V^i(\mathcal{F}) \subseteq \{0\} \quad \text{for all } i \geq 0.$$

By base change one obtains that  $R^i \Phi_P \mathcal{F}$  is supported at most at  $0 \in \widehat{A}$  for  $0 \leq i \leq g$ . It remains to show that  $R^i \Phi_P \mathcal{F} = 0$  for  $i \neq g$ . Note that

$$H^j(\widehat{A}, R^i \Phi_P \mathcal{F} \otimes \alpha) = 0 \quad \text{for all } j > 0, 0 \leq i \leq g, \text{ and } \alpha \in \text{Pic}^0(\widehat{A}),$$

and so by base change we have

$$R^j \Psi_P(R^i \Phi_P \mathcal{F}) = 0 \quad \text{for all } j > 0 \text{ and } 0 \leq i \leq g.$$

By an easy argument involving the spectral sequence of the composition of two functors, since  $\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^*[-g]$ , it then follows that  $R^0 \Psi_P(R^i \Phi_P \mathcal{F}) = \mathcal{H}^i((-1_A)^* \mathcal{F}[-g])$ , and so in particular

$$R^0 \Psi_P(R^i \Phi_P \mathcal{F}) = 0 \quad \text{for } i < g.$$

But then  $\mathbf{R}\Psi_P(R^i \Phi_P \mathcal{F}) = 0$  for  $i < g$ , and hence  $R^i \Phi_P \mathcal{F} = 0$  for  $i < g$ .

□

**How to check the GV property.** A very useful tool for detecting generic vanishing is a cohomological criterion introduced in [Hac, Corollary 3.1]. Before stating it, we recall that an ample line bundle  $N$  on an abelian variety  $B$  induces an isogeny

$$\varphi_N : B \longrightarrow \widehat{B}, \quad x \rightarrow t_x^* N \otimes N^{-1},$$

where  $t_x$  denotes translation by  $x \in B$ .

**THEOREM 3.1.7.** *A coherent sheaf  $\mathcal{F}$  on  $A$  is a GV-sheaf if and only if given any sufficiently large power  $M$  of an ample line bundle on  $\widehat{A}$ , one has*

$$H^i(A, \mathcal{F} \otimes R^g \Psi_P(M^{-1})) = 0 \quad \text{for all } i > 0.$$

If  $\varphi_M : \widehat{A} \rightarrow A$  is the isogeny induced by  $M$ , this is also equivalent to

$$H^i(\widehat{A}, \varphi_M^* \mathcal{F} \otimes M) = 0 \quad \text{for all } i > 0.$$

**REMARK 3.1.8.** Note that since  $M$  is ample,  $H^i(\widehat{A}, M^{-1} \otimes \alpha) = 0$  for all  $i < g$  and  $\alpha \in \text{Pic}^0(\widehat{A}) \simeq A$ , and so  $R^i \Psi_P(M^{-1}) = 0$  for  $i \neq g$ . If we denote  $R^g \Psi_P(M^{-1}) = \widehat{M}^{-1}$ , then by [Mu, Proposition 3.11] we have  $\varphi_M^* \widehat{M}^{-1} \simeq M^{\oplus h^0(M)}$ , hence the second assertion.

**Direct images of pluricanonical bundles.** The main generic vanishing result we need to use for our Kodaira dimension problem is the following partial generalization of Theorem 3.1.1(1) to pluricanonical bundles.

**THEOREM 3.1.9** ([PS2, Theorem 1.10]). *If  $f : X \rightarrow A$  is a morphism from a smooth projective variety to an abelian variety, then  $f_* \omega_X^{\otimes m}$  is a GV-sheaf for every  $m \geq 1$ .*

**PROOF.** Let  $M = L^{\otimes d}$ , where  $L$  is an ample and globally generated line bundle on  $\widehat{A}$ , and  $d$  is an integer that can be chosen arbitrarily large. Let  $\varphi_M : \widehat{A} \rightarrow A$  be the isogeny induced by  $M$ . According to Theorem 3.1.7, it is enough to show that

$$H^i(\widehat{A}, \varphi_M^* f_* \omega_X^{\otimes m} \otimes M) = 0 \quad \text{for all } i > 0.$$

Equivalently, we need to show that

$$H^i(\widehat{A}, h_* \omega_{X_1}^{\otimes m} \otimes L^{\otimes d}) = 0 \quad \text{for all } i > 0,$$

where  $h : X_1 \rightarrow \widehat{A}$  is the base change of  $f : X \rightarrow A$  via  $\varphi_M$ , as in the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \downarrow h & & \downarrow f \\ \widehat{A} & \xrightarrow{\varphi_M} & A \end{array}$$

We can conclude immediately if we know that there exists a bound  $d = d(g, m)$ , i.e. depending only on  $g = \dim A$  and  $m$ , such that the vanishing in question holds for any morphism  $h$ . But Theorem 2.3.5 tells us that this indeed happens whenever  $d \geq m(g+1) - g$ , and we are done.  $\square$

It is interesting to note that, due to results in the theory of  $M$ -regularity on abelian varieties, Theorem 3.1.9 implies that in this setting we in fact have much stronger statements than in Theorem 2.4.4 and Theorem 2.3.5.

**COROLLARY 3.1.10** ([PS2, Corollary 5.4]). *If  $f : X \rightarrow A$  is a morphism from a smooth projective variety to an abelian variety, for every ample line bundle  $L$  on  $A$  and every  $m \geq 1$  one has:*

- (i)  $f_*\omega_X^{\otimes m}$  is a nef sheaf on  $A$ .
- (ii)  $H^i(A, f_*\omega_X^{\otimes m} \otimes L) = 0$  for all  $i > 0$ .
- (iii)  $f_*\omega_X^{\otimes m} \otimes L^{\otimes 2}$  is globally generated.

**REMARK 3.1.11** (Higher direct images). In [PS2], the question is raised whether the higher direct images  $R^j f_*\omega_X^{\otimes m}$  have the GV property as well for  $j > 0$  and  $m \geq 2$ . This is shown by Shibata [Shi] to be false. More precisely, he shows that there exist smooth projective varieties  $X$  with  $\omega_X^{-1}$  big, and with a nontrivial morphism  $f : X \rightarrow A$ ; for every such  $f$ , there exist  $j > 0$  and  $m \geq 2$  such that  $R^j f_*\omega_X^{\otimes m}$  is not a GV-sheaf. On the other hand, Shibata shows that if  $\dim X = 2$  and  $\kappa(X) \geq 0$ , then the question has a positive answer. It is natural to ask whether this is the case in arbitrary dimension, under the assumption  $\kappa(X) \geq 0$ .

We will also use the following partial generalization of Theorem 3.1.1(2), due to Chen-Hacon [CH, §3] and Lai [Lai, Theorem 3.5].

**THEOREM 3.1.12.** *If  $X$  is a smooth projective variety,  $V^0(\omega_X^{\otimes m}) \subseteq \text{Pic}^0(X)$  is a finite union of abelian subvarieties translated by points of finite order for every  $m \geq 1$ .*

This theorem implies that if  $f : X \rightarrow A$  is an algebraic fiber space to an abelian variety, then  $V^0(A, f_*\omega_X^{\otimes m})$  is also a finite union of abelian subvarieties translated by points of finite order, the reason being that the pullback morphism  $f^* : \text{Pic}^0(A) \rightarrow \text{Pic}^0(X)$  is injective.

### 3.2. The Iitaka conjecture over an abelian base

Using analytic techniques, Cao and Păun [CP] proved the subadditivity of the Kodaira dimension in the case when the base is an abelian variety  $A$ . Since  $\kappa(A) = 0$ , this amounts to the following:

**THEOREM 3.2.1.** *Let  $f : X \rightarrow A$  be an algebraic fiber space, with  $X$  a smooth projective variety and  $A$  an abelian variety over the complex numbers. If  $F$  is the general fiber of  $f$ , then*

$$\kappa(X) \geq \kappa(F).$$

Here I will present a simplified proof of this theorem obtained in [HPS], still depending on some the analytic techniques in [CP], but with more algebraic input from generic vanishing theory, as discussed in the previous section. In fact *loc. cit.* addresses a more general statement, for which we need the following:

**DEFINITION 3.2.2.** A smooth projective variety  $Y$  is said to be of *maximal Albanese dimension* if its Albanese map  $a: Y \rightarrow \text{Alb}(Y)$  is generically finite onto its image. By the universal property of the Albanese, this is equivalent to asking that there exist a morphism  $Y \rightarrow A$  to an abelian variety, which is generically finite onto its image; in particular subvarieties of abelian varieties are obvious such examples.

