# Hodge theory and singularities

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## Introduction

These are notes for the Harvard course Math 265, held in Fall 2024.

CHAPTER 1

# Background

In this chapter we will review a number of fundamental facts regarding the topology and Hodge theory of smooth complex projective varieties, as well as some standard classes of singularities in birational geometry.

## 1.1. Review of the Hodge theory of smooth projective varieties

All the varieties we consider in this section are over the complex numbers. Many of the results in this section work in the more general setting of compact Kähler manifolds.

**Hodge decomposition and Hodge filtration.** The most fundamental result in this area is the Hodge decomposition of the singular cohomology with complex coefficients. Most of [**GH**, Ch.0], for instance, is devoted to the proof of this theorem.

THEOREM 1.1.1. *If $X$ is a compact Kähler manifold, then there exists a decomposition*

$$H^i(X, \mathbf{C}) \simeq \bigoplus_{p+q=i} H^{p,q}(X)$$

*with:*

- $H^{p,q}(X) = \overline{H^{q,p}(X)}$
- $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

As $H^{p,q}(X)$ usually denotes the cohomology group obtained from $(p,q)$-forms via the $\overline{\partial}$ operator, the second bullet is in fact the independent *Dolbeaut Theorem*. In particular, the $H^{q,0}(X)$ spaces can be seen as the global sections of the various bundles of holomorphic forms on $X$, or after conjugation as the cohomology groups of the structure sheaf. As such, they are birational invariants.

We use the standard notation

$$b_i(X) = \dim_{\mathbf{C}} H^i(X, \mathbf{C}),$$

the $i$-th *Betti number* of $X$, and

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p),$$

the $(p,q)$-*Hodge number* of $X$. We obviously have $h^{p,q}(X) = h^{q,p}(X)$, and the Hodge decomposition theorem implies the identity

$$b_i(X) = \sum_{p+q=i} h^{p,q}(X).$$

As a first example of application, this gives an immediate obstruction to a complex manifold being Kähler:

COROLLARY 1.1.2. *If $X$ is compact Kähler and $i$ is an odd integer, then $b_i(X)$ is even.*

The *Serre Duality* theorem applied to the vector bundle $E = \Omega_X^p$ gives the isomorphism (note that $\Omega_X^{n-p} \simeq \omega_X \otimes (\Omega_X^p)^\vee$):

$$H^q(X, \Omega_X^p) \simeq H^{n-q}(X, \Omega_X^{n-p})^\vee,$$

for every $p$ and $q$, or equivalently $H^{p,q}(X) \simeq H^{n-p,n-q}(X)^\vee$. In particular, we have

$$h^{p,q}(X) = h^{n-p,n-q}(X).$$

Putting everything together, we obtain the identities

$$h^{p,q}(X) = h^{q,p}(X) = h^{n-p,n-q}(X) = h^{n-q,n-p}(X) \tag{1.1.1}$$

for all $p$ and $q$.

DEFINITION 1.1.3 (**Hodge diamond**). We usually collect the Hodge numbers of a compact Kähler manifold $X$ in a *Hodge diamond*, as represented below.

$$\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & \iddots & & \vdots & & \ddots & \\
h^{n,0} & h^{n-1,1} & & \cdots & & h^{1,n-1} & h^{0,n} \\
 & \ddots & & \vdots & & \iddots & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}$$

This diamond has a few symmetries, thanks to (1.1.1): it is invariant under reflection across the middle column, it is also invariant reflection across the middle row, and it has diagonal symmetry (in other words it is left invariant under rotation by 180°).

When $X$ is *projective*, it contains closed subvarieties of any dimension; for such a subvariety $Z$ of codimension $p$, it is a standard fact that the Poincaré dual of $[Z]$ gives a class

$$0 \neq \eta_Z \in H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X) \subseteq H^{2p}(X, \mathbf{C}).$$

Therefore $H^{p,p}(X) \neq 0$ for all $p$, hence we have

$$b_{2p}(X) \geq h^{p,p} > 0.$$

No other Hodge spaces need be non-trivial; for instance the Hodge diamond of $\mathbf{P}^n$ has 1's in the middle column, and 0's everywhere else (it is "minimal" among all algebraic varieties). (Exercise.)

Moreover, when $X$ is projective, $H_{\mathbb{Z}} := H^k(X, \mathbb{Z})$ is (the main[1]) example of a *polarized pure Hodge structure of weight* $k$, meaning that:

- $H_{\mathbb{Z}}$ is a finitely generated free abelian group, together with a decomposition

$$H_{\mathbf{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbf{C} = \bigoplus_{p+q=k} H^{p,q}$$

with $H^{p,q} = \overline{H^{q,p}}$. (This is the definition of a pure Hodge structure of weight $k$.) This is equivalent to the data of the Hodge filtration

$$F^\ell H_{\mathbf{C}} = \bigoplus_{p+q=k,\, p\geq \ell} H^{p,q},$$

together with the properties $F^\ell \cap \overline{F^{k-\ell+1}} = \{0\}$ and $F^\ell \oplus \overline{F^{k-\ell+1}} = H_{\mathbf{C}}$ for all $\ell$.

- There is a bilinear form (the *polarization*)

$$Q \colon H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to \mathbb{Z}$$

satisfying the following properties (the "Hodge-Riemann bilinear relations"):

(1) $Q$ is symmetric when $i$ is even, and skew-symmetric when $i$ is odd.
(2) The decomposition is orthogonal with respect to $i^k \cdot S(\alpha, \beta)$ (extended to $H_{\mathbf{C}}$), where $S(\alpha, \beta) := Q(\alpha, \bar{\beta})$.
(3) We have $i^{p-q-k} \cdot (-1)^{k(k-1)/2} \cdot S(\alpha, \alpha) > 0$ for all $0 \neq \alpha \in H^{p,q}$.

The polarization is in fact naturally obtained on *primitive* cohomology, and then assembled on $H^i(X, \mathbb{Z})$ by means of the Lefschetz decomposition, which splits it into the direct sum of (images of) various primitive spaces.

Going back to the Hodge decomposition, a somewhat weaker but still useful statement is the degeneration of the Hodge-to-de Rham spectral sequence. Consider the de Rham complex of $X$:

$$\Omega_X^\bullet \colon \qquad 0 \to \mathcal{O}_X \to \Omega_X^1 \to \cdots \to \Omega_X^n \to 0$$

placed in degrees $0, \cdots, n$, with the usual differential $d$ on forms. It has a "stupid filtration" by truncation, a decreasing filtration given by the complexes

$$F^p\Omega_X^\bullet = \Omega_X^{\geq p} = [0 \to \Omega_X^p \to \cdots \to \Omega_X^n \to 0]$$

so that

$$\mathrm{gr}_F^p \, \Omega_X^\bullet = F^p\Omega_X^\bullet / F^{p+1}\Omega_X^\bullet \simeq \Omega_X^p[-p].$$

The usual spectral sequence associated to a filtered complex is, in this case, called the *Hodge-to-de Rham* spectral sequence, as looks as follows:

$$E_1^{pq} = \mathbb{H}^{p+q}(X, \mathrm{gr}_F^p \, \Omega_X^\bullet) \implies H^{p+q}(X, \Omega_X^\bullet).$$

Using the description above and the de Rham theorem, this becomes

$$E_1^{pq} = H^q(X, \Omega_X^p) \implies H^{p+q}(X, \mathbf{C}).$$

[1] Strictly speaking, the main example is *primitive* cohomology.

The main result is that this spectral sequence degenerates at $E_1$; this follows from the Hodge decomposition, but can also be proved algebraically.[2]

**The Lefschetz theorems.** The key result about the topology of projective manifolds is the following, sometimes also called the Weak Lefschetz theorem:

Theorem 1.1.4 (**Lefschetz hyperplane theorem**). *Let $X$ be a smooth complex projective variety of dimension $n$, and let $D$ be an effective ample divisor on $X$. Then the restriction map*

$$r_i : H^i(X,\mathbb{Z}) \longrightarrow H^i(D,\mathbb{Z})$$

*is an isomorphism for $i \le n-2$, and injective for $i = n-1$.*

A conceptual approach is via the following theorem essentially saying that complex affine manifolds have only half as much topology as one might expect:[3]

Theorem 1.1.5 (**Andreotti-Frankel**). *Let $Y \subset \mathbf{C}^r$ be a closed $n$-dimensional complex submanifold. Then $Y$ has the homotopy type of a $CW$ complex of real dimension $\le n$. As a consequence*

$$H^i(Y,\mathbb{Z}) = 0 \text{ and } H_i(Y,\mathbb{Z}) = 0 \text{ for } i > n.$$

Let's sketch how this leads to the proof of the theorem on hyperplane sections. Since $D$ is ample, for some $m \gg 0$ we have that $mD$ is very ample, and therefore there exists an embedding $X \subset \mathbf{P}^N$ and a hyperplane $H$ in $\mathbf{P}^N$ such that $mD = X \cap H$. This implies that $Y = X \smallsetminus D = X \smallsetminus mD$ is a smooth affine complex variety of dimension $n$. The Andreotti-Frankel theorem implies then that $H_j(Y,\mathbb{Z}) = 0$ for $j > n$. On the other hand, for all $j$ one has by Alexander-Lefschetz duality[4]

$$H_j(Y,\mathbb{Z}) \simeq H^{2n-j}(X,D;\mathbb{Z})$$

and therefore $H^i(X,D;\mathbb{Z}) = 0$ for $i < n$. This is equivalent to the desired conclusion, by the long exact sequence of (relative) cohomology

$$\cdots \longrightarrow H^i(X,D;\mathbb{Z}) \longrightarrow H^i(X,\mathbb{Z}) \longrightarrow H^i(D,\mathbb{Z}) \longrightarrow H^{i+1}(X,D;\mathbb{Z}) \longrightarrow \cdots$$

The proof of Theorem 1.1.5 is a very nice application of basic Morse theory; see e.g. [**La**, §3.1.A].

Given the Hodge decomposition, there is a holomorphic version of the Lefschetz Hyperplane Theorem 1.1.4. Note that when $D$ is smooth, the restriction maps in that theorem are morphisms of pure Hodge structures.

Corollary 1.1.6. *Let $X$ be a smooth complex projective variety of dimension $n$, and let $D$ be a smooth effective ample divisor on $X$. Then the restriction maps*

$$r_{p,q} : H^q(X,\Omega_X^p) \longrightarrow H^q(D,\Omega_D^p)$$

[2] It is not known however how to go from here to a splitting as in the Hodge decomposition, without the choice of a Kähler metric and the notion of harmonic forms.

[3] Note that every $\mathcal{C}^\infty$ manifold of real dimension $2n$ has the homotopy type of a $CW$ complex of real dimension $\le 2n$.

[4] See e.g. [**Hat**] §3.3; note that Poincaré duality is the special case $D = 0$.

*are isomorphisms for $p+q \leq n-2$, and injective for $p+q = n-1$.*

One Hodge space that has a special interpretation is $H^{1,1}(X)$. Every integral class here is known to be analytic, in other words it comes from a divisor on $X$. For a proof of the theorem below see e.g. [**GH**] p.163.

THEOREM 1.1.7 (**Lefschetz theorem on** $(1,1)$**-classes**). *Let $X$ be a smooth projective complex variety of dimension $n$. Then every class in $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ (i.e. integral $(1,1)$-class) is the first Chern class of a line bundle on $X$, or equivalently the Poincaré dual of $[D] \in H_{2n-2}(X, \mathbb{Z})$ for some divisor $D$ on $X$.*

**Further results on Hodge numbers.** We record a few more basic facts regarding the Hodge numbers of smooth projective varieties.

THEOREM 1.1.8. *Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism of complex varieties (or a proper submersion of compact Kähler manifolds). Then the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$.*

This is an immediate consequence of two fundamental facts: one is Ehresmann's theorem, stating that $\pi$ is a $C^\infty$ locally trivial fibration (so in particular the fibers have the same topology), and the other is the upper semicontinuity of Hodge numbers in smooth families. See [**Vo**, §9.1].

Another point is that the Hodge numbers $h^{p,0}$ are well behaved under birational equivalence:

PROPOSITION 1.1.9. *If $X$ and $Y$ are birational smooth projective varieties over an algebraically closed field, then*

$$h^{p,0}(X) = h^{p,0}(Y) \text{ for all } p.$$

PROOF. Exercise. (Hint: use Hartogs' theorem, or an algebraic analogue.) □

REMARK 1.1.10. Another way to think about the equality above is in the conjugate form $h^{0,p}(X) = h^{0,p}(Y)$, i.e.

$$h^p(X, \mathcal{O}_X) = h^p(Y, \mathcal{O}_Y).$$

This can also be shown using the Leray spectral sequence and the following fundamental (and nontrivial) fact about birational morphisms:

EXERCISE 1.1.11. Let $f : X \to Y$ be a birational morphism between smooth varieties. Then

$$f_*\mathcal{O}_X \simeq \mathcal{O}_Y \quad \text{and} \quad R^i f_*\mathcal{O}_X = 0 \text{ for } i > 0.$$

(Hint: try first the case of a blow-up in a smooth center.)

EXAMPLE 1.1.12. It is not true in general that all Hodge numbers are birational invariants. The simplest example is that of blow-ups. Let $X$ be a smooth projective variety (or more generally complex manifold) of dimension $n$, and let $\tilde{X} = \mathrm{Bl}_x(X)$ be the blow-up of $X$ at an arbitrary point $x$. Denote by $E \simeq \mathbf{P}^{n-1}$ the exceptional divisor. This divisor introduces its own new cycles (including $E$ itself) to $H_i(\tilde{X}, \mathbb{Z})$. More precisely, for each $i > 0$ one has

$$H_i(\tilde{X}, \mathbb{Z}) \simeq H_i(X, \mathbb{Z}) \oplus H_i(E, \mathbb{Z}).$$

This reflects in the Hodge decomposition. Since all analytic cycles are in $H^{p,p}$ for various $p$, we obtain

$$H^{p,p}(\tilde{X}) \simeq H^{p,p}(X) \oplus \mathbf{C}$$

for all $0 < p < n$, so the non-extremal Hodge numbers on the middle column go up by 1.

Exercise 1.1.13. Verify carefully all the statements made in the Example above.

## 1.2. Rational singularities

Definition 1.2.1. A *simple normal crossing (SNC)* divisor on a smooth variety $X$ of dimension $n$ is an effective divisor $E = \sum E_i$ such that each component $E_i$ is a smooth codimension 1 subvariety of $X$, and in local coordinates in the neighborhood of each point in $x \in E$ we can write $E = (x_1 \cdot \ldots \cdot x_m = 0)$ with $m \leq n$. (If only the second condition is satisfied, we say that $E$ is a *normal crossing (NC)* divisor.

Definition 1.2.2. A *resolution (of singularities)* of a variety $X$ is a proper birational morphism $f\colon \widetilde{X} \to X$ with $\widetilde{X}$ smooth. A *log resolution* of $X$ is a resolution such that the exceptional locus $\mathrm{Exc}(f)$ (i.e. the union of the hypersurfaces contracted by $f$) is a divisor on $\widetilde{X}$ with SNC support.

More generally, if $D$ be an effective $\mathbb{Q}$-divisor on X, a log resolution of the pair $(X, D)$ is a proper birational morphism $f\colon \widetilde{X} \to X$, where $\widetilde{X}$ is a smooth variety and the divisor $f^{-1}D + \mathrm{Exc}(f)$ has simple normal crossing (SNC) support.

By Hironaka's famous theorem, log resolutions exist in characteristic 0. Moreover, his result shows that they can be chosen with the following properties:

- $f$ is an isomorphism away from $\mathrm{Sing}(X)$ (or $\mathrm{Sing}(X) \cup \mathrm{Supp}(D)$); such an $f$ is called a *strong log resolution*.
- $f$ is a composition of blow-ups with smooth centers.

Example 1.2.3. Pictures at the board.

**Assumption:** From now on we restrict to complex (or more generally characteristic 0) varieties.

Definition 1.2.4. A variety $X$ has *rational singularities* if for every resolution of singularities $f\colon \widetilde{X} \to X$ we have

$$f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X \quad \text{and} \quad R^i f_*\mathcal{O}_{\widetilde{X}} = 0 \ \text{ for } \ i > 0.$$

Equivalently, in $\mathbf{D}^b_{\mathrm{coh}}(X)$ (the bounded derived category of coherent sheaves on $X$) we have $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$.

Note that the first condition is equivalent to $X$ being normal, by the Zariski Main Theorem.

The next exercise contains two well-known and very important (nontrivial) facts about birational morphisms between smooth varieties.

EXERCISE 1.2.5. (i) Show that if $f\colon Y \to X$ is a proper birational map of smooth varieties, then

$$R^i f_* \mathcal{O}_Y = 0 \quad \text{for} \quad i > 0.$$

(Hint: try first the case of blow-ups at smooth centers, then use Hironaka.) Thus smooth varieties do indeed have rational singularities.

(ii) (**Grauert-Riemenschneider Vanishing.**) Show that if $f\colon Y \to X$ is a generically finite surjective (e.g. birational) morphism, with $Y$ smooth, then

$$R^i f_* \omega_Y = 0 \quad \text{for} \quad i > 0.$$

The following Proposition provides another useful characterization of rational singularities. Recall first that every $X$ has a dualizing complex $\omega_X^\bullet$, which is an object in $\mathbf{D}^b_{\text{coh}}(X)$, whose cohomologies live in degrees (at most) $-n, \ldots, 0$. If we have an embedding $X \subseteq Y$ in a smooth variety, then

$$\omega_X^\bullet \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y)[-\dim Y].$$

Thus standard commutative algebra shows that $\omega_X^\bullet$ is supported only in degree $-n$ if and only if $X$ is Cohen-Macaulay, in which case $\omega_X^\bullet \simeq \omega_X[n]$. Here

$$\omega_X \simeq \mathcal{H}^{-n}\omega_X^\bullet \simeq \mathcal{E}xt^r(\mathcal{O}_X, \omega_Y),$$

where $r = \operatorname{codim}_Y(X)$, is the *dualizing sheaf* of $X$ (see [**Ha**, Ch.III, §7]).

PROPOSITION 1.2.6. *Let $f\colon \widetilde{X} \to X$ be a resolution. Then the following are equivalent:*

*(i)* $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$.

*(ii) $X$ is Cohen-Macaulay, and* $f_*\omega_{\widetilde{X}} \simeq \omega_X$.

PROOF. You can find a proof of this statement that does not use "heavy" duality theory in [**KM**, §5.1]. Here we give a quick proof using the dualizing complex and Grothendieck duality.

The Grothendieck duality theorem (see ???) gives us the isomorphism

$$\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om(\omega_{\widetilde{X}}, \omega_{\widetilde{X}}) \simeq \mathbf{R}\mathcal{H}om(\mathbf{R}f_*\omega_{\widetilde{X}}, \omega_X^\bullet[-n]).$$

Here $n = \dim X$, and note that $\omega_{\widetilde{X}}^\bullet = \omega_{\widetilde{X}}[n]$ since $\widetilde{X}$ is smooth.

Thus $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$ is equivalent (applying duality one more time) to

$$\mathbf{R}f_*\omega_{\widetilde{X}} \simeq \omega_X^\bullet[-n].$$

Note however that $\mathbf{R}f_*\omega_{\widetilde{X}} = f_*\omega_{\widetilde{X}}$ by the Grauert-Riemenschneider vanishing theorem; see Exercise 1.2.5(ii), so in fact we have

$$f_*\omega_{\widetilde{X}} \simeq \omega_X^\bullet[-n].$$

The left hand side is a sheaf in degree 0, while the right hand side is supported in non-negative degrees, starting with the dualizing sheaf $\omega_X$. It follows that this is equivalent to $\omega_X^\bullet \simeq \omega_X[n]$, i.e. $X$ is Cohen-Macaulay, together with the isomorphism $f_*\omega_{\widetilde{X}} \simeq \omega_X$. □

Corollary 1.2.7. *The following are equivalent:*

*(i) $X$ has rational singularities.*

*(ii) There exists a resolution $f\colon \widetilde{X} \to X$ such that $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$.*

Proof. Condition (i) means that (ii) holds for every resolution. To show that (ii) implies (i), by Proposition 1.2.6 $X$ is Cohen-Macaulay, and it suffices to show that if the condition $f_*\omega_{\widetilde{X}} \simeq \omega_X$ holds for one resolution, then it holds for every resolution.

Take any other resolution $g\colon Y \to X$. A standard consequence of Hironaka's theorem is that there is a third resolution $h\colon Z \to X$ dominating both $Y$ and $\widetilde{X}$, i.e. factoring through $f$ and $g$. But for a birational morphism $\varphi\colon Z \to Y$ between smooth varieties we have $\mathbf{R}\varphi_*\omega_Z \simeq \omega_Y$ (for instance use Exercise 1.2.5(i) and the proof of Proposition 1.2.6), so together with Grauert-Riemenschneider vanishing we obtain

$$g_*\omega_Y \simeq h_*\omega_X \simeq f_*\omega_{\widetilde{X}} \simeq \omega_X.$$

□

Remark 1.2.8. Rational singularities can be defined analogously in the analytic category. By the GAGA principle however, a complex variety $X$ has rational singularities if and only if $X^{\mathrm{an}}$ has rational singularities. Indeed, if $f\colon Y \to X$ is a proper morphism, and $\mathscr{F}$ is a coherent sheaf on $Y$, then $(R^if_*\mathscr{F})^{\mathrm{an}} \simeq R^if_*^{\mathrm{an}}\mathscr{F}^{\mathrm{an}}$.

**Note.** There are some immediate benefits when one is in the presence of rational singularities:

• For every resolution $f\colon \widetilde{X} \to X$, we have

$$H^i(X, \mathcal{O}_X) \simeq H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \quad \text{for all} \quad i.$$

• Kodaira Vanishing continues to hold if $X$ is projective, i.e. for every ample line bundle on $X$, we have:

$$H^i(X, \omega_X \otimes L) = 0 \quad \text{for all} \quad i > 0.$$

Indeed, if $f\colon \widetilde{X} \to X$ is a resolution, then

$$H^i(X, \omega_X \otimes L) \simeq H^i(X, \mathbf{R}f_*\omega_{\widetilde{X}} \otimes L) \simeq H^i(\widetilde{X}, \omega_{\widetilde{X}} \otimes f^*L).$$

where the second isomorphism follows from the projection formula. Now since $f$ is birational and $L$ is ample, we have that $f^*L$ is big and nef, and therefore Kawamata-Viehweg vanishing (see [**La**, Theorem 4.3.1]) on $\widetilde{X}$ gives the statement.

Here are some first examples of rational singularities.

Example 1.2.9. A curve with rational singularities is smooth, since it is normal.

EXAMPLE 1.2.10. Rational singularities appear in any dimension at least two; the simplest examples are cones over hypersurfaces. Let $Y \subset \mathbf{P}^n$ be a smooth hypersurface of degree $d$, with $n \geq 2$, and let

$$X = C(Y) \subset \mathbf{A}^{n+1}$$

be the affine cone over $Y$.