**THEOREM 3.2.3.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space with general fiber  $F$ . If  $Y$  has maximal Albanese dimension, then*

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

This includes of course the case where  $Y$  is a curve of genus  $\geq 1$ , where the following result was first proved by Kawamata [Ka2, Theorem 2].

**Analytic input.** A crucial ingredient in this story is the notion of a singular hermitian metric on a torsion-free sheaf, appearing in an important result of Păun and Takayama. In order to state it, recall that to a singular hermitian metric  $h$  on a line bundle  $L$ , one associates the multiplier ideal sheaf  $\mathcal{I}(h) \subseteq \mathcal{O}_X$ , consisting of those functions that are locally square-integrable with respect to  $h$ .

**THEOREM 3.2.4** ([PT, Theorem 3.3.5]). *Let  $f: X \rightarrow Y$  be a projective morphism of smooth varieties, and let  $(L, h)$  be a line bundle on  $X$  with a singular hermitian metric of semi-positive curvature. Then the torsion-free sheaf  $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$  has a canonical singular hermitian metric with semi-positive curvature.*

For the relevant definitions and the proof, one can consult the survey given in [HPS]. Another key statement, due to Cao and Păun [CP, Corollary 2.9 and Theorem 5.23] is that singular hermitian metrics behave much like smooth metrics with Griffiths semi-positive curvature: if the determinant line bundle  $\det \mathcal{F}$  has trivial first Chern class, then  $\mathcal{F}$  is actually a hermitian flat bundle. This is (a) below; part (b) is an additional result shown in [HPS].

**THEOREM 3.2.5.** *Let  $f: X \rightarrow Y$  be a surjective morphism of smooth projective varieties. Let  $(L, h)$  be a line bundle on  $X$  with a singular hermitian metric of semi-positive curvature, and define  $\mathcal{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ .*

- (a) *If  $c_1(\det \mathcal{F}) = 0$  in  $H^2(Y, \mathbb{R})$ , then the torsion-free sheaf  $\mathcal{F}$  is locally free, and the singular hermitian metric in Theorem 3.2.4 is smooth and flat.*
- (b) *Every nonzero morphism  $\mathcal{F} \rightarrow \mathcal{O}_Y$  is split surjective.*

The application of these results to Theorem 3.2.3 stems from the fact that the sheaves  $\mathcal{F}_m$  with  $m \geq 2$  naturally fit into this framework. Indeed, for every  $m \in \mathbb{N}$  such that  $P_m(F) \neq 0$ , where  $F$  is the general fiber of  $f$ , the spaces of  $m$ -canonical forms on the smooth fibers of  $f$  induce in a canonical way a singular hermitian metric with semi-positive curvature on  $\omega_{X/Y}$ , called the  *$m$ -th Narasimhan-Simha metric* (for  $m = 1$  this is the Hodge metric). If  $h$  the induced singular hermitian metric on the line bundle  $L = \omega_{X/Y}^{\otimes(m-1)}$ , the inclusion

$$f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)) \subseteq f_*(\omega_{X/Y} \otimes L) = f_*\omega_{X/Y}^{\otimes m}$$

is generically an isomorphism, and so Theorem 3.2.4 and Theorem 3.2.5 apply.

**COROLLARY 3.2.6.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space.*

- (a) *For any  $m \in \mathbb{N}$ , the torsion-free sheaf  $f_*\omega_{X/Y}^{\otimes m}$  has a canonical singular hermitian metric with semi-positive curvature.*
- (b) *If  $c_1(\det f_*\omega_{X/Y}^{\otimes m}) = 0$  in  $H^2(Y, \mathbb{R})$ , then  $f_*\omega_{X/Y}^{\otimes m}$  is locally free, and the singular hermitian metric on it is smooth and flat.*
- (c) *Every nonzero morphism  $f_*\omega_{X/Y}^{\otimes m} \rightarrow \mathcal{O}_Y$  is split surjective.*

**Proof of Theorem 3.2.3.** Recall first that Kawamata proved one of the early basic results about morphisms to abelian varieties, conjectured by Ueno.

**THEOREM 3.2.7** ([Ka1, Theorem 1]). *If  $X$  is a smooth projective variety with  $\kappa(X) = 0$ , then its Albanese map  $a: X \rightarrow A = \text{Alb}(X)$  is an algebraic fiber space.*

**REMARK 3.2.8.** This theorem admits various numerical strengthenings. For instance...

Moving on to the proof of the main result, we first note that we can perform a useful reduction.

**LEMMA 3.2.9.** *To prove Theorem 3.2.3, it is enough to assume that  $\kappa(X) = 0$  and that  $Y$  is an abelian variety.*

**PROOF.** We begin by showing that if  $\kappa(X) = -\infty$ , then  $\kappa(F) = -\infty$ . If this were not the case, then we could pick some  $m > 0$  such that  $P_m(F) > 0$  and hence  $f_*\omega_X^{\otimes m} \neq 0$ . Let  $Y \rightarrow A$  be the Albanese morphism of  $Y$ , and  $g: X \rightarrow A$  the induced morphism. Since  $F$  is an irreducible component of the general fiber of  $X \rightarrow g(X) \subseteq A$ , it follows that  $g_*\omega_X^{\otimes m} \neq 0$ . By Theorem 3.1.9,  $g_*\omega_X^{\otimes m}$  is a GV-sheaf, hence Lemma 3.1.4 implies that the set

$$V^0(g_*\omega_X^{\otimes m}) = \{ P \in \text{Pic}^0(A) \mid H^0(A, g_*\omega_X^{\otimes m} \otimes P) \neq 0 \}$$

is non-empty. Now by Theorem 3.1.12 and the comment immediately after,  $V^0(g_*\omega_X^{\otimes m})$  contains a torsion point  $P \in \text{Pic}^0(A)$ , i.e. there is an integer  $k > 0$  such that  $P^{\otimes k} \simeq \mathcal{O}_A$ . But then  $h^0(X, \omega_X^{\otimes m} \otimes g^*P) = h^0(A, g_*\omega_X^{\otimes m} \otimes P) \neq 0$  and so

$$h^0(X, \omega_X^{\otimes km}) = h^0(X, (\omega_X^{\otimes m} \otimes P)^{\otimes k}) \neq 0.$$

This contradicts the assumption  $\kappa(X) = -\infty$ .

Assume now that  $\kappa(X) \geq 0$ . We will first prove the statement in the case that  $\kappa(Y) = 0$ . By Theorem 3.2.7, since  $Y$  is of maximal Albanese dimension, it is in fact birational to its Albanese variety and so we may assume that  $Y$  is an abelian variety. Let  $h: X \rightarrow Z$  the Iitaka fibration of  $X$ . Since we are allowed to work birationally, we can assume that  $Z$  is smooth. We denote by  $G$  its general fiber, so that in particular  $\kappa(G) = 0$ . By the same result of Kawamata, the Albanese map of  $G$  is surjective, so we deduce that  $B = f(G) \subseteq Y$  is an abelian subvariety. If  $G \rightarrow B' \rightarrow B$  is the Stein factorization, then  $B' \rightarrow B$  is an étale map of abelian varieties. We thus have an induced fiber space

$$G \longrightarrow B'$$

over an abelian variety, with  $\kappa(G) = 0$ , and whose general fiber is  $H = F \cap G$ . Assuming that Theorem 3.2.3 holds for algebraic fiber spaces of Kodaira dimension zero over abelian varieties, we obtain  $\kappa(H) = 0$ . Note however that  $H$  is also an irreducible component of the general fiber of

$$h|_F: F \longrightarrow h(F).$$

Considering the Stein factorization of this morphism, the Easy Addition formula implies that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F).$$

(Note that we can assume that  $g(F)$  is smooth, by passing to a birational model.) Since  $\dim h(F) \leq \dim Z = \kappa(X)$ , we obtain the required inequality  $\kappa(F) \leq \kappa(X)$ .

Finally we prove the general case. Since  $Y$  has maximal Albanese dimension, after replacing it by a resolution of singularities of an étale cover of its Stein factorization, and  $X$  by a resolution of the corresponding fiber product, by [Kai, Theorem 13] we may assume that  $Y = Z \times K$  where  $Z$  is of general type and  $K$  is an abelian variety. In particular  $\kappa(Y) = \dim Z = \kappa(Z)$ . If  $E$  is the general fiber of the induced morphism  $X \rightarrow Z$ , then the induced morphism  $E \rightarrow K$  has general fiber isomorphic to  $F$ . By what we have proven above, we deduce that  $\kappa(E) \geq \kappa(F)$ . We then have the required inequality

$$\kappa(X) = \kappa(Z) + \kappa(E) \geq \kappa(Y) + \kappa(F),$$

where the first equality is due to Theorem 2.6.1, since  $Z$  is of general type.  $\square$

We may therefore proceed by assuming that  $f: X \rightarrow A$  is a fiber space onto an abelian variety, and  $\kappa(X) = 0$ . Note that this last condition means that we have  $h^0(X, \omega_X^{\otimes m}) = 1$  for all sufficiently divisible integers  $m > 0$ . The task at hand is to show that  $\kappa(F) = 0$  (since if  $\kappa(F) = -\infty$ , then  $\kappa(X) = -\infty$  as well). We show in fact the following more precise statement:

**THEOREM 3.2.10.** *If  $f: X \rightarrow A$  is an algebraic fiber space over an abelian variety, with  $\kappa(X) = 0$ , then we have*

$$\mathcal{F}_m = f_* \omega_X^{\otimes m} \simeq \mathcal{O}_A$$

for every  $m \in \mathbb{N}$  such that  $P_m(X) \neq 0$ .

**PROOF.** We fix an  $m$  such that  $H^0(A, \mathcal{F}_m) = H^0(X, \omega_X^{\otimes m}) \neq 0$ . In particular  $\mathcal{F}_m$  is a non-trivial GV-sheaf on  $A$ , by Theorem 3.1.9. Since  $\kappa(X) = 0$ , we have  $h^0(A, \mathcal{F}_m) = 1$ , and in particular  $0 \in V^0(\mathcal{F}_m)$ . We claim that

$$V^0(\mathcal{F}_m) = \{0\},$$

which will then imply that  $\mathcal{F}_m$  is unipotent by Proposition 3.1.6.

To see this, note first that by Theorem 3.1.12 and the comment immediately after,  $V^0(\mathcal{F}_m)$  is a union of torsion translates of abelian subvarieties of  $\text{Pic}^0(A)$ . If there were two distinct points  $P, Q \in V^0(\mathcal{F}_m)$ , we could therefore assume that they are both torsion of the same order  $k$ . Since  $f$  is a fiber space, the mapping

$$f^*: \text{Pic}^0(A) \longrightarrow \text{Pic}^0(X)$$

is injective, and so  $f^*P$  and  $f^*Q$  are distinct as well. Now if  $P \in V^0(\mathcal{F}_m)$ , then

$$H^0(X, \omega_X^{\otimes m} \otimes f^*P) \simeq H^0(A, \mathcal{F}_m \otimes P) \neq 0,$$

and similarly for  $Q$ . Let  $D \in |mK_X + f^*P|$  and  $G \in |mK_X + f^*Q|$ , so that  $kD, kG \in |mkK_X|$ . Since  $h^0(X, \omega_X^{\otimes mk}) = 1$ , it follows that  $kD = kG$ , and hence  $f^*P = f^*Q$ , which gives a contradiction.

Now since  $\mathcal{F}_m$  is unipotent, we have

$$\det \mathcal{F}_m \simeq \mathcal{O}_A.$$

Corollary 3.2.6 implies then that  $\mathcal{F}_m$  has a smooth hermitian metric that is flat. Thus  $\mathcal{F}_m$  is a successive extension of trivial bundles  $\mathcal{O}_A$  that can be split off as direct summands with the help of the flat metric. It follows that in fact  $\mathcal{F}_m \simeq \mathcal{O}_A^{\oplus r}$ , the trivial bundle of some rank  $r \geq 1$ . But then, since

$$h^0(A, \mathcal{F}_m) = h^0(X, \omega_X^{\otimes m}) = 1,$$

we obtain that  $r = 1$ , which is the statement of the theorem.  $\square$

REMARK 3.2.11. Note that the hypothesis of the theorem is not quite equivalent to having  $\mathcal{F}_m \neq 0$ . In fact this is the case if and only if there exists  $P \in \text{Pic}^0(A)$  such that  $H^0(X, \omega_X^{\otimes m} \otimes f^*P) \neq 0$ , which by the projection formula is equivalent to  $V^0(\mathcal{F}_m) \neq \emptyset$ . Since  $\mathcal{F}_m$  is a  $GV$ -sheaf, this is in turn equivalent to  $\mathcal{F}_m \neq 0$  by Lemma 3.1.4.

### 3.3. Further results and conjectures

A refinement of the methods developed in the previous section leads to a more precise statement than Iitaka's conjecture, almost realizing the ideal situation in ??? in the case of an abelian variety base.

THEOREM 3.3.1 ([LPS, Theorem B]). *Let  $f: X \rightarrow A$  be a morphism from a smooth projective variety to an abelian variety. Then there exists an isogeny  $\varphi: A' \rightarrow A$  such that  $f'_*\omega_{X'}^{\otimes m} \simeq \varphi^*(f_*\omega_X^{\otimes m})$  is globally generated for every  $m \geq 1$ , where  $X'$  is the fiber product of  $X$  and  $A'$  over  $A$ , as in the diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow f' & & \downarrow f \\ A' & \xrightarrow{\varphi} & A \end{array}$$

I will eventually include a proof.

COROLLARY 3.3.2. *If  $f: X \rightarrow A$  is an algebraic fiber space over an abelian variety, with general fiber  $F$ , then there exists an étale cover  $X' \rightarrow X$  such that*

$$P_m(X') \geq P_m(F) \quad \text{for all } m \geq 1.$$

Note that this immediately implies Theorem 3.2.1, since then

$$\kappa(X) = \kappa(X') \geq \kappa(F).$$

Theorem 3.3.1 has a companion showing that the situation is even more precise for smooth morphisms. Besides the story above, this requires some of the techniques discussed in the next chapter.

**THEOREM 3.3.3** ([MP, Theorem B]). *Let  $f: X \rightarrow A$  be a surjective morphism from a smooth projective variety to an abelian variety. Assume that  $f$  is smooth away from a closed set of codimension at least 2 in  $A$ , and denote its general fiber by  $F$ . Then for every  $m \geq 1$  we have*

$$f_*\omega_X^{\otimes m} \simeq \bigoplus_{i=1}^{P_m(F)} \alpha_i,$$

where  $\alpha_i \in \text{Pic}^0(A)$  are (possibly repeated) torsion line bundles. In particular, if  $f_*\omega_X^{\otimes m}$  is globally generated for some  $m$ , then

$$f_*\omega_X^{\otimes m} \simeq \mathcal{O}_A^{\oplus P_m(F)}.$$

**COROLLARY 3.3.4.** *If  $f: X \rightarrow A$  is an algebraic fiber space over an abelian variety, with general fiber  $F$ , and smooth away from a closed set of codimension at least 2 in  $A$ , then there exists an étale cover  $X' \rightarrow X$  such that*

$$P_m(F) = P_m(X') \geq P_m(X) \quad \text{for all } m \geq 1.$$

Consequently  $\kappa(X) = \kappa(F)$ .

The last assertion follows in combination with Corollary 3.3.2.

There is a fundamental problem about fiber spaces over abelian varieties, going beyond Iitaka's conjecture (but phrased around the same time), that is still open even when the base is an elliptic curve.

**CONJECTURE 3.3.5 (Ueno's Conjecture K).** *Let  $f: X \rightarrow A$  be the Albanese map of a smooth projective variety with  $\kappa(X) = 0$  (hence  $f$  is an algebraic fiber space by ???). Then there exists an isogeny  $A' \rightarrow A$  such that*

$$X \times_A A' \sim F \times A',$$

*i.e.  $X$  becomes birational to a product after an étale base change.*

The same conclusion is predicted in [MP, Conjecture C] for any fiber space  $f: X \rightarrow A$ , with  $\kappa(F) \geq 0$ , which is smooth away from a closed set of codimension at least 2 in  $A$ .

Finally, note that Ueno's conjecture is a slightly stronger statement than saying that if  $\kappa(X) = 0$ , then  $\text{Var}(f) = 0$ ; this in turn is a special case of the  $C_{n,m}^+$  conjecture discussed in a later section. I would like to propose a generalization of this form of the conjecture to an arbitrary fiber space over an abelian variety.