A log resolution of the pair $(\mathbf{A}^{n+1}, X)$ is obtained by blowing up the origin $0 \in \mathbf{A}^{n+1}$. Denote this by $f\colon \widetilde{\mathbf{A}^{n+1}} \to \mathbf{A}^{n+1}$. We have the standard formulas

$$K_{\widetilde{\mathbf{A}^{n+1}}} = f^* K_{\mathbf{A}^{n+1}} + nE$$

and

$$f^* X = \widetilde{X} + dE,$$

where $E \simeq \mathbf{P}^n$ is the exceptional divisor, and $\widetilde{X}$ is the proper transform of $X$. Recall that $F = \tilde{X} \cap E \simeq X \subset \mathbf{P}^n$, as the projectivized tangent cone to the singularity at the vertex.

Therefore we have

$$K_{\widetilde{\mathbf{A}^{n+1}}} + \widetilde{X} = f^*(K_{\mathbf{A}^{n+1}} + X) + (n-d)E.$$

From this, using adjunction for both $X$ and $\widetilde{X}$, we deduce that

$$\omega_{\widetilde{X}} \simeq f^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}((n-d)F)).$$

Now we have $f_*\mathcal{O}_{\widetilde{X}}((n-d)F)) \subseteq \mathcal{O}_X$, with equality if and only if $n \geq d$. (Exercise!). Therefore

$$f_*\omega_{\widetilde{X}} \simeq \omega_X \iff n \geq d.$$

Since $X$ is in any case Cohen-Macaulay, since it is a hypersurface, the conclusion is:

$$X \text{ has rational singularities} \iff d \leq n.$$

Hence the simplest rational singularity is the cone $x_1^2 + x_2^2 + x_3^2 = 0$ over a conic in $\mathbf{P}^2$. The cone over a cubic (elliptic curve) or higher is not a rational singularity; you see here a hint of why the word "rational" is used in the definition.

EXAMPLE 1.2.11. The previous example can be generalized higher codimension subvarieties in $\mathbf{P}^n$, and to abstract cones. Let $Y \subset \mathbf{P}^n$ be a subvariety in projective space, and consider first the classical cone $X = C(Y)$ as discussed above. Recall that $X$ is normal if and only if $Y \subset \mathbf{P}^n$ is projectively normal; this means that the coordinate ring $S(Y) = k[X_0, \ldots, X_n]/I(Y)$ is normal, or equivalently that $Y$ is normal and the restriction maps

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d)) \to H^0(Y, \mathcal{O}_Y(d))$$

are surjective for all $d \geq 0$ (i.e. all linear systems of hypersurfaces in $\mathbf{P}^n$ cut out *complete* linear systems on $Y$).

Given an ample line bundle $L$ on $Y$, we can also consider the abstract cone

$$Z_L = C(Y, L) := \operatorname{Spec}\Big(\bigoplus_{m \geq 0} H^0(Y, L^{\otimes m})\Big).$$

There is by definition an affine morphism $\pi\colon Z \to Y$.

EXERCISE 1.2.12. If $L = \mathcal{O}_Y(1)$ and $Y$ is normal, then $Z_L$ is the normalization of the classical cone $C(Y)$.

EXERCISE 1.2.13. If $Y$ is smooth, then $Z_L$ has rational singularities if and only if

$$H^i(Y, L^{\otimes m}) = 0 \quad \text{for all} \quad i > 0, m \geq 0.$$

Note that this happens automatically when $Y$ is Fano, i.e. $\omega_Y^{-1}$ is ample, by Kodaira Vanishing, as we can write

$$L^{\otimes m} = \omega_Y \otimes A$$

with $A = L^{\otimes m} \otimes \omega_Y^{-1}$, which is ample for all $m \geq 0$.

A valuable tool for getting more information and more examples is a characterization of rational singularities due to Kovács:

THEOREM 1.2.14 ([**Kov2**, Theorem 1]). *Let $f \colon Y \to X$ be a morphism of varieties, such that $Y$ has rational singularities and the natural morphism $\varphi \colon \mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_Y$ has a splitting in $\mathbf{D}^b_{\mathrm{coh}}(X)$ (i.e. there is a morphism $\psi \colon \mathbf{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$ such that $\psi \circ \varphi$ is a quasi-isomorphism of $\mathcal{O}_X$ to itself). Then $X$ has rational singularities.*

PROOF. We consider compatible resolutions for $X$ and $Y$, i.e. sitting in a commutative diagram:

(1.2.1)
$$\begin{CD} \widetilde{Y} @>{\tilde{f}}>> \widetilde{X} \\ @V{\sigma}VV @VV{\pi}V \\ Y @>{f}>> X \end{CD}$$

We get a commutative diagram of morphisms in the derived category

(1.2.2)
$$\begin{CD} \mathcal{O}_X @>{\varphi}>> \mathbf{R}f_*\mathcal{O}_Y \\ @V{\alpha}VV @VV{\beta}V \\ \mathbf{R}\pi_*\mathcal{O}_{\widetilde{X}} @>{\gamma}>> \mathbf{R}(\pi \circ \tilde{f})_*\mathcal{O}_{\widetilde{Y}} \end{CD}$$

Since $Y$ has rational singularities, $\beta$ is an isomorphism, hence $\gamma \circ \alpha$ has a splitting as well, and therefore so does $\alpha$. Thus we may assume from the beginning that $Y$ is smooth and $f$ is a resolution of $X$.

Now applying the functor $\mathbf{R}\mathcal{H}om(\,\cdot\,, \omega_X^\bullet)$ to the quasi-isomorphism

$$\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_Y \to \mathcal{O}_X$$

leads to a quasi-isomorphism

$$\omega_X^\bullet \to f_*\omega_Y[n] \to \omega_X^\bullet,$$

with $n = \dim X$, where for the middle term we used Grothendieck duality as in the proof of Proposition 1.2.6, and the Grauert-Riemenschneider vanishing theorem. This implies immediately that $\mathcal{H}^i\omega_X^\bullet = 0$ for $i \neq -n$, i.e. $X$ is Cohen-Macaulay. Furthermore, it gives a split isomorphism

$$\omega_X \to f_*\omega_Y \to \omega_X.$$

The second map is the canonical one, which is always injective since $f_*\omega_Y$ is torsion-free. Since the splitting implies that it is also surjective, we obtain $f_*\omega_Y \simeq \omega_X$, and therefore $X$ has rational singularities by Proposition 1.2.6. □

Corollary 1.2.15. *Let $f\colon Y \to X$ be a finite surjective morphism, such that $Y$ has rational singularities and $X$ is normal. Then $X$ has rational singularities.*

Proof. Finite morphisms of normal varieties in characteristic 0 have a trace map

$$\mathrm{tr}\colon f_*\mathcal{O}_Y \to \mathcal{O}_X$$

which splits the natural morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$; see e.g. [**KM**, 5.6 and 5.7]. Therefore one can apply Theorem 1.2.14. □

Exercise 1.2.16. Show the following corollary of Theorem 1.2.14 from [**Kov2**]: if $f\colon Y \to X$ is a surjective morphism with connected fibers, and $Y$ has rational singularities, then $X$ has rational singularities if and only if $X$ is Cohen-Macaulay and $R^{\dim Y - \dim X} f_*\omega_Y \simeq \omega_X$.

Example 1.2.17. Recall that a variety $X$ has *quotient singularities* if it can be written (locally analytically) as $X = Y/G$, where $Y$ is smooth and $G$ is a finite group acting on $Y$. Corollary 1.2.15 and Remark 1.2.8 (and the fact that the arguments above work in the analytic category) imply that every variety with quotient singularities has rational singularities.

Note for later discussion that quotient singularities don't have to be Gorenstein. For instance, this is the case for the quotient of $\mathbf{C}^3$ by the group $\mathbb{Z}/2\mathbb{Z}$ generated by the involution $x \to -x$. (Exercise!)

More generally, quotients of smooth varieties by reductive group actions have rational singularities. This is a theorem of Boutot [**Bou**]. It is again a quick consequence of Kovács' characterization as well, since the required splitting is given by the Reynolds operator.

Example 1.2.18. Toric (or more generally toroidal) varieties have rational singularities; see [?]. Note that a better behaved class of toric varieties, the so called *simplicial* ones, have quotient singularities.

Example 1.2.19. Given suitable integers $m, n, k$, the generic determinantal variety $X^k_{m,n}$ is defined as the zero locus of the collection of $k \times k$ minors of an $m \times n$ matrix of variables. This is an affine variety in $\mathbf{A}^{m \times n}$, known to have rational singularities by a theorem of Kempf [**Ke**].

When $m = n = k$, they are hypersurfaces given by the vanishing of the generic determinant. For instance, when $m = n = k = 2$, we obtain a quadric cone in $\mathbf{A}^4$; when $m = n = k = 3$ we obtain a cubic in $\mathbf{A}^9$, etc.

**Singularities of the MMP.** Let $X$ be a normal variety such that $K_X$ is $\mathbb{Q}$-Cartier, i.e. there exists an integer $m > 0$ such that $mK_X$ is Cartier. If $f\colon \widetilde{X} \to X$ is a resolution of singularities, we can write

$$K_{\widetilde{X}} \sim_{\mathbb{Q}} f^*K_X + \sum_i a_i E_i,$$

where $E_i$ are the exceptional divisors of $f$, and $a_i \in \mathbb{Q}$. According to standard definitions in birational geometry, we say that $X$ has

- *terminal* singularities if $a_i > 0$ for all $i$.
- *canonical* singularities if $a_i \geq 0$ for all $i$.
- *log terminal* (or *klt*) singularities if $a_i > -1$ for all $i$.
- *log canonical* (or *klt*) singularities if $a_i \geq -1$ for all $i$.

For details about these classes of singularities (including indepence of the choice of reslolution) and their uses, an excellent source is [**KM**]. Terminal singularities are the singularities of minimal models, while canonical singularities are those of canonical models.

Remark 1.2.20. If $X$ is Gorenstein, then the notions of canonical and log terminal singularities coincide. Moreover, under the same assumption, rational singularities are canonical. Indeed, otherwise some $a_i$ would be a negative integer, and therefore by the projection formula we would have

$$f_*\omega_{\widetilde{X}} \simeq \omega_X \otimes \mathcal{J},$$

with $\mathcal{J} \subsetneq \mathcal{O}_X$ a nontrivial ideal sheaf, contradicting one of the charaterizations of rational singularities.

One of the basic results of the theory is the following theorem, due to Elkik:

Theorem 1.2.21 ([**El**]). *If $X$ has klt singularities, then $X$ has rational singularities.*

A more general result, due to Fujita and Kawamata-Matsuda-Matsuki (see [**KMM**] or [**KM**, Theorem 5.22]), states that if $(X, D)$ is a klt pair, then $X$ has rational singularities. I will skip this for the moment, as we have not discussed singularities of pairs. The approach in [**Kov2**] leads to a simple proof of Theorem 1.2.21, which I will sketch next.

Proof. We first reduce to the case when $K_X$ is Cartier,[5] so in particular the singularities of $X$ are canonical. To this end, note that locally on $X$ we can also pass to an index 1 cover, meaning a finite cover $Y \to X$ (which is étale in codimension 1) such that $K_Y$ is Cartier; see for instance [**Re**, Corollary 1.9]. We then use the following:

Exercise 1.2.22. Show that if $X$ is klt, then its index 1 cover is klt as well (hence canonical). More generally, this happens for any surjective morphism of normal varieties that is étale in codimension one.