**CONJECTURE 3.3.6.** *Let  $f: X \rightarrow A$  be an algebraic fiber space, with  $X$  a smooth projective variety and  $A$  an abelian variety. Then*

$$\dim V^0(A, f_*\omega_X^{\otimes m}) \geq \text{Var}(f)$$

*for every integer  $m \geq 2$  such that  $f_*\omega_X^{\otimes m} \neq 0$ .*

Note that when  $\kappa(X) = 0$ , Theorem 3.2.10 says that  $f_*\omega_X^{\otimes m} \simeq \mathcal{O}_A$  for every  $m$  such that  $P_m(X) \neq 0$ , in which case  $V^0(A, f_*\omega_X^{\otimes m})$  is just a point.



## CHAPTER 4

### Further conjectures

This chapter discusses the last (and most hectic) part of the course, where we reviewed a number of other conjectures regarding the Kodaira dimension: Viehweg's  $C_{n,m}^+$  and hyperbolicity conjectures, and superadditivity conjectures. Currently, Iitaka's subadditivity conjecture is just the beginning of a long story.

#### 4.1. Log Kodaira dimension

The notion of Kodaira dimension can be extended to the setting of quasi-projective varieties, which plays an important role in what follows.

Using Hironaka's theorem, every complex smooth quasi-projective variety  $V$  admits a smooth projective compactification  $Y$  such that the boundary  $D = Y \setminus V$  is an SNC divisor.

DEFINITION 4.1.1. With the notation above, the *log Kodaira dimension* of  $V$  is

$$\kappa(V) := \kappa(Y, \omega_Y(D)),$$

i.e. the Iitaka dimension of the line bundle  $\log$  canonical line bundle  $\omega_Y(D)$ .

EXERCISE 4.1.2. Show that the definition above is independent of the choice of compactification with SNC boundary, and therefore  $\kappa(V)$  is a well defined invariant of  $V$ .

EXAMPLE 4.1.3. (1) Say  $Y = \mathbf{P}^1$ . Then:

- If  $V = \mathbf{C} = \mathbf{P}^1 \setminus \text{pt}$ , then  $\kappa(V) = -\infty$ .
- If  $V = \mathbf{C}^* = \mathbf{P}^1 \setminus 2\text{pts}$ , then  $\kappa(V) = 0$ .
- If  $V = \mathbf{P}^1 \setminus \Gamma$ , with  $\#\Gamma \geq 3$ , then  $\kappa(V) = 1$ .

(2) If  $V = \mathbf{C}^2 \setminus \mathbf{C}$ , then we can write

$$V = \mathbf{P}^2 \setminus (L_1 \cup L_2),$$

where  $L_1$  and  $L_2$  are lines, and so  $\kappa(V) = -\infty$ .

DEFINITION 4.1.4. We say that  $V$  is of *log general type* if  $\kappa(V) = \dim V$ .

EXAMPLE 4.1.5. If  $V = \mathbf{P}^1 \setminus \Gamma$ , with  $\#\Gamma \geq 3$ , then  $V$  is of log general type. The same holds if  $V$  is an elliptic curve with any non-empty finite set of points removed.

## 4.2. Superadditivity conjectures

Recall that Iitaka’s subadditivity conjecture predicts that for an algebraic fiber space  $f: X \rightarrow Y$  of smooth projective varieties we have

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

Here we start by discussing a complementary “superadditivity” statement in the same setting; this is [Po, Conjecture 2.1].

**CONJECTURE 4.2.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space between smooth projective varieties, and let  $V \subseteq Y$  be the open subset over which  $f$  is smooth. Then*

$$\kappa(F) + \kappa(V) \geq \kappa(X).$$

Recall that the *log Kodaira dimension*  $\kappa(V)$  can be defined as follows: after a birational base change we can assume that the complement  $D = Y \setminus V$  is a simple normal crossing (SNC) divisor. One then defines

$$\kappa(V) := \kappa(Y, \omega_Y(D)),$$

the Iitaka dimension of  $\omega_Y(D)$ , which is easily checked to be independent of the choice of compactification of  $V$  with simple normal crossing boundary.

**REMARK 4.2.2 (Obvious cases).** The conjecture clearly holds when  $\kappa(X) = -\infty$  (so in particular when  $\kappa(F) = -\infty$ ). It also holds when  $\kappa(V) = \dim Y$ , i.e.  $V$  is of log general type, because of the Easy Addition formula. It is worth noting that an important class of fiber spaces for which  $V$  is of log general type is that of “moduli” families; more generally, by [PS4, Theorem A], relying also on important work on Viehweg’s hyperbolicity conjecture in [VZ2], [CaP], this holds for every family with maximal variation such that  $F$  admits a good minimal model.

**REMARK 4.2.3 (Smooth case).** In the smooth case, when combined with Iitaka’s subadditivity, Conjecture 4.2.1 leads to the following additivity formula, also generalized later from a different point of view by Conjecture 4.2.9:

**CONJECTURE 4.2.4.** *If  $f: X \rightarrow Y$  is smooth algebraic fiber space between smooth projective varieties, with general fiber  $F$ , then*

$$\kappa(X) = \kappa(F) + \kappa(Y).$$

**REMARK 4.2.5 (Domain of general type).** It is also amusing to spell out the special case when  $X$  is of general type. When  $f$  is smooth (i.e.  $V = Y$ ), or when  $Y$  is not uniruled, this was proved in [PS5]; the full statement is a consequence of a more general result by Park, see [Pa, Corollary 1.6].

**THEOREM 4.2.6.** *With the notation in Conjecture 4.2.1, if  $X$  is of general type, then  $V$  is of log general type.*

The non-obvious cases of Conjecture 4.2.1 and Conjecture 4.2.4 are summarized in the next two theorems.

**THEOREM 4.2.7.** *Conjecture 4.2.1 holds when:*

- (1)  $Y$  is an abelian variety, or more generally a variety of maximal Albanese dimension.<sup>1</sup>
- (2)  $Y$  is a curve.
- (3)  $f$  is smooth, and either the general fiber of the Iitaka fibration of  $Y$  admits a good minimal model,<sup>2</sup> or  $Y$  is uniruled. In particular, it holds when  $f$  is smooth and  $Y$  is a surface or a threefold.
- (4)  $f$  is smooth, and  $\kappa(Y) \geq \dim Y - 3 \geq 0$ .
- (5)  $F$  is of general type, so in particular when  $X$  is of general type.
- (6)  $F$  has semiample canonical bundle.

PROOF. Part (1) is shown in [MP], using techniques from [LPS].

Part (2) is clear when  $g(Y) \geq 2$ . When  $Y = \mathbf{P}^1$  it follows from [VZ1, Theorem 0.2], while when  $Y$  is elliptic it follows from (1).

Part (3) is established in [PS5]; see Theorem C and Corollaries E, F in *loc. cit.*

Part (4) follows by applying Lemma ?? below to a model  $g: Y' \rightarrow Z$  of the Iitaka fibration of  $Y$ , with  $Y'$  and  $Z$  smooth. Its general fiber  $G$  has dimension at most 3, hence (2) and (3) apply.

Parts (5) and (6) follow from more general results in [PS5], [Pa] for (5), and [Cam] for both, explained in the next section.  $\square$

THEOREM 4.2.8. *Conjecture 4.2.4 holds when*

- (1)  $f$  is a fiber bundle.
- (2)  $Y$  is of general type.
- (3)  $Y$  is an abelian variety, or more generally a variety of maximal Albanese dimension.
- (4)  $Y$  is a curve.
- (5)  $Y$  is a surface.
- (6)  $Y$  is uniruled.
- (7)  $Y$  is a good minimal model with  $\kappa(Y) = 0$ .
- (8)  $X$  is a good minimal model with  $\kappa(X) = 0$ .
- (9)  $F$  is of general type.
- (10)  $F$  has semiample canonical bundle.
- (11) we assume the conjectures of the MMP.

PROOF. Part (1) is one of the original results on the Iitaka conjecture, obtained (in a more general setting) in [NU].

For (2), (3), (4), and (5), since we know either from Remark 4.2.2 or from the previous statement that Conjecture 4.2.1 is settled in these cases, Conjecture 4.2.4 follows

<sup>1</sup>This means that  $Y$  admits a generically finite (not necessarily surjective) morphism to an abelian variety.

<sup>2</sup>More precisely one needs to assume a conjecture of Campana-Peternell, which is in turn a consequence of the existence of good minimal models.

as in Remark 4.2.3, as in all these cases we know that Iitaka's conjecture holds (see [Ka2], [Vi1], [Ka3], [CP], [HPS]).

For part (6), see [PS5, Proposition G]; the main input is the case  $Y = \mathbf{P}^1$ , proved in [VZ1, Theorem 0.2].

Part (7) is [PS3, Theorem H(ii)]. On the other hand, if  $X$  is a good minimal model with  $\kappa(X) = 0$ , then so is  $Y$  by [TZ], hence (8) also follows.

Parts (9) and (10) follow from (5) and (6) in Theorem 4.2.7 and the fact that Iitaka's conjecture holds when  $F$  is of general type by [Ko], and when  $F$  has semiample canonical bundle by [Ka3].

Part (11) is [PS5, Corollary D]; a more careful explanation of what exactly is needed is given in *loc. cit.*  $\square$

**The most general conjecture.** An even stronger proposal about projective morphisms is made in [Po, Conjecture 3.1], namely:

CONJECTURE 4.2.9. *If  $f : U \rightarrow V$  is smooth projective algebraic fiber space between smooth quasi-projective varieties, with general fiber  $F$ , then*

$$\kappa(U) = \kappa(F) + \kappa(V).$$

In other words, in the presence of smooth projective morphisms of *quasi-projective* varieties, subadditivity in the log version of Iitaka's conjecture should become *additivity*. As noted earlier, this is quite open even when  $U$  and  $V$  are projective. If  $f : X \rightarrow Y$  is an algebraic fiber space with  $X$  and  $Y$  projective, and  $V$  is the locus over which  $f$  is smooth, then for  $U = f^{-1}(V)$  we obviously have  $\kappa(U) \geq \kappa(X)$ , hence Conjecture 4.2.9 implies Conjecture 4.2.1.

A first remark is that the conjecture is known to hold when  $V$  is of log general type, i.e.  $\kappa(V) = \dim V$ , without any smoothness hypothesis on  $f$ ; this is really a result about subadditivity, due to Kawamata [Ka1, Theorem 30] when  $\kappa(U) \geq 0$  and Maehara [Ma, Corollary 2] in general, since the other inequality follows from Easy Addition.

Here are some sample recent results on other cases. The first regards base spaces that compactify to abelian varieties:

THEOREM 4.2.10 ([MP, Theorem A]). *Let  $f : X \rightarrow A$  be an algebraic fiber space, with  $X$  a smooth projective variety and  $A$  an abelian variety. Assume that  $f$  is smooth over an open set  $V \subseteq A$ , and denote  $U = f^{-1}(V)$  and the general fiber of  $f$  by  $F$ . Then  $\kappa(U) = \kappa(V) + \kappa(F)$ .*

Another is that the conjecture holds when  $U$  is of log general type; this is a recent theorem of Park. Extending a result shown in [PS5] in the projective case, he proves the following more general fact:

THEOREM 4.2.11 ([Pa, Theorem 1.5]). *In the situation of Conjecture 4.2.9, assume that  $\kappa(F) \geq 0$ . Then*

$$V \text{ is of log general type} \iff \kappa(U) = \kappa(F) + \dim V.$$

*In particular, if  $U$  is of log general type, then  $V$  is of log general type.*

In [Pa, Theorem 1.10], Park also completes the proof of the conjecture when  $V$  is a curve; due to results in [VZ1] and [MP], the cases that need to be established are those when  $V$  is  $\mathbf{P}^1$  minus one or two points.

Moreover, the methods of [PS4] show that the conjecture holds when  $F$  is of general type, assuming that  $\kappa(V) \geq 0$ ; see Remark 4 in *loc. cit.* However, more generally and quite importantly, Campana [Cam] has shown that this last assumption can be removed, and furthermore:

**THEOREM 4.2.12** ([Cam, Theorem 1]). *Conjecture 4.2.9 holds when  $F$  has semiample canonical bundle.*

Let's finish by summarizing this discussion:

**THEOREM 4.2.13.** *Conjecture 4.2.9 holds when:*

- (1)  $V$  is of log general type [Ka1], [Ma].
- (2)  $V$  compactifies to an abelian variety [MP].
- (3)  $V$  is a curve [VZ1], [MP], [Pa].
- (4)  $U$  is of log general type [Pa].
- (5)  $F$  is of general type [PS5], [Cam].
- (6)  $F$  has semiample canonical bundle [Cam].

### 4.3. Some proofs in the case of smooth morphisms

In class we proved two of the results described in the previous section, in order to give a flavor of the main techniques. Here are the statements, both answering special cases of Conjecture 4.2.4 regarding additivity for smooth morphisms of projective varieties. The first is a weaker version of an older result of Viehweg-Zuo [VZ1, Theorem 0.2], while the second is a result from [PS5].

**THEOREM 4.3.1.** *If  $f: X \rightarrow \mathbf{P}^1$  is a smooth projective morphism, then  $\kappa(X) = -\infty$ .*

**REMARK 4.3.2.** This result can be improved in various ways. What Viehweg and Zuo actually showed is that if  $\kappa(X) \geq 0$ , then  $f$  has at least 3 singular fibers. This fits in a bigger picture that will be discussed later. Moreover, using symplectic methods, Pieloch showed that under the hypothesis of the theorem,  $f$  has a section, and  $X$  is uniruled.

**THEOREM 4.3.3.** *If  $f: X \rightarrow Y$  is a smooth fiber space, with  $X$  and  $Y$  smooth and projective, and we assume that  $\kappa(F) \geq 0$  for the general fiber  $F$  of  $f$ , then*

$$Y \text{ is of general type} \iff \kappa(X) = \kappa(F) + \dim Y.$$

*In particular, if  $X$  is of general type, then  $Y$  is of general type.*

Both of these results are consequences of a similar Hodge-theoretic construction that has its origins in work of Viehweg and Zuo, and was refined in the setting of Hodge

modules in joint work with Schnell, combined with a criterion due to Campana and Păun for a variety to be of general type, based on the positivity of its cotangent bundle.

**The technical construction.** Let us fix a surjective morphism  $f: X \rightarrow Y$  between two smooth projective varieties, and denote by  $D$  the proper closed subset of  $Y$  over which  $f$  is not smooth. We also fix a line bundle  $L$  on  $X$ , and consider on  $Y$  the line bundle

$$B = \omega_{Y/X} \otimes f^* L^{-1}.$$

We assume that the following condition holds:

$$(4.3.1) \quad H^0(Y, B^{\otimes m}) \neq 0 \quad \text{for some } m \geq 1.$$

We also denote by  $\mathcal{A}_Y$  the sheaf of graded algebras  $\text{Sym } \mathcal{T}_Y$ , the symmetric algebra of the tangent bundle  $\mathcal{T}_Y$ .

**THEOREM 4.3.4** ([PS4, Theorem 2.2]). *Assuming (4.3.1), one can find a graded  $\mathcal{A}_Y$ -module  $\mathcal{G}_\bullet$  that is coherent over  $\mathcal{A}_Y$  and has the following properties:*

- (i) *One has  $\mathcal{G}_0 \simeq L \otimes f_* \mathcal{O}_X$ .*
- (ii) *Each  $\mathcal{G}_k$  is torsion-free on the open subset  $X \setminus D$ .*
- (iii) *There exists a regular holonomic  $\mathcal{D}$ -module  $\mathcal{M}$  with good filtration  $F_\bullet \mathcal{M}$ , and an inclusion of graded  $\mathcal{A}_Y$ -modules  $\mathcal{G}_\bullet \subseteq \text{gr}_\bullet^F \mathcal{M}$ .*
- (iv) *The filtered  $\mathcal{D}$ -module  $(\mathcal{M}, F_\bullet \mathcal{M})$  underlies a polarizable Hodge module  $M$  on  $Y$  with strict support  $Y$ , and  $F_k \mathcal{M} = 0$  for  $k < 0$ .*

The proof of this theorem is outside of the scope of these notes, so we will take it for granted in what follows.

When  $f$  is a smooth morphism (i.e.  $D = \emptyset$ ), we use the construction in the theorem above via the following special case of [PS4, Theorem 3.5].