We may therefore assume that $X$ has canonical singularities, hence by Grauert-Riemenschneider we have $\mathbf{R}f_*\omega_{\widetilde{X}} \simeq \omega_X$, and we also assume that this is a line bundle. Moreover, the fact that all $a_i \geq 0$ implies that there exists a nontrivial morphism $\mathbf{L}f^*\omega_X \simeq f^*\omega_X \to \omega_{\widetilde{X}}$. Applying $\mathbf{R}\mathcal{H}om(\,\cdot\,, \omega_{\widetilde{X}})$ to this morphism, and then $\mathbf{R}f_*$, we obtain

$$\mathbf{R}f_*\mathbf{R}\mathcal{H}om(\omega_{\widetilde{X}}, \omega_{\widetilde{X}}) \to \mathbf{R}f_*\mathbf{R}\mathcal{H}om(\mathbf{L}f^*\omega_X, \omega_{\widetilde{X}}) \simeq \mathbf{R}\mathcal{H}om(\omega_X, \mathbf{R}f_*\omega_{\widetilde{X}}),$$

[5] In a previous version of the notes I incorrectly wrote that we are reducing to the case when $X$ is Gorenstein, but this is not the case since we don't yet know that $X$ is Cohen-Macaulay. Thanks to S. Kovács for pointing this out.

where the last isomorphism is simply adjunction between $f^*$ and $f_*$ in the derived category. By the previous comments, this amounts to a morphism

$$\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X,$$

and it is easily checked that the composition with the natural map $\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_{\widetilde{X}}$ is the identity. This implies that $X$ has rational singularities by Theorem 1.2.14. ☐

## 1.3. Du Bois singularities; preliminary definition

Let's go back to Example 1.2.10. Consider the case of the cone $X = C(Y) \subset \mathbf{A}^3$ over a smooth plane curve $Y \subset \mathbf{P}^2$ of degree $d$. We have seen that $X$ has rational singularities if and only if $d \leq 2$.

Let's consider now the next case, when $d = 3$, i.e. the case of a plane elliptic curve. We know that $f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$, since $X$ is normal, but $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \neq \mathcal{O}_X$. In fact it is not hard to see from our earlier calculation that

$$R^1 f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_{\{0\}},$$

where $0 \in \mathbf{A}^3$ is the origin.

What if instead of considering just $\widetilde{X}$, we actually looked at the entire preimage of $X$, more precisely at

$$G := \widetilde{X} \cup E = f^{-1}(X)_{\mathrm{red}}?$$

What is $\mathbf{R}f_*\mathcal{O}_G$? To answer this question, start with the standard short exact sequence

$$0 \to \mathcal{O}_G \to \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_E \to \mathcal{O}_F \to 0, \tag{1.3.1}$$

where $F = \widetilde{X} \cap E \simeq Y$. We have

$$\mathbf{R}f_*\mathcal{O}_E \simeq \mathcal{O}_{\{0\}},$$

since $E \simeq \mathbf{P}^2$, while

$$f_*\mathcal{O}_F \simeq \mathcal{O}_{\{0\}} \quad \text{and} \quad R^1 f_*\mathcal{O}_F \simeq \mathcal{O}_{\{0\}},$$

since $F$ is an elliptic curve (hence $H^1(F, \mathcal{O}_F) \simeq \mathbf{C}$). Pushing forward (1.3.1) and studying the differentials carefully (exercise!), we obtain

$$\mathbf{R}f_*\mathcal{O}_G \simeq f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X.$$

Thus looking at the total preimage this time, we obtain a property similar to that in the definition of rational singularities.

Definition 1.3.1 (**Provisional definition of Du Bois singularities**). Let $X$ be a complex variety, embedded in a smooth variety $Y$. We say that $X$ has *Du Bois singularities* if there exists a strong log resolution of the pair $(Y, X)$, with $G = f^{-1}(X)_{\mathrm{red}}$, such that the natural morphism $\mathcal{O}_X \to \mathbf{R}f_*\mathcal{O}_G$ is an isomorphism.

We will later improve this by showing that if this property holds for one such resolution, then it holds for every such resolution, by showing that the object $\mathbf{R}f_*\mathcal{O}_G$ is always isomorphic to a Hodge-theoretic object, the 0-th Du Bois complex $\underline{\Omega}^0_X$ of $X$ (which features in the original definition of Du Bois singularities).

Using this definition, the discussion about cones over cubics, and Example 1.2.10, can be extended to the following:

EXERCISE 1.3.2. Let $X \subset \mathbf{A}^{n+1}$ be the cone over a smooth hypersurface of degree $d$ in $\mathbf{P}^n$. Show that $X$ has Du Bois singularities if and only if $d \leq n+1$.

CHAPTER 2

# The filtered de Rham complex and the Du Bois complexes

## 2.1. Hyperresolutions and cubical resolutions

Throughout this section we allow our varieties to be reducible.

DEFINITION 2.1.1. Let $X$ be a complex variety of dimension $n$. A *hyperresolution* (or *simplicial resolution*) $\varepsilon_\bullet \colon X_\bullet \to X$ of $X$ is a diagram of varieties and morphisms

$$\cdots \longrightarrow X_2 \overset{\varepsilon_{2,0}}{\underset{\varepsilon_{2,2}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\varepsilon_{2,1}}}}}} X_1 \overset{\varepsilon_{1,0}}{\underset{\varepsilon_{1,1}}{\rightrightarrows}} X_0 \xrightarrow{\varepsilon_{0,0}} X.$$

where:

(1) $X_i$ is a smooth (not necessarily connected) variety for each $i \geq 0$.

(2) $\varepsilon_{i,j} \colon X_i \to X_{i-1}$, with $i \geq 0$ and $j = 0, \ldots, i$, are proper morphisms such that

$$\varepsilon_{i,j} \circ \varepsilon_{i+1,j} = \varepsilon_{i,j} \circ \varepsilon_{i+1,j+1}.$$

(As a consequence, it is straightforward to check that there exists a unique morphism

$$\varepsilon_i \colon X_i \to X$$

defined by the diagram, as a composition of any of the $\varepsilon_{i,j}$.)

(3) $\varepsilon_\bullet \colon X_\bullet \to X$ satisfies *cohomological descent*, in the sense that there is an isomorphism in the derived category of sheaves of abelian. groups on $X$:

$$\mathbb{Z}_X \xrightarrow{\simeq} \left[\mathbf{R}\varepsilon_{0*}\mathbb{Z}_{X_0} \xrightarrow{\varphi_1} \mathbf{R}\varepsilon_{1*}\mathbb{Z}_{X_1} \xrightarrow{\varphi_2} \mathbf{R}\varepsilon_{2*}\mathbb{Z}_{X_2} \xrightarrow{\varphi_3} \cdots\right]$$

where the right hand side is the (iterated) cone of the morphisms $\varphi_i$ is induced by the morphism

$$\varepsilon_{i,0} - \varepsilon_{i,1} + \varepsilon_{i,2} - \cdots$$

for each $i$.

The last condition should be thought of as saying that the resolution "approximates" $X$ topologically, via an inclusion-exclusion principle using the cohomologies of the $X_i$.

EXAMPLE 2.1.2. Consider the case of a two-term hyperresolution:

$$X_1 \overset{\varepsilon_{1,0}}{\underset{\varepsilon_{1,1}}{\rightrightarrows}} X_0 \xrightarrow{\varepsilon_{0,0}} X.$$

This induces

$$\begin{array}{ccccc} & & & & \mathbf{R}(\varepsilon_{00}\circ\varepsilon_{1,0})_*\mathbb{Z}_{X_1} = \mathbf{R}\varepsilon_{1*}\mathbb{Z}_{X_1} \\ & & & \overset{\varepsilon_{10}}{\nearrow} & \Big\downarrow = \\ \mathbb{Z}_X & \longrightarrow & \mathbf{R}\varepsilon_{0*}\mathbb{Z}_{X_0} & & \\ & & & \underset{\varepsilon_{11}}{\searrow} & \\ & & & & \mathbf{R}(\varepsilon_{00}\circ\varepsilon_{1,1})_*\mathbb{Z}_{X_1} = \mathbf{R}\varepsilon_{1*}\mathbb{Z}_{X_1} \end{array}$$

The map $\varphi_1\colon \mathbf{R}\varepsilon_{0*}\mathbb{Z}_{X_0} \to \mathbf{R}\varepsilon_{1*}\mathbb{Z}_{X_1}$ considered in the definition of cohomological descent is the difference $\varphi_1 = \varepsilon_{1,0} - \varepsilon_{1,1}$.

EXAMPLE 2.1.3. Let $X = (y^2 - x^3 = 0) \subset \mathbf{C}^2$ be a cuspidal curve. The normalization map $f\colon \widetilde{X} \to X$ is already a hyperresolution, as it is easy to check that $f_*\mathbb{Z}_{\widetilde{X}} \simeq \mathbb{Z}_X$.

EXAMPLE 2.1.4. Let $X = (y^2 - x^2 - x^3 = 0) \subset \mathbf{C}^2$ be a nodal curve. The normalization map $f\colon \widetilde{X} \to X$ doesn't work anymore, meaning $f_*\mathbb{Z}_{\widetilde{X}} \neq \mathbb{Z}_X$, as one can easily check by applying Mayer-Vietoris in a neighborhood of the node and its preimages. This also suggests what to do instead. Denote the node by $p$, and let $f^{-1}(p) = \{r, s\}$. Consider the diagram

$$\begin{array}{ccc} \{r,s\} & \xrightarrow{\ i\ } & \widetilde{X} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \{p\} & \xrightarrow[\ i\ ]{} & X \end{array}$$

where $i$ denotes the obvious inclusions. Now define

$$X_0 := \widetilde{X} \sqcup \{p\} \quad \text{and} \quad X_1 := \{r, s\}.$$

We then have a simplicial resolution where all the maps are induced from the diagram above:

$$X_1 \underset{f}{\overset{i}{\rightrightarrows}} X_0 \xrightarrow{(f,i)} X.$$

This satisfies cohomological descent, since using Mayer-Vietoris as above, we can patch everything into an exact sequence of singular cohomology

$$0 \to H^0(X,\mathbb{Z}) \to H^0(\widetilde{X},\mathbb{Z}) \oplus H^0(\{p\},\mathbb{Z}) \to H^0(\{r\}\sqcup\{s\},\mathbb{Z}) \to H^1(X,\mathbb{Z}) \to H^1(\widetilde{X},\mathbb{Z}) \to 0.$$

Even better we have a short exact sequence of sheaves

$$0 \to \mathbb{Z}_X \to f_*\mathbb{Z}_{\widetilde{X}} \oplus i_*\mathbb{Z}_{\{p\}} \xrightarrow{i-f} f_*\mathbb{Z}_{\{r\}} \oplus f_*\mathbb{Z}_{\{s\}} \to 0.$$

EXAMPLE 2.1.5. Let $X \subset \mathbf{C}^3$ be the cone over a smooth curve $C$ in $\mathbf{P}^2$, and let $f\colon \widetilde{X} \to X$ be the resolution we considered before (by blowing-up the vertex), where

$f^{-1}(\{0\}) = C$. It is again easy to see from Mayer-Vietoris that $f$ does not satisfy cohomological descent. On the other hand, we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \{p\} & \xrightarrow[i]{} & X \end{array}$$

where $i$ denotes the obvious inclusions. We define

$$X_0 := \widetilde{X} \sqcup \{p\} \quad \text{and} \quad X_1 := C.$$

We then have a simplicial resolution where all the maps are induced from the diagram above:

$$X_1 \underset{f}{\overset{i}{\rightrightarrows}} X_0 \xrightarrow{(f,i)} X.$$

It is not hard to check that this satisfies cohomological descent (exercise!).

These examples fit in a general framework, which is what is needed in order to establish the existence of hyperresolutions in general:

Exercise 2.1.6. Consider a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{i} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

where $E = f^{-1}(Z)_{\mathrm{red}}$, and $f$ induces an isomorphism $\widetilde{X} \smallsetminus E \simeq X \smallsetminus Z$. If we set

$$X_1 = E \quad \text{and} \quad X_0 = \widetilde{X} \sqcup Z,$$

then the diagram

$$X_1 \underset{f}{\overset{i}{\rightrightarrows}} X_0 \xrightarrow{(f,i)} X,$$

with the obvious morphisms, satisfies cohomological descent; in other words, we have an exact triangle

$$\mathbb{Z}_X \longrightarrow \mathbf{R}\varepsilon_{0*}\mathbb{Z}_{X_0} \xrightarrow{\varphi_1} \mathbf{R}\varepsilon_{1*}\mathbb{Z}_{X_1} \xrightarrow{+1} .$$

Therefore, if $\widetilde{X}$, $Z$ and $E$ are smooth, then this gives a hyperresolution of $X$. (Cf. also [**PS**, §5.1 and §5.2].)