**THEOREM 4.3.5.** *Let  $Y$  be a smooth projective variety, and let  $M$  be a pure Hodge module with strict support  $Y$  and underlying filtered  $\mathcal{D}_Y$ -module  $(\mathcal{M}, F_\bullet \mathcal{M})$ , which is generically a variation of Hodge structure of weight  $k$ . Assume that there exists a graded  $\mathcal{A}_Y$ -submodule  $\mathcal{G}_\bullet \subseteq \text{gr}_\bullet^F \mathcal{M}$  satisfying the following two properties:*

- *There is an inclusion of a line bundle  $L \hookrightarrow \mathcal{G}_0$ , and  $\mathcal{G}_k = 0$  for  $k < 0$ .*
- *Each  $\mathcal{G}_k$  is torsion free.*

*Then at least one of the following holds:*

- (i)  *$L^{-1}$  is pseudo-effective.*
- (ii) *There exist an integer  $1 \leq s \leq k$ , a weakly positive reflexive sheaf  $\mathcal{W}$  on  $Y$ , and an inclusion*

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes s}.$$

**PROOF.** The graded  $\mathcal{A}_Y$ -module structure induces a chain of homomorphisms of coherent  $\mathcal{O}_Y$ -modules

$$\mathcal{G}_0 \xrightarrow{\vartheta_0} \mathcal{G}_1 \otimes \Omega_Y^1 \xrightarrow{\vartheta_1 \circ \text{id}} \mathcal{G}_2 \otimes (\Omega_Y^1)^{\otimes 2} \longrightarrow \dots$$

We denote

$$K_k := \ker (\vartheta_k : \mathcal{G}_k \longrightarrow \mathcal{G}_{k+1} \otimes \Omega_Y^1).$$

and also

$$K_k(M) := \ker (\mathrm{gr}_k^F \mathcal{M} \longrightarrow \mathrm{gr}_{k+1}^F \mathcal{M} \otimes \Omega_Y^1).$$

The latter is the kernel of the “generalized Kodaira-Spencer map” induced by the mappings  $F_k \mathcal{M} \rightarrow F_{k+1} \mathcal{M} \otimes \Omega_Y^1$  given by the filtered  $\mathcal{D}_Y$ -module structure.

The key point here is the fact that for any pure polarizable Hodge module  $M$  with strict support  $Y$ , and for any integer  $k$ , the dual  $K_k(M)^\vee$  of such a kernel is a weakly positive sheaf (automatically reflexive); in this context, this is shown in [PW], but the result goes back to work of Zuo [Zuo], cf. also Brunenbarbe [Br], in the context of Deligne canonical extensions (this in turn goes back to Griffiths’ work on the Hodge metric on Hodge bundles associated to polarizable variations of Hodge structures).

Since there are obvious inclusions

$$K_k \hookrightarrow K_k(M),$$

Lemma 2.4.17 implies that  $K_k^\vee$  are also weakly positive for all  $k$ . To make use of this property, note that given the inclusion of  $L$  into  $\mathcal{G}_0$ , there are two possibilities:

The first is that the induced homomorphism

$$L \longrightarrow \mathcal{G}_1 \otimes \Omega_Y^1$$

is the zero map, hence  $L \subseteq K_0$ . This induces a non-trivial homomorphism

$$K_0^\vee \longrightarrow L^{-1}.$$

Since  $K_0^\vee$  is weakly positive, using again Lemma 2.4.17 we get that  $L^{-1}$  is pseudo-effective, i.e. the condition in (i).

As  $\mathcal{G}_1$  is torsion free, the only other possibility is that we have an inclusion

$$L \hookrightarrow \mathcal{G}_1 \otimes \Omega_Y^1.$$

We can then repeat the same argument via the morphisms  $\vartheta_s \circ \mathrm{id}$  with  $s \geq 1$ . The next thing to note however is that  $\mathcal{G}_p = 0$  for  $p > k$ . Indeed, note that an inclusion  $\mathcal{G}_p \subseteq \mathrm{gr}_p^F \mathcal{M}$  with  $\mathcal{G}_p \neq 0$  can only hold as long as  $\mathrm{gr}_p^F \mathcal{M}$  is not a torsion sheaf. Recall however that  $(\mathcal{M}, F_\bullet \mathcal{M})$  underlies an extension of a variation of Hodge structure  $\mathbf{V}$  of weight  $k$  on an open set  $U \subset X$ . Thus over  $U$  the sheaves  $\mathrm{gr}_p^F \mathcal{M}$  coincide with Hodge bundles of  $\mathbf{V}$ , and therefore are non-zero only for  $p \leq k$ .

Therefore there is an  $s \leq k$  where the inclusions will have to stop, i.e. such that

$$L \subseteq \mathcal{G}_s \otimes (\Omega_Y^1)^{\otimes s} \quad \text{and} \quad L \not\subseteq \mathcal{G}_{s+1} \otimes (\Omega_Y^1)^{\otimes s+1}.$$

This implies that

$$L \subseteq K_s \otimes (\Omega_Y^1)^{\otimes s},$$

hence there exists a nontrivial homomorphism

$$K_s^\vee \otimes L \longrightarrow (\Omega_Y^1)^{\otimes s}.$$

Denoting by  $\mathcal{W} \otimes L$  the saturation of the image of this homomorphism, we have a short exact sequence

$$0 \longrightarrow \mathcal{W} \otimes L \longrightarrow (\Omega_Y^1)^{\otimes s} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Using the weak positivity of  $K_s^\vee$  and Lemma 2.4.17 we deduce that  $\mathcal{W}$  is weakly positive. Moreover, since  $\mathcal{Q}$  is torsion free, while  $(\Omega_Y^1)^{\otimes s}$  is locally free, it is easy to check that  $\mathcal{W} \otimes L$  is reflexive. Hence we obtain the condition in (ii).  $\square$

Combining Theorem 4.3.5 with an important result of Campana and Păun leads to the following consequence:

**COROLLARY 4.3.6.** *Under the hypothesis of Theorem 4.3.5, if  $L$  is a big line bundle, then  $Y$  is of general type.*

**PROOF.** If  $L$  is big, then  $L^{-1}$  cannot be pseudo-effective, as otherwise  $\mathcal{O}_Y = L \otimes L^{-1}$  would be big, a contradiction. Thus (ii) in the conclusion of the theorem has to hold, in other words we have an inclusion

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes s},$$

where  $s$  is a positive integer and  $\mathcal{W}$  is a weakly positive reflexive sheaf. Since  $\mathcal{W}$  is locally free away from a closed subset of codimension at least 2, it follows that we also have an inclusion

$$N := \det \mathcal{W} \otimes L^{\otimes r} \hookrightarrow (\mathcal{W} \otimes L)^{\otimes r},$$

where  $r$  is the generic rank of  $\mathcal{W}$ . By Lemma 2.4.17,  $\det \mathcal{W}$  is a weakly positive, i.e. pseudo-effective, line bundle, and since  $L$  is big, we finally deduce that  $N$  is a big line bundle, and there exists  $M > 0$  such that there is an inclusion

$$N \hookrightarrow (\Omega_Y^1)^{\otimes M}.$$

This is precisely the hypothesis of [CaP, Theorem 7.11] (in the special case of empty boundary), and its conclusion is that  $Y$  is of general type.  $\square$

**Proof of the two results.** In both cases we will use a particular line bundle on  $Y$ , and an integer  $m \geq 1$ , for which (4.3.1) holds.

In the setting of Theorem 4.3.1, where  $Y = \mathbf{P}^1$ , we take  $L = \mathcal{O}_{\mathbf{P}^1}(2) \simeq \omega_{\mathbf{P}^1}^{-1}$ . Let's assume that the conclusion is false, so that  $\kappa(X) \geq 0$ . This means that there exists an integer  $m \geq 1$  such that  $H^0(X, \omega_X^{\otimes m}) \neq 0$ , which is immediately seen to be equivalent to

$$H^0(Y, B^{\otimes m}) \neq 0 \quad \text{where} \quad B = \omega_{X/Y} \otimes f^* L^{-1}.$$

The discussion above applies, and note that in this case  $L$  is ample; but then Corollary 4.3.6 leads to the absurd conclusion that  $\mathbf{P}^1$  is of general type.

**REMARK 4.3.7.** The proof of Theorem 4.3.1 works identically if we replace  $\mathbf{P}^1$  by any smooth projective variety  $Y$  such that  $\omega_Y^{-1}$  is big, for instance a Fano variety.

Let's now switch to the setting of Theorem 4.3.3. If  $Y$  is of general type, then we actually have  $\kappa(X) = \kappa(F) + \dim Y$  for any algebraic fiber space over  $Y$ , by Theorem 2.6.1(ii). Thus the content is the opposite implication.

Let now  $A$  be an ample line bundle on  $Y$ . According to [Mo, Proposition 1.14], the condition that  $\kappa(X) = \kappa(F) + \dim Y$  is equivalent to having

$$H^0(X, \omega_X^{\otimes m} \otimes f^* A^{-1}) \neq 0$$

for some  $m \geq 1$ . Using the projection formula, this is equivalent to the existence of a nontrivial (hence injective) morphism

$$(4.3.2) \quad L := A \otimes (\omega_Y^{-1})^{\otimes m} \hookrightarrow \mathcal{F}_m = f_* \omega_{X/Y}^{\otimes m}.$$

Recall that by the invariance of plurigenera, the sheaf  $\mathcal{F}_m$  is locally free of rank  $P_m(F)$ .

We now use Viehweg's fiber product trick, in the form provided in Proposition 2.4.2 (with  $s = m$ ): let  $X^{(m)}$  denote the  $m$ -fold fiber product  $X \times_Y \cdots \times_Y X$ , and the induced smooth algebraic fiber space  $f^{(m)} : X^{(m)} \rightarrow Y$ . Then

$$f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m} \simeq \mathcal{F}_m^{\otimes m},$$

which combined with (4.3.2) leads to an inclusion

$$L^{\otimes m} \hookrightarrow f_*^{(m)} \omega_{X^{(m)}/Y}^{\otimes m}.$$

By replacing  $X$  by  $X^{(m)}$ , and the fiber space  $f$  by  $f^{(m)}$ , we are therefore allowed to assume from the beginning that we are working with a smooth fiber space  $f$  such that

$$H^0(X, B^{\otimes m}) \neq 0 \quad \text{where} \quad B = \omega_{X/Y} \otimes f^* L^{-1}.$$

Hence we are again in a position to apply Theorem 4.3.4 and Theorem 4.3.5, and we deduce that at least one of the following holds:

- (i)  $L^{-1}$  is pseudo-effective.
- (ii) There exist an integer  $1 \leq s \leq k$ , a weakly positive reflexive sheaf  $\mathcal{W}$  on  $Y$ , and an inclusion

$$\mathcal{W} \otimes L \hookrightarrow (\Omega_Y^1)^{\otimes s}.$$

In the first case, we immediately get that  $\omega_Y$  is big, since

$$\omega_Y^{\otimes m} \simeq L^{-1} \otimes A.$$

In the second case, we argue as follows. We rewrite the inclusion above as

$$(4.3.3) \quad \mathcal{W} \otimes A \hookrightarrow (\Omega_Y^1)^{\otimes N} \otimes \omega_Y^{\otimes m}.$$

Now  $\det \Omega_Y^1 = \omega_Y$ , hence there exists also a (split) inclusion

$$\omega_Y^{\otimes m} \hookrightarrow (\Omega_Y^1)^{\otimes m \cdot \dim Y}.$$

Putting everything together, we deduce just as in the proof of Corollary 4.3.6 the existence of an inclusion

$$N \hookrightarrow (\Omega_Y^1)^{\otimes M}$$

for some  $M > 0$ , where  $N$  is a big line bundle. Therefore  $Y$  is of general type by the Campana-Păun result.

#### 4.4. Variation of a morphism and the $C_{n,m}^+$ conjecture

This section introduces an important concept due to Viehweg. Its intuitive interpretation is a measure of the “birational moduli” of a family.

Let  $f : X \rightarrow Y$  be an algebraic fiber space, with  $X$  and  $Y$  smooth projective varieties over  $\mathbf{C}$ , and let  $\eta$  be the generic point of  $Y$ . We use the notation  $X_\eta$  for the generic fiber of  $f$ , and  $X_{\bar{\eta}}$  for the generic geometric fiber, i.e.

$$X_{\bar{\eta}} \simeq X_\eta \times_{k(\eta)} \overline{k(\eta)}.$$

**DEFINITION 4.4.1 (Variation).** The *variation* of  $f$ , denoted  $\text{Var}(f)$ , is the smallest integer  $\ell$  such that there exists an algebraically closed subfield

$$K \subseteq \overline{K(Y)} = \overline{k(\eta)} \quad \text{with} \quad \text{trdeg}_{\mathbf{C}} K = \ell,$$

and a smooth projective variety  $T$  defined over  $K$ , such that

$$T \times_K \overline{K(Y)} \sim X_{\bar{\eta}}.$$

(Here  $\sim$  means birational.) Note that

$$0 \leq \text{Var}(f) \leq \dim Y,$$

and if  $\text{Var}(f) = \dim Y$  we say that  $f$  has *maximal variation*; this last condition means that there exists a nonempty open set  $V \subseteq Y$  such that the fiber of  $f$  over any  $y \in U$  is birational to at most countably many other fibers over points in  $U$ .

**REMARK 4.4.2.** The definition in fact makes sense if  $f$  is a projective morphism, but  $X$  and  $Y$  are arbitrary (i.e. not necessarily projective) varieties.

The study of arbitrary families is sometimes reduced to that of families of maximal variation by means of the following useful result, left as an exercise for the moment.

**PROPOSITION 4.4.3.** *Let  $f : X \rightarrow Y$  be a fiber space of smooth projective varieties. Then there exists another fiber space  $f'' : X'' \rightarrow Y''$  of smooth projective varieties, with*

$$\text{Var}(f) = \text{Var}(f'') = \dim Y'' \quad \text{and} \quad X_{\bar{\eta}} \simeq X''_{\bar{\eta}''} \times_{\overline{k(\eta'')}} \overline{k(\eta)},$$

*and another smooth projective variety  $Y'$  with a generically finite map  $\tau : Y' \rightarrow Y$ , and a map  $\rho : Y' \rightarrow Y''$  such that  $X \times_Y Y'$  and  $X'' \times_{Y''} Y'$  are birationally isomorphic over  $Y'$  (meaning that the birational isomorphism respects the projections onto  $Y'$ ).*

Denoting by  $X'$  a common resolution of  $X \times_Y Y'$  and  $X'' \times_{Y''} Y'$  in Proposition 4.4.3, we obtain a commutative diagram

$$(4.4.1) \quad \begin{array}{ccccc} X & \xleftarrow{\tau'} & X' & \xrightarrow{\rho'} & X'' \\ \downarrow f & & \downarrow f' & & \downarrow f'' \\ Y & \xleftarrow{\tau} & Y' & \xrightarrow{\rho} & Y'' \end{array}$$

EXAMPLE 4.4.4. An isotrivial family has variation equal to 0. More generally, one has  $\text{Var}(f) = 0$  if and only if  $f$  is *birationally isotrivial*, i.e. there exists a generically finite cover  $\tau : Y' \rightarrow Y$  such that the fiber product  $X \times_Y Y'$  is birational to  $Y' \times F$ , where  $F$  is the general fiber of  $f$ . Indeed, in this case  $Y''$  in the proposition above is just a point.

EXAMPLE 4.4.5. Say  $f : X \rightarrow Y$  is a family of stable curves with general member a smooth curve of genus  $g \geq 2$ , induced by pullback from the moduli space  $\overline{\mathcal{M}}_g$  via a morphism  $\varphi : Y \rightarrow \overline{\mathcal{M}}_g$ . Then

$$\text{Var}(f) = \dim \varphi(Y).$$

In particular,  $f$  has maximal variation if and only if  $\varphi$  is a generically finite onto its image. Given our discussion of positivity coming from the moduli space of curves at the end of Section 2.2, in this case we have that

$$\det f_* \omega_{X/Y}^{\otimes m}$$

is a big and nef line bundle for  $m \gg 0$ . In general one focuses on this property even when there is no obvious moduli space involved.

The example above, and similar consideration involving other parameter spaces (like period domains), suggests that when the family has non-trivial variation there is extra positivity in the sheaves  $f_* \omega_{X/Y}^{\otimes m}$ , which may lead to even better bounds for  $\kappa(X)$  than what is predicted by Iitaka's conjecture. This was formalized by Viehweg [?]:

CONJECTURE 4.4.6 (**Viehweg's  $C_{n,m}^+$  conjecture**). *Let  $f : X \rightarrow Y$  be an algebraic fiber space between smooth projective varieties, with  $\kappa(Y) \geq 0$ , and denote by  $F$  the generic fiber of  $f$ . Then*

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Y), \text{Var}(f)\}.$$

EXAMPLE 4.4.7 (**Counterexample with  $\kappa(Y) = -\infty$** ). Let  $f : S \rightarrow \mathbf{P}^1$  be an elliptic K3 surface such that the elliptic fibration is non-isotrivial. Then  $\kappa(S) = 0$ ,  $\kappa(F) = 0$  and  $\text{Var}(f) = 1$ , hence the inequality in Conjecture 4.4.6 does not hold. In view of other considerations in this chapter, it is also worth noting that if  $V \subset \mathbf{P}^1$  is the open set over which  $f$  is smooth, then  $V$  is the complement of at least three points, hence  $\kappa(V) \geq 0$ . Thus it is not possible to replace the assumption  $\kappa(Y) \geq 0$  by  $\kappa(V) \geq 0$  in the conjecture.