Note that in this story we have to consider reducible varieties as well. Here is the simplest example:

Example 2.1.7. Let $X = (xy = 0) \subset \mathbf{C}^2$, with irreducible components $E_1$ and $E_2$, both isomorphic to $\mathbf{P}^1$. It is then a simple check using Mayer-Vietoris that we get a hyperresolution for $X$ using

$$X_0 = E_1 \sqcup E_2 \quad \text{and} \quad X_1 = E_1 \cap E_2 = \{0\},$$

where all the maps are given by the natural inclusions.

This generalizes to arbitrary SNC. divisors:

EXAMPLE 2.1.8 (**SNC divisors**). Let $E = E_1 + \cdots + E_n$ be an SNC divisor in some smooth variety (where we may have $E_j = 0$ for some $j$). We obtain a hyperresolution of $E$ by taking

$$X_k := \bigsqcup_{i_0 < \cdots < i_k} E_{i_0} \cap E_{i_1} \cap \cdots \cap E_{i_k}$$

The map $\varepsilon_{kj} \colon X_k \to X_{k-1}$, with $0 \leq j \leq k$ is induced by the inclusions

$$E_{i_0} \cap E_{i_1} \cap \cdots \cap E_{i_k} \hookrightarrow E_{i_0} \cap E_{i_1} \cap \cdots \cap \widehat{E_j} \cap \cdots \cap E_{i_k}.$$

Cohomological descent is checked by using Mayer-Vietoris successively, and applying the inclusion-exclusion principle.

**Cubical resolutions.** In all of the examples above we considered more than just a hyperresolution; we actually got it from some commutative "square", in which we separated the components of each term in the hyperresolution. To understand this better, we consider the concept of a cubical variety, starting with some examples:

(0) A 0-cubical variety is simply a variety $X = X_\emptyset$.

(1) A 1-cubical variety is a morphism of varieties $f \colon X_{\{0\}} \to X_\emptyset$.

(2) A 2-cubical variety is a commutative diagram of morphisms of varieties

$$\begin{array}{ccc} X_{\{0,1\}} & \longrightarrow & \widetilde{X}_{\{1\}} \\ \downarrow & & \downarrow \\ X_{\{0\}} & \longrightarrow & X_\emptyset \end{array}$$

(3) A 3-cubical variety is q commutative cube of morphisms of varieties. (INCLUDE PICTURE.)

Etc. The indexing is explained by the general definition we give next.

DEFINITION 2.1.9. A *cubical variety* is a contravariant functor

$$F \colon \square_n \to \mathrm{Var}_{\mathbf{C}},$$

where $\mathrm{Var}_{\mathbf{C}}$ is the category of complex varieties and $\square_n$ is the cubical category, i.e. the category whose objects are (ordered) subsets $I$ of the set $\{0, \ldots, n-1\}$, including the empty set $\emptyset$, and the morphisms are given by

$$\mathrm{Hom}(I, J) = \begin{cases} *, & \text{if } I \subseteq J. \\ \emptyset, & \text{if } I \not\subseteq J. \end{cases}$$

Here $*$ means the set with one element. We denote $X_I := F(I)$.

Next we note that a $(k+1)$-cubical variety gives rise to a semisimplicial variety of length $k$.

EXAMPLE 2.1.10. The 2-cubical variety above leads to a semisimplicial variety with $X = X_\emptyset$ and

$$X_1 = X_{\{0,1\}} \quad \text{and} \quad X_0 = X_{\{0\}} \sqcup X_{\{1\}}.$$

The maps are the obvious ones induced from the cubical diagram.

EXAMPLE 2.1.11. The 3-cubical variety above leads to a semisimplicial variety with $X = X_\emptyset$ and

$$X_2 = X_{\{0,1,2\}} \quad \text{and} \quad X_1 = X_{\{0,1\}} \sqcup X_{\{0,2\}} \sqcup X_{\{1,2\}} \quad \text{and} \quad X_0 = X_{\{0\}} \sqcup X_{\{1\}} \sqcup X_{\{2\}}.$$

The maps are again the obvious ones induced from the cubical diagram. The two maps from $X_1$ to $X_0$ are given by the two possible inclusions on each term, in lexicographical order.

DEFINITION 2.1.12. A *cubical resolution* of a variety $X$ is a cubical variety with $X_\emptyset = X$, such that all the $X_I$ are smooth and all the morphisms are proper, and such that the associated semisimplicial variety is a hyperresolution (i.e. satisfies cohomological descent).

EXERCISE 2.1.13. Describe the passage from a cubical resolution to a hyperresolution in categorical/combinatorial language.

**Existence.** The existence of hyperresolutions was first established by Deligne in [?]. Later on, Guillén, Navarro Aznar, Pascal-Guainza, and Puerta, proved the following more precise result, using the theory of cubical resolutions:

THEOREM 2.1.14 ([**GNPP**, Theorem 2.15]). *There exists a hyperresolution $\varepsilon_\bullet \colon X_\bullet \to X$ such that $\dim X_i \le n - i$ for all $i \ge 0$. (In particular, such a hyperresolution has length at most $n$.) In fact there exists an $(n+1)$-cubical resolution of $X$ such that $\dim X_I \le n - |I| + 1$ for all subsets $I \subseteq \{0, \ldots, n\}$.*

I sketched the proof in class, but at the moment I'm not sure I can explain it any better than it is done in [**PS**, Theorem 5.26], so I will just refer you to that source. Let's instead look at another example, which this time is not a cone, and contains the idea of the general procedure.

EXAMPLE 2.1.15. Consider the isolated surface singularity $X = (x^2 + y^2 + z^3 = 0) \subset \mathbf{C}^3$, so called of type $A_2$. Let

$$f = \mathrm{Bl}_{\{0\}} \colon \widetilde{\mathbf{C}^3} \to \mathbf{C}^3$$

be the blow-up at the origin, and let $f^{-1}(X) = \widetilde{X} \cup F$, where $\widetilde{X}$ is the proper transform of $X$, and $F$ is the (reduced) exceptional divisor. We have:

EXERCISE 2.1.16. We have that $\widetilde{X}$ is smooth, while $E := F \cap \widetilde{X}$ is an SNC divisor given as the union of two smooth rational curves $E = E_1 \cup E_2$.

Assuming this exercise, the restriction $f \colon \widetilde{X} \to X$ is a log resolution of $X$, but with reducible exceptional divisor, hence we cannot directly apply Exercise 2.1.6. Instead, we denote $D = \{0\}$ (the discriminant locus of this map), so that $E = f^{-1}(D)$, and we can think of the restriction $f\colon E \to D$ as being the 1-cubical variety $Z$ that is the "singular locus" of the 1-cubical variety $f\colon \widetilde{X} \to X$.

INCLUDE PICTURE.

## 2.2. The filtered de Rham complex

Recall that if $f\colon Z \to Y$ is a proper morphism of smooth varieties, we have a canonical induced morphism

$$\Omega_Y^\bullet \to \mathbf{R}f_*\Omega_Z^\bullet,$$

adjoint to the morphism $f^*\Omega_Y^\bullet \to \Omega_Z^\bullet$, and compatible with the filtration by truncation. Note that these objects are morphism live in the derived category of filtered differential complexes: the complexes consist of coherent sheaves, but their differentials are given by differential operators of order $\leq 1$. We apply this functoriality in the smooth case to a hyperresolution in order to construct the filtered de Rham complex of a singular variety.

Let $X$ be a complex variety of dimension $n$, and fix a hyperresolution $\varepsilon_\bullet\colon X_\bullet \to X$.

Definition 2.2.1. The *filtered de Rham complex* of $X$ is

$$\underline{\Omega}_X^\bullet := \mathbf{R}\varepsilon_{\bullet *}\Omega_{X_\bullet}^\bullet,$$

defined as an object in the derived category of filtered differential complexes on $X$. What does this mean concretely? Just as in the cohomological descent condition, we have maps between objects on $X$ given by

$$\mathbf{R}\varepsilon_{0*}\Omega_{X_0}^\bullet \xrightarrow{\varphi_1} \mathbf{R}\varepsilon_{1*}\Omega_{X_1}^\bullet \xrightarrow{\varphi_2} \mathbf{R}\varepsilon_{2*}\Omega_{X_2}^\bullet \xrightarrow{\varphi_3} \cdots ,$$

where $\varphi_i$ is induced by the morphism

$$\varepsilon_{i,0} - \varepsilon_{i,1} + \varepsilon_{i,2} - \cdots ,$$

and $\underline{\Omega}_X^\bullet$ is the iterated cone of these maps (placed as before so that its cohomologies live in non-negative degrees). This gives us the object.

To construct the filtration, recall that each $\Omega_{X_i}^\bullet$ is filtered by $F^p\Omega_{X_i}^\bullet = \Omega_{X_i}^{\geq p}$, with $\mathrm{gr}_F^p\,\Omega_{X_i}^\bullet \simeq \Omega_{X_i}^p[-p]$. Therefore $\mathbf{R}\varepsilon_{i*}\Omega_{X_i}^\bullet$ is filtered by $\mathbf{R}\varepsilon_{i*}\Omega_{X_i}^{\geq p}$, with associated graded (cone) $\mathbf{R}\varepsilon_{i*}\Omega_{X_i}^p$, and by functoriality these filtrations are compatible via the maps $\varphi_i$. This induces a filtration

$$F^p\underline{\Omega}_X^\bullet := \mathbf{R}\varepsilon_{\bullet *}\Omega_{X_\bullet}^{\geq p}.$$

Definition 2.2.2. The *p-th Du Bois complex* of $X$ is

$$\underline{\Omega}_X^p := \mathrm{gr}_F^p\,\underline{\Omega}_X^\bullet[p] \simeq \mathbf{R}\varepsilon_{\bullet *}\Omega_{X_\bullet}^p[p].$$

It can be seen as the (iterated) cone of the sequence of morphisms

$$\mathbf{R}\varepsilon_{0*}\Omega_{X_0}^p \xrightarrow{\varphi_1} \mathbf{R}\varepsilon_{1*}\Omega_{X_1}^p \xrightarrow{\varphi_2} \mathbf{R}\varepsilon_{2*}\Omega_{X_2}^p \xrightarrow{\varphi_3} \cdots . \tag{2.2.1}$$

A basic result, conjectured by Deligne and proved by Du Bois, is the following:

THEOREM 2.2.3 ([**DB**, Theorem 3.11]). *A morphism of hyperresolutions* $X'_\bullet \to X_\bullet$ *induces a quasi-isomorphism* $\underline{\Omega}^p_{X'} \to \underline{\Omega}^p_X$*. Consequently* $\underline{\Omega}^p_X$ *is well defined for all* $p$*, and therefore so is* $\underline{\Omega}^\bullet_X$*.*

I will assume this here, and proceed with discussing some first features and examples of Du Bois complexes.

LEMMA 2.2.4. *We have* $\underline{\Omega}^p_X \in \mathbf{D}^b_{\mathrm{coh}}(X)$ *for all* $p$*, meaning that:*

*(i)* $\mathcal{H}^q\Omega^p_X$ *are coherent sheaves for all* $p$ *and* $q$*.*

*(ii)* $\underline{\Omega}^p_X$ *is represented by a complex with* $\mathcal{O}_X$*-linear differentials.*

PROOF. This is basically immediate from the description in (2.2.1), since the $\varepsilon_i$ are proper. To be more precise, note that by general homological algebra (the spectral sequence relating a double complex to the cohomology of the total complex), we have a spectral sequence

$$E_1^{ji} = R^i\varepsilon_j\Omega^p_{X_j} \implies \mathcal{H}^{i+j}\underline{\Omega}^p_X.$$

□

We can also produce concrete double complexes of flasque sheaves whose total complexes represent $\underline{\Omega}^\bullet_X$ and $\underline{\Omega}^p_X$ respectively. For this, recall two basic facts in the theory of manifolds (see [**GH**, Ch.0]):

(1) For a smooth manifold $X$ of real dimension $d$, by the ordinary Poincaré Lemma the complex

$$0 \to \mathcal{A}^0_X \to \mathcal{A}^1_X \to \cdots \to \mathcal{A}^n_X \to 0$$

consisting of the sheaves of smooth forms on $X$, with the exterior derivative $d$ as differential, is quasi-isomorphic to the constant sheaf $\mathbb{R}_X$. Hence if $X$ is a complex manifold of dimension $d = 2n$, its complexification is quasi-isomorphic to the constant sheaf $\mathbf{C}_X$.

(2) For a complex manifold $X$ of dimension $n$, and an integer $0 \le p \le n$, by the $\bar\partial$-Poincaré Lemma the complex

$$0 \to \mathcal{A}^{p,0}_X \to \mathcal{A}^{p,1}_X \to \cdots \to \mathcal{A}^{p,n}_X \to 0$$

consisting of the sheaves of (complexified) smooth $(p,q)$-forms on $X$, with the operator $\bar\partial$ as differential, is quasi-isomorphic to the vector bundle $\Omega^p_X$ of holomorphic $p$-forms on $X$.

Both of these resolutions are by fine, and therefore acyclic, sheaves. This implies that $\underline{\Omega}^\bullet_X$ can be seen as being represented by the total complex of the double complex:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^0_{X_0,\mathbf{C}} & \xrightarrow{d} & \varepsilon_*\mathcal{A}^1_{X_0,\mathbf{C}} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{2n}_{X_0,\mathbf{C}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon_{10}-\varepsilon_{11}} & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^0_{X_1,\mathbf{C}} & \longrightarrow & \varepsilon_*\mathcal{A}^1_{X_1,\mathbf{C}} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{2n}_{X_1,\mathbf{C}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon_{20}-\varepsilon_{21}+\varepsilon_{22}} & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^0_{X_n,\mathbf{C}} & \longrightarrow & \varepsilon_*\mathcal{A}^1_{X_n,\mathbf{C}} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{2n}_{X_n,\mathbf{C}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

Analogously, $\underline{\Omega}^p_X$ can be seen as being represented by the total complex of the double complex:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^{p,0}_{X_0} & \xrightarrow{\overline{\partial}} & \varepsilon_*\mathcal{A}^{p,1}_{X_0} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{p,n}_{X_0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon_{10}-\varepsilon_{11}} & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^{p,0}_{X_1} & \longrightarrow & \varepsilon_*\mathcal{A}^{p,1}_{X_1} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{p,n}_{X_1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon_{20}-\varepsilon_{21}+\varepsilon_{22}} & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \varepsilon_*\mathcal{A}^{p,0}_{X_n} & \longrightarrow & \varepsilon_*\mathcal{A}^{p,1}_{X_n} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_*\mathcal{A}^{p,n}_{X_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & &
\end{array}
$$

Here we have to be careful to shift the entire double complex to the left $p$ times, in order to have the first column be placed in degree 0, consistent to our convention for $\underline{\Omega}^p_X$.

EXAMPLE 2.2.5. If $X$ is smooth, then $\mathrm{id}\colon X \to X$ is a hyperresolution, hence $\underline{\Omega}^\bullet_X = \Omega^\bullet_X$, and $\underline{\Omega}^p_X = \Omega^p_X$ for all $p$.

EXAMPLE 2.2.6. If $X = (y^2 - x^3 = 0) \subset \mathbf{C}^2$ is the standard cusp, recall from Example 2.1.3 that the normalization map $f\colon \widetilde{X} \to X$ is already a hyperresolution. Therefore

$$\underline{\Omega}^0_X \simeq f_*\mathcal{O}_{\widetilde{X}} \not\simeq \mathcal{O}_X.$$

(Exercise: specify the cokernel of the canonical map $\mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}}$.) and

$$\underline{\Omega}^1_X \simeq f_*\omega_{\widetilde{X}} \not\simeq \omega_X.$$

The second non-equality statement follows from the first by duality (note that $X$ is Gorenstein, hence it has a (locally free) dualizing sheaf).

EXAMPLE 2.2.7. If $X = (y^2 - x^2 - x^3 = 0) \subset \mathbf{C}^2$ is the standard node, we use the hyperresolution in Example 2.1.4.

According to the definition, we have a quasi-isomorphism

$$\underline{\Omega}^0_X \simeq [f_*\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\{p\}} \to f_*\mathcal{O}_{\{r,s\}}].$$

Note however that we have a short exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\{p\}} \to f_*\mathcal{O}_{\{r,s\}} \to 0$$

given by the same map (exercise!), and therefore

$$\underline{\Omega}^0_X \simeq \mathcal{O}_X.$$

Note that in this case this answer is better, even though the hyperresolution is a bit more complicated than in the case of a cusp. Moreover, again by definition, we have (since $\Omega^1_Z = 0$ for a finite set $Z$):

$$\underline{\Omega}^1_X \simeq f_*\omega_{\widetilde{X}} \not\simeq \omega_X.$$

The second non-isomorphism statement follows for instance from the fact that $X$ is Cohen-Macaulay but does not have rational singularities, or can be checked directly.

EXAMPLE 2.2.8. Let $X = C(Y) \subset \mathbf{C}^3$ be the cone over a smooth conic in $\mathbf{P}^2$, i.e. an $A_1$-singularity. We use the hyperresolution in Example 2.1.5. We have by definition:

$$\underline{\Omega}^0_X \simeq [\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\{p\}} \to \mathbf{R}f_*\mathcal{O}_C].$$

Now $X$ has rational singularities, so $\mathbf{R}f_*\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_X$. On the other hand, $C \simeq \mathbf{P}^1$, so we have $\mathbf{R}f_*\mathcal{O}_C \simeq \mathcal{O}_{\{p\}}$. Thus the map becomes

$$f_*\mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\{p\}} \xrightarrow{\varepsilon_{10}-\varepsilon_{11}} \mathcal{O}_{\{p\}}$$

and it's not hard to check that its kernel is $\mathcal{O}_X$. We thus have

$$\underline{\Omega}^0_X \simeq \mathcal{O}_X.$$

Let's first jump to

$$\underline{\Omega}^2_X \simeq [\mathbf{R}f_*\Omega^2_{\widetilde{X}} \oplus \Omega^2_{\{p\}} \to \mathbf{R}f_*\Omega^2_C].$$

Since $\Omega^2_{\{p\}} = \Omega^2_C = 0$ for dimension reasons, we obtain

$$\underline{\Omega}^2_X \simeq \mathbf{R}f_*\omega_{\widetilde{X}} \simeq f_*\omega_{\widetilde{X}} \simeq \omega_X.$$

The next to last isomorphism holds by Grauert-Riemenschneider, while the last holds because $X$ has rational singularities. Finally, let's look at

$$\underline{\Omega}^1_X \simeq [\mathbf{R}f_*\Omega^1_{\widetilde{X}} \oplus \Omega^1_{\{p\}} \to \mathbf{R}f_*\Omega^1_C].$$

Here we obviously have $\Omega^1_{\{p\}} = 0$, and we obtain a long exact sequence

$$0 \to \mathcal{H}^0\underline{\Omega}^1_X \to f_*\Omega^1_{\widetilde{X}} \to f_*\omega_C \to \mathcal{H}^1\underline{\Omega}^1_X \to R^1f_*\Omega^1_{\widetilde{X}} \to R^1f_*\omega_C \to \mathcal{H}^2\underline{\Omega}^2_X \to 0.$$

Since $C \simeq \mathbf{P}^1$, we have $f_*\omega_C = 0$ and $R^1f_*\omega_C \simeq \mathcal{O}_{\{p\}}$. Moreover, we have that

$$R^1f_*\Omega^1_{\widetilde{X}} \to R^1f_*\omega_C$$

is an isomorphism (exercise!). We conclude that

$$\mathcal{H}^0\underline{\Omega}^1_X \simeq f_*\Omega^1_{\widetilde{X}}, \quad \text{and} \quad \mathcal{H}^i\underline{\Omega}^1_X = 0 \;\; \text{for} \;\; i > 0.$$

One can show that $f_*\Omega^1_{\widetilde{X}}$ is a reflexive sheaf, different from the Kähler differentials $\Omega^1_X$.

EXERCISE 2.2.9. Compute the Du Bois complexes of the cone over a smooth plane cubic.

EXAMPLE 2.2.10. Let $X = (x^2 + y^2 + z^3 = 0) \subset \mathbf{C}^3$ be an $A_2$-singularity. The answer tuns out to be exactly as in Example 2.2.8. TO ADD.

EXERCISE 2.2.11. Let's take this temporarily as an exercise, until we learn more about the Du Bois complexes of cones. Let $X = C(Y) \subset \mathbf{C}^3$ be the cone over a smooth curve of degree $d \geq 4$ in $\mathbf{P}^2$. Show that in this case we have

$$\mathcal{H}^0\underline{\Omega}^0_X \simeq \mathcal{O}_X, \quad \mathcal{H}^0\underline{\Omega}^1_X \simeq f_*\Omega^1_{\widetilde{X}}, \quad \mathcal{H}^0\underline{\Omega}^2_X \simeq f_*\omega_{\widetilde{X}}$$

but also

$$\mathcal{H}^1\underline{\Omega}^0_X \neq 0 \quad \text{and} \quad \mathcal{H}^1\underline{\Omega}^1_X \neq 0.$$

## 2.3. Basic properties of the Du Bois complexes

The Du Bois complexes of a variety $X$ have a number of general, and important, properties, that follow almost directly from the definition. We study them here, while other, more subtle, properties will be discussed later.