EXAMPLE 4.4.8. Let  $f : S \rightarrow E$  be a surjective morphism from a smooth projective surface to an elliptic curve, with general fiber  $F$  satisfying  $g(F) \geq 2$ . There are two main possibilities, according to the two possible values 0 and 1 for  $\text{Var}(f)$ :

(i)  $f$  is isotrivial, meaning a product  $E \times F$  at least after passing to a finite cover of  $E$ . In this case it is not hard to show that  $\kappa(S) = 1 = \kappa(F) + \kappa(E)$ , so in particular  $S$  also has an elliptic fibration. (Note that in the case of families of smooth projective curves isotrivial and birationally isotrivial is essentially the same thing, since such curves do not have other smooth birational models.)

(ii)  $f$  is not isotrivial. In this case one can check that  $f$  cannot also have an elliptic fibration, and since in any case  $\kappa(S) \geq 1$ , it means that  $S$  must be of general type. (As

mentioned in Example 1.5.19, surfaces of general type with  $q(S) = 1$  do exist.) Note that in this case

$$\kappa(S) = \kappa(F) + \text{Var}(f) > \kappa(F) + \kappa(E).$$

Conjecture 4.4.6 is clear when  $X$  is of general type, as  $\text{Var}(f)$  cannot go beyond the dimension of  $Y$ . Also, it is equivalent to the usual  $C_{n,m}$  conjecture when  $Y$  is of general type, and we have seen that this is known to be true.

There are two main results regarding the conjecture that are treated in the literature, summarized in the following theorem. The second is in fact now known to imply the first.

**THEOREM 4.4.9.** *Conjecture 4.4.6 holds when:*

- (1)  $F$  is of general type (due to Kollár [Ko]).<sup>3</sup>
- (2)  $F$  has a good minimal model (due to Kawamata [Ka3]).

The standard line of attack is to prove an even stronger statement; Viehweg showed that the  $C_{n,m}^+$  conjecture is a consequence of another conjecture regarding direct images of relative pluricanonical bundles for families with maximal variation.

**CONJECTURE 4.4.10 (Viehweg’s  $Q_{n,m}$  conjecture).** *Let  $f : X \rightarrow Y$  be an algebraic fiber space between smooth projective varieties, such that  $\kappa(F) \geq 0$  and  $\text{Var}(f) = \dim Y$ . Then  $\det f_* \omega_{X/Y}^{\otimes m}$  is a big line bundle on  $Y$  for some  $m > 0$ .*

**THEOREM 4.4.11 (Viehweg [Vi1, Theorem II]).** *The  $Q_{n,m}$  conjecture implies the  $C_{n,m}^+$  conjecture.*

**REMARK 4.4.12.** It is interesting to note that, unlike all other conjectures discussed in these notes, even the case of Conjecture 4.4.6 when  $Y$  is a curve is not yet settled. More precisely, the conjecture has not been established when  $Y$  is an elliptic curve. This is the case even when  $\kappa(X) = 0$ , when Theorem ?? says that  $\kappa(F) = 0$  as well; here the conjecture predicts that one should have  $\text{Var}(f) = 0$  as well. In fact in this case there is a celebrated, and a priori stronger, earlier prediction made by Ueno.

**CONJECTURE 4.4.13 (Ueno’s Conjecture K).** *Let  $a : X \rightarrow A$  be the Albanese map of a smooth projective variety with  $\kappa(X) = 0$ . Then there exists an isogeny of abelian varieties  $B \rightarrow A$  such that*

$$X \times_A B \sim B \times F,$$

where  $F$  is the general fiber of  $f$ . In other words,  $X$  becomes birational to a product (over the base) after an étale base change.

## 4.5. Hyperbolicity conjectures

There is a different type of conjectures involving the variation, which loosely go under the name of *hyperbolicity*. This goes back to (part of) a conjecture of Shafarevich about one-parameter families of curves, which was extended to arbitrary dimension by

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<sup>3</sup>You may consult my other notes for (most of) the proof of this theorem.

Viehweg. His original conjecture was about families of canonically polarized varieties, but it is now understood that it should hold in higher generality, as follows:

**CONJECTURE 4.5.1.** *Let  $f: X \rightarrow Y$  be an algebraic fiber space with  $X$  and  $Y$  smooth and projective, and with general fiber  $F$  such that  $\kappa(F) \geq 0$ . Denote by  $V$  the open subset of  $Y$  over which  $f$  is smooth. If  $\text{Var}(f) = \dim Y$ , then  $\kappa(V) = \dim Y$ , i.e.  $V$  is of log general type.*

When  $F$  is canonically polarized, i.e.  $\omega_F$  is ample, the conjecture was settled by Campana and Păun [**CaP**], relying also on major input from work of Viehweg and Zuo, after previous special cases were obtained by Kebekus-Kovács and Patakfalvi. It is now known that, in the general form stated here, it follows from the conjectures of the MMP. More precisely, we have:

**THEOREM 4.5.2** ([**PS3**, Theorem A]). *Conjecture 4.5.1 holds when  $F$  is assumed to have a good minimal model.*

In particular, the conjecture holds when  $F$  is of general type.

Moreover, Kebekus and Kovács have noted that Viehweg's conjecture can be appropriately extended to fiber spaces with arbitrary variation. Again, the statement was originally phrased for families of canonically polarized varieties, but makes sense in the general setting:

**CONJECTURE 4.5.3** ([**KK**, Conjecture 1.6]). *Let  $f: X \rightarrow Y$  be an algebraic fiber space with  $X$  and  $Y$  smooth and projective, and with general fiber  $F$  such that  $\kappa(F) \geq 0$ . Denote by  $V$  the open subset of  $Y$  over which  $f$  is smooth. Then:*

- (1) *If  $\kappa(V) \geq 0$ , then  $\kappa(V) \geq \text{Var}(f)$ .*
- (2) *If  $\kappa(V) = -\infty$ , then  $\dim Y > \text{Var}(f)$ .*

Part (2) is of course a special case of Conjecture 4.5.1. The conjecture has been verified when the general fiber of  $f$  is of general type by Taji [**Ta**] and Wei-Wu [**WW**]; the paper [**Ta**] also addresses the case when  $\omega_F$  is semiample.

**REMARK 4.5.4 (Superadditivity and (generalized)  $C_{n,m}^+$  imply Kebekus-Kovács).** At least when  $\kappa(Y) \geq 0$ , note that putting together Conjecture 4.2.1 and Conjecture 4.4.6 we obtain  $\kappa(V) \geq \text{Var}(f)$ , i.e. part (1) in Conjecture 4.5.3.

Note however that it makes sense to conjecture the following strengthening of  $C_{n,m}^+$  (I am not aware of this having been stated in the literature):

**CONJECTURE 4.5.5 (Generalized  $C_{n,m}^+$  conjecture).** *Let  $f: U \rightarrow V$  be a projective algebraic fiber space, with  $U$  and  $V$  smooth quasi-projective varieties and  $\kappa(V) \geq 0$ . If  $F$  is the generic fiber of  $f$ , then*

$$\kappa(U) \geq \kappa(F) + \max\{\kappa(V), \text{Var}(f)\}.$$
<sup>4</sup>

<sup>4</sup>A further generalization is possible for non-projective morphisms, whenever the notion of variation is defined in the context of log pairs.

Using this instead of Conjecture 4.4.6, in combination with the additivity Conjecture 4.2.9, yields the full Kebekus-Kovács conjecture.

REMARK 4.5.6 (**Further consequences of superadditivity plus  $C_{n,m}^+$** ). If  $\kappa(V) = \text{Var}(f)$ , the same combination of conjecture yields

$$\kappa(X) = \kappa(F) + \kappa(V) = \kappa(F) + \text{Var}(f).$$

Moreover, any fiber space with  $\kappa(V) = 0$  should satisfy  $\text{Var}(f) = 0$ , i.e. birational isotriviality, as well as  $\kappa(X) = \kappa(F)$ . As discussed in Remark 4.4.12, this is not known even when  $V$  is an elliptic curve.

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