(1) **Functoriality:** For every morphism of varieties $f\colon Y \to X$, there is an induced filtered morphism

$$\underline{\Omega}^\bullet_X \to \mathbf{R}f_*\underline{\Omega}^\bullet_Y,$$

and therefore also induced morphisms

$$\underline{\Omega}^p_X \to \mathbf{R}f_*\underline{\Omega}^p_Y \quad \text{for all } p.$$

PROOF. The procedure that gives the existence of cubical resolutions also resolves cubical varieties. In particular, one can construct hyperresolutions compatible with the morphism $f$, i.e. a commutative diagram

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{f_\bullet} & X_\bullet \\ {\scriptstyle \varepsilon^Y_\bullet}\downarrow & & \downarrow{\scriptstyle \varepsilon^X_\bullet} \\ Y & \xrightarrow{f} & X \end{array}$$

where $\varepsilon_\bullet^Y : Y_\bullet \to Y$ and $\varepsilon_\bullet^X : X_\bullet \to X$ are hyperresolutions. Note that for each $i$ we have filtered morphisms

$$\Omega^\bullet_{X_i} \to \mathbf{R}f_{i*}\Omega^\bullet_{Y_i},$$

by functoriality. Therefore we obtain an induced filtered morphism

$$\underline{\Omega}^\bullet_X \to \mathbf{R}\varepsilon^X_{\bullet\,*}\Omega^\bullet_{X_\bullet} \to \mathbf{R}\varepsilon^X_{\bullet\,*}\mathbf{R}f_{\bullet *}\Omega^\bullet_{Y_\bullet} \simeq \mathbf{R}f_*\mathbf{R}\varepsilon^Y_{\bullet\,*}\Omega^\bullet_{Y_\bullet} = \mathbf{R}f_*\underline{\Omega}^\bullet_Y.$$

□

(2) **Local nature:** For every open set $U \subseteq X$, we have

$$\underline{\Omega}^p_{X|U} \simeq \underline{\Omega}^p_U \quad \text{for all } p.$$

More generally, $\underline{\Omega}^\bullet_X$ is local with respect to the étale topology: if $f\colon Y \to X$ is an étale morphism, then there exists a natural filtered isomorphism

$$f^*\underline{\Omega}^\bullet_X \simeq \underline{\Omega}^\bullet_Y.$$

Proof. Let $\varepsilon^X_\bullet : X_\bullet \to X$ be a hyperresolution for $X$. For every $i$ we can form the fiber diagram

$$\begin{array}{ccc} Y_i := Y \times_X X_i & \xrightarrow{f_i} & X_i \\ \downarrow{\scriptstyle \varepsilon_i^Y} & & \downarrow{\scriptstyle \varepsilon_i^X} \\ Y & \xrightarrow{f} & X \end{array}$$

Since $f$ is étale, $Y_i$ is smooth, and it's not hard to see that the resulting $\varepsilon^Y_\bullet : Y_\bullet \to Y$ is a hyperresolution. We then have

$$f^*\underline{\Omega}^\bullet_X \simeq f^*\mathbf{R}\varepsilon^X_{\bullet\,*}\Omega^\bullet_{X_\bullet} \simeq \mathbf{R}\varepsilon^Y_{\bullet\,*}f^*_\bullet\Omega^\bullet_{X_\bullet} \simeq \mathbf{R}\varepsilon^Y_{\bullet\,*}\Omega^\bullet_{Y_\bullet} = \underline{\Omega}^\bullet_Y.$$

The last two isomorphisms follow from the push-pull formula, and from the fact that $f_i^*\Omega^\bullet_{X_i} \simeq \Omega^\bullet_{Y_i}$, since the $f_i$ are also étale. □

(3) **Comparison with Kähler differentials:** For every $p \geq 0$, there exists a natural morphism

$$\Omega^p_X \to \underline{\Omega}^p_X,$$

where $\Omega^p_X := \wedge^p\Omega^1_X$ is the $p$-th sheaf of Kähler differentials on $X$. Moreover, if $X$ is smooth, this morphism is an isomorphism.

Proof. If $f\colon Y \to X$ is any morphism, then we have a canonical morphism $f^*\Omega^p_X \to \Omega^p_Y$, has by the adjoint property a morphism $\Omega^p_X \to f_*\Omega^p_Y$. Therefore, given a hyperresolution $\varepsilon_\bullet\colon X_\bullet \to X$, we have a natural morphism

$$\Omega^p_X \to \varepsilon_{0*}\Omega^p_{X_0},$$

which induces by composition the desired $\Omega^p_X \to \underline{\Omega}^p_X$. □

Note that even in the singular case we can consider the Kähler-de Rham complex, with the filtration by truncation, and in the same vein we have a filtered morphism

$$\Omega^\bullet_X \to \underline{\Omega}^\bullet_X.$$

(4) **Topological and Hodge-theoretic properties:** There is a natural morphism $\mathbf{C}_X \to \underline{\Omega}_X^\bullet$, which is an isomorphism. In particular, we have

$$H^i(X, \mathbf{C}) \simeq \mathbb{H}^i(X, \underline{\Omega}_X^\bullet) \quad \text{for all } i.$$

Moreover, there exists a Hodge-to-de Rham-type spectral sequence

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \implies H^{p+q}(X, \mathbf{C}),$$

which degenerates at $E_1$ if $X$ is projective.

Proof. Let $\varepsilon_\bullet \colon X_\bullet \to X$ be a hyperresolution. For each $i$, we have a quasi-isomorphism

$$\mathbf{C}_{X_i} \to \Omega_{X_i}^\bullet$$

given by the Poincaré Lemma. Therefore we obtain a quasi-isomorphism

$$\mathbf{C}_X \simeq \mathbf{R}\varepsilon_{\bullet *}\mathbf{C}_{X_\bullet} \to \mathbf{R}\varepsilon_{\bullet *}\Omega_{X_\bullet}^\bullet \simeq \underline{\Omega}_X^\bullet,$$

where the first isomorphism is a consequence of the cohomological descent property. Note in particular the following consequences:

(1) $\mathcal{H}^i\underline{\Omega}_X^\bullet = 0$ for all $i \neq 0$.
(2) The composition $\mathbf{C}_X \to \mathscr{O}_X \to \mathcal{H}^0\underline{\Omega}_X^\bullet$ is an isomorphism.
(3) $H^k(X, \mathbf{C}) \simeq \mathbb{H}^k(X, \underline{\Omega}_X^\bullet)$ for all $k \geq 0$.

Moreover, the spectral sequence of a filtered complex is in this case

$$E_1^{pq} = \mathbb{H}^{p+q}(X, \operatorname{gr}_p^F \underline{\Omega}_X^\bullet) \implies \mathbb{H}^{p+q}(X, \underline{\Omega}_X^\bullet),$$

or equivalently, in the Hodge-to-de Rham form

$$E_1^{pq} = \mathbb{H}^q(X, \underline{\Omega}_X^p) \implies H^{p+q}(X, \mathbf{C}).$$

It remains to check that the differentials

$$d \colon \mathbb{H}^q(X, \underline{\Omega}_X^p) \to \mathbb{H}^q(X, \underline{\Omega}_X^{p+1})$$

are all 0. Note however that we have spectral sequences

$$E_1^{ij} = H^j(X, \mathbf{R}\varepsilon_{i*}\Omega_{X_i}^p) \simeq H^j(X_i, \Omega_{X_i}^p) \implies \mathbb{H}^{i+j}(X, \underline{\Omega}_X^p),$$

and so these differentials are built from the differentials

$$d \colon H^j(X_i, \Omega_{X_i}^p) \to H^j(X_i, \Omega_{X_i}^{p+1}),$$

which are all 0 by classical Hodge theory. □

(5) **Vanishing and dimension of support:** We have

$$\underline{\Omega}_X^p = 0 \quad \text{for} \quad p < 0 \quad \text{and} \quad p > n,$$

where $n = \dim X$. For every $p$, we also have

$$\mathcal{H}^q\underline{\Omega}_X^p = 0 \quad \text{for} \quad q < 0 \quad \text{and} \quad q > n.$$

More precisely, we have

$$\operatorname{codim} \operatorname{Supp}(\mathcal{H}^q\underline{\Omega}_X^p) \geq q \quad \text{for all} \quad q. \tag{2.3.1}$$

Proof. The first statement is clear, since $\underline{\Omega}_X^p$ is defined in via complexes containing only terms of the form $\mathbf{R}\varepsilon_{i*}\underline{\Omega}_{X_i}^p$. For the same reason, and since these terms start in degree 0, we have the vanishing $\mathcal{H}^q\underline{\Omega}_X^p = 0$ for $q < 0$. We are left with showing (2.3.1).

To this end, recall that there is a spectral sequence

$$E_1^{ji} = R^i\varepsilon_j\Omega_{X_j}^p \implies \mathcal{H}^{i+j}\underline{\Omega}_X^p.$$

Moreover, we can consider a hyperresolution such that $\dim X_j \le n - j$ for each $j$. It suffices to show that in this case $\dim E_1^{ij} \le n - q$ whenever $i + j = q$. But this is a consequence of the general statement in the exercise below. ☐

Exercise 2.3.1. If $f\colon Z \to W$ is a proper morphism, with $\dim Z \le a$, and if $\mathscr{F}$ is a coherent sheaf on $Z$, then for any $b \ge 0$ we have

$$\dim \operatorname{Supp} R^b f_*\mathscr{F} \le a - b.$$

Remark 2.3.2. By (3) we know that all $\mathcal{H}^i\underline{\Omega}_X^p$ with $i > 0$ are supported at most on $X_{\rm sing}$. Therefore one can often do better than (2.3.1).

(6) **Top Du Bois complex:** We have

$$\underline{\Omega}_X^n \simeq f_*\omega_{\widetilde{X}}$$

for every resolution of singularities $f\colon \widetilde{X} \to X$.

Proof. This is clear by definition: if we consider a hyperresolution $\varepsilon_\bullet\colon X_\bullet \to X$ that starts with the resolution $f$, then $\widetilde{X}$ is part of $X_0$, and the only component of any $X_i$ that has dimension $n$. Therefore from the defining complex for $\underline{\Omega}_X^n$ using $X_\bullet$, we are only left with $\mathbf{R}f_*\omega_{\widetilde{X}}$, which is the same as $f_*\omega_{\widetilde{X}}$ by Grauert-Riemenschneider. ☐

Example 2.3.3 (**Hodge-Du Bois numbers**).

## 2.4. Less basic properties of the Du Bois complexes

There are other general properties of Du Bois complexes that do not follow so easily from the definition. I will list a few of them, but for now I will skip most of the proofs. MORE DETAILS TO BE ADDED.

(7) **Steenbrink vanishing:** A more subtle, but still completely general, vanishing result for the cohomologies of Du Bois complexes is the following:

Theorem 2.4.1 ( [**St**, (4.1)]). *If $X$ is an $n$-dimensional complex variety, then*

$$\mathcal{H}^q\underline{\Omega}_X^p = 0 \quad \text{if} \quad p + q > n.$$

(8) **More vanishing:** In general, Steenbrink's vanishing theorem in (7) is the best possible result, in the sense that there exist examples (see e.g. [**MOPW**, Example 1.7]) showing that we may have $\mathcal{H}^q\underline{\Omega}_X^p \ne 0$ when $p + q = n$. However, we have:

PROPOSITION 2.4.2. *If $k < n$ and $\mathcal{H}^{n-p-1}\underline{\Omega}_X^p = 0$ for all $p \le k-1$, then*

$$\mathcal{H}^{n-k}\underline{\Omega}_X^k = 0.$$

*In particular $\mathcal{H}^n\underline{\Omega}_X^0 = 0$.*

PROOF. Consider the (local) spectral sequence of a filtered complex, associated to the Hodge filtration on the filtered de Rham complex:

$$E_1^{p,q} := \mathcal{H}^q\underline{\Omega}_X^p \implies \mathcal{H}^{p+q}\underline{\Omega}_X^\bullet.$$

Since $\underline{\Omega}_X^\bullet$ is quasi-isomorphic to $\mathbf{C}_X$, the spectral sequence converges to $\mathbf{C}_X$, placed in cohomological degree 0. Note that for any $\ell \ge 1$, the term $E_{\ell+1}^{k,n-k}$ is obtained as the cohomology of the complex

$$E_\ell^{k-\ell,n-k+\ell-1} \to E_\ell^{k,n-k} \to E_\ell^{k+\ell,n-k-\ell+1},$$

and the right hand side is 0 by Theorem 2.4.1, while the left hand side is 0 by assumption. Therefore

$$\mathcal{H}^{n-k}\underline{\Omega}_X^k = E_1^{k,n-k} = E_\infty^{k,n-k} = 0.$$

□

EXAMPLE 2.4.3. When $X$ is a surface, by Theorem 2.4.1 and Proposition 2.4.2 the only nontrivial higher cohomologies of the Du Bois complexes of $X$ are $\mathcal{H}^1\underline{\Omega}_X^0$ and $\mathcal{H}^1\underline{\Omega}_X^1$. Moreover, the Proposition says that if $\mathcal{H}^1\underline{\Omega}_X^0 = 0$, then we also have $\mathcal{H}^1\underline{\Omega}_X^1 \neq 0$. It would be interesting to give an example where the converse implication does not hold.

REMARK 2.4.4. Note that the result does not hold for $k = n$, as we've seen in (6) that for any variety $X$ we have $\underline{\Omega}_X^n \simeq \mathcal{H}^0\underline{\Omega}_X^n \simeq \pi_*\omega_{\widetilde{X}}$, where $\pi\colon \widetilde{X} \to X$ is a resolution.

(9) **Torsion-freeness of the 0-th cohomology:** For every resolution of singularities $f\colon \widetilde{X} \to X$, and every $p$, we have an inclusion

$$\mathcal{H}^0\underline{\Omega}_X^p \hookrightarrow f_*\Omega_{\widetilde{X}}^p.$$

In particular, $\mathcal{H}^0\underline{\Omega}_X^p$ is a torsion-free sheaf.

(10) **Rational singularities and reflexivity:** In the setting of (7), if $X$ has rational singularities, we have

$$\mathcal{H}^0\underline{\Omega}_X^p \simeq f_*\Omega_{\widetilde{X}}^p \simeq f_*\Omega_{\widetilde{X}}^p(\log E) \simeq \Omega_X^{[p]},$$

for all $p$, where $\Omega_X^{[p]}$ is the reflexive hull of the sheaf of Kähler differentials $\Omega_X^p$, and $E$ is the reduced exceptional divisor on the resolution $\widetilde{X}$. (This is by far the deepest of all the properties listed so far, and in this generality it is due to Kebekus and Schnell [**KS**].)

(11) **Simple normal crossing divisors:** Let $E = E_1 + \cdots + E_k$ be a SNC divisor in a smooth variety $Y$. Recall that to $E$ we can associate, for each $p$, the locally free sheaf $\Omega_Y^p(\log E)$ of $p$-forms with log poles along $E$. If $z_1, \ldots, z_n$ are local coordinates on $Y$ such that $E_i = (z_i = 0)$ for $i = 1, \ldots, k$, then this sheaf is generated by the forms

$\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$. It is straighforward to check that we have canonical inclusions of locally free sheaves of rank $n$:

$$\Omega_Y^p(\log E)(-E) \hookrightarrow \Omega_Y^p \hookrightarrow \Omega_Y^p(\log E).$$

The Du Bois complexes of $E$ are given by the following:

PROPOSITION 2.4.5. *For every $p \geq 0$ we have that $\underline{\Omega}_E^p$ is a sheaf, sitting in the short exact sequence*

$$0 \longrightarrow \Omega_Y^p(\log E)(-E) \xrightarrow{i} \Omega_Y^p \longrightarrow \underline{\Omega}_E^p \longrightarrow 0,$$

*where $i$ is the canonical inclusion. In particular $\underline{\Omega}_E^0 \simeq \mathcal{O}_E$.*

This follows by putting together the description of $\underline{\Omega}_E^p$ we obtained previously, and a well-known independent computation of the cokernel of $i$. On the one hand, let's first look at the standard cubical resolution of $E$ described in Example 2.1.8. It tells us that by definition the Du Bois complex $\underline{\Omega}_E^p$ is computed by the complex

$$0 \longrightarrow \bigoplus_i \Omega_{E_i}^p \longrightarrow \bigoplus_{i<j} \Omega_{E_i \cap E_j}^p \longrightarrow \bigoplus_{i<j<k} \Omega_{E_i \cap E_j \cap E_k}^p \longrightarrow \cdots \tag{2.4.1}$$

where the differentials are given as always by the alternating sums of restriction maps.

On the other hand, there are well-known long exact sequences involving forms the sheaves of forms with log poles. The one of interest to us is:

LEMMA 2.4.6. *In the setting above, there is a long exact sequence*

$$0 \longrightarrow \Omega_Y^p(\log E)(-E) \xrightarrow{i} \Omega_Y^p \xrightarrow{i} \bigoplus_i \Omega_{E_i}^p \longrightarrow \bigoplus_{i<j} \Omega_{E_i \cap E_j}^p \longrightarrow \bigoplus_{i<j<k} \Omega_{E_i \cap E_j \cap E_k}^p \longrightarrow \cdots$$

*where $i$ is the canonical inclusion, $r$ is the component-wise restriction map, and the rest of the differentials are precisely those appearing in (2.4.1).*

The case $k = 1$, i.e. that when $E$ is a smooth divisor, appears in [**La**, Lemma 4.2.4]. More on this appears in [**EV**, §2]. The general case is an application of the exactness of the Koszul complex. (Find a general reference.)

EXERCISE 2.4.7. Check that the subsheaf $\mathrm{tor}(\Omega_E^p)$ of $\Omega_E^p$ generated by torsion sections is the subsheaf of forms supported on the singular locus of $E$, and show the isomorphism

$$\underline{\Omega}_E^p \simeq \Omega_E^p / \mathrm{tor}(\Omega_E^p).$$

(12) **Mayer-Vietoris triangle:** The Mayer-Vietoris-type cohomological descent property in our story is inherited at the level of Du Bois complexes as well:

PROPOSITION 2.4.8. *Let $Z \subseteq X$ be a closed subset, and consider a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

*where $f$ is a proper morphism, and $E \subseteq Y$ is a closed subset such that $f$ induces an isomorphism $Y \smallsetminus E \simeq X \smallsetminus Z$. Then for each $p$ there exists an exact triangle in $\mathbf{D}^b_{\mathrm{coh}}(X)$:*

$$\underline{\Omega}^p_X \longrightarrow \mathbf{R}f_*\underline{\Omega}^p_Y \oplus \underline{\Omega}^p_Z \longrightarrow \mathbf{R}f_*\underline{\Omega}^p_E \xrightarrow{+1} .$$

Note that the statement is clear by the definition of $\underline{\Omega}^p_X$ when $Y$, $Z$ and $E$ are smooth, since in that case the diagram is a cubical resolution of $X$. In general one needs to do a tedious induction (to be added later).

Most of the time this is used when $f\colon \widetilde{X} \to X$ is a strong log resolution of $X$, with $Y = \widetilde{X}$, $Z = X_{\mathrm{sing}}$ and $E = f^{-1}(Z)_{\mathrm{red}}$.

Example 2.4.9 (**Isolated singularities**). Let $x \in X$ be an isolated singularity, and $f\colon \widetilde{\to} X$ a strong log resolution of $X$, with $E = f^{-1}(x)_{\mathrm{red}}$. For $p > 0$, the exact triangle in Proposition 2.4.8, combined with Proposition 2.4.5, gives

$$\underline{\Omega}^p_X \simeq \mathbf{R}f_*\Omega^p_{\widetilde{X}}(\log E)(-E).$$

For $p = 0$, using the same ingredients, we obtain an exact triangle

$$\mathbf{R}f_*\mathcal{O}_{\widetilde{X}}(-E) \longrightarrow \underline{\Omega}^0_X \longrightarrow \mathcal{O}_{\{x\}} \xrightarrow{+1} .$$

(12) **Steenbrink's triangle:** Another way to take advantage of the triangle in (11) is to embed $X$ in a smooth variety and consider a log resolution of the pair. This was first considered by Steenbrink in [**St**], in the more general form:

Proposition 2.4.10. *Let $X \subseteq Y$ be a closed subvariety of an algebraic variety, and let $f\colon \widetilde{Y} \to Y$ be a proper birational morphism with $\widetilde{Y}$ smooth and $E = f^{-1}(X)_{\mathrm{red}}$ an SNC divisor, such that $f$ induces an isomorphism $\widetilde{Y} \smallsetminus E \simeq Y \smallsetminus X$. Then we have an exact triangle in in $\mathbf{D}^b_{\mathrm{coh}}(Y)$:*

$$\mathbf{R}f_*\Omega^p_{\widetilde{Y}}(\log E)(-E) \longrightarrow \underline{\Omega}^p_Y \longrightarrow \underline{\Omega}^p_X \xrightarrow{+1} .$$

This follows from Proposition 2.4.8, Proposition 2.4.5, and a simple application of the octahedral axiom.

## 2.5. The Du Bois complexes of special classes of varieties

This section is devoted to the study of the Du Bois complexes of a few important classes of singular varieties, discussed in Rosie's lectures.

**Quotient singularities.** Whenever a finite group $G$ acts on an affine variety $Y = \mathrm{Spec}(A)$, we can form the quotient $X = Y/G = \mathrm{Spec}(A^G)$ as an affine algebraic variety, due to the well-known fact that the algebra of invariants $A^G$ is finitely generated. This can be extended to arbitrary varieties using covers. We say that a variety $X$ has *quotient singularities* if it is (étale) locally the quotient of a smooth $Y$ by the action of a finite group.

Theorem 2.5.1 ([**DB**, Theorem 5.3]). *If $X$ has quotient singularities, then*

$$\underline{\Omega}^p_X \simeq \Omega^{[p]}_X \quad \text{for all} \quad p \geq 0.$$

(Will include the proof later.)

In other words, all higher cohomologies $\mathcal{H}^i\underline{\Omega}_X^p$ with $i > 0$ are 0, while the 0-th cohomologies $\mathcal{H}^0\underline{\Omega}_X^p$ are isomorphic to the reflexive differentials. This latter can now also be seen as a consequence of the fact that quotient singularities are always rational by Boutot's theorem, given the result by Kebekus-Schnell in §2.4(10).

REMARK 2.5.2 (**Geometric quotients**). The statement of Theorem 2.5.1 can be extended to actions of reductive groups $G$, as long as $X = Y//G$ is a geometric quotient. The reason for this is Luna's Slice Theorem, which says that étale locally such quotients look like quotients by finite groups.

On the other hand, there do exist GIT quotients with non-vanishing higher cohomology sheaves for certain Du Bois complexes; see [?].

**Abstract cones over smooth varieties.** Here we consider cones over smooth subvarieties in $\mathbf{P}^n$, and more generally abstract cones as in the setting of Example 1.2.11.

Let $Y \subset \mathbf{P}^n$ be a subvariety in projective space, endowed with an ample line bundle $L$. We consider the abstract cone

$$X = C(Y, L) := \mathrm{Spec}\Big(\bigoplus_{m\geq 0} H^0(Y, L^{\otimes m})\Big),$$

which comes together with the affine morphism $\pi\colon X \to Y$.

We aim to describe the cohomologies of the Du Bois complexes of $X$. Since $X$ is affine, it suffices to describe $\Gamma(X, \mathcal{H}^i\underline{\Omega}_X^p)$ for each $i$ and $p$. These are given by the following:

PROPOSITION 2.5.3. *We have $\mathcal{H}^0\underline{\Omega}_X^0 \simeq \mathscr{O}_X$, while for every other $p \geq 0$ and $i \geq 0$ we have*

$$\Gamma(X, \mathcal{H}^i\underline{\Omega}_X^p) \simeq \bigoplus_{m\geq 1} \left(H^i(Y, \Omega_Y^p \otimes L^{\otimes m}) \oplus H^i(Y, \Omega_Y^{p-1} \otimes L^{\otimes m})\right).$$

PROOF. Consider the blow-up $f\colon \widetilde{X} \to X$ at the vertex 0, with exceptional divisor $E \simeq X$. We obtain a commutative diagram

$$\begin{array}{ccc} E \simeq Y & \xrightarrow{\;i\;} & \widetilde{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \{0\} & \xrightarrow[\;i\;]{} & X \end{array}$$

In the case $p = 0$, we have the Mayer-Vietoris triangle

$$\mathscr{O}_X \longrightarrow \mathbf{R}f_*\mathscr{O}_{\widetilde{X}} \oplus \mathscr{O}_{\{0\}} \xrightarrow{\varphi} \mathbf{R}f_*\mathscr{O}_E \xrightarrow{+1},$$

while in the case $p \geq 1$ we have the Mayer-Vietoris triangle

$$\underline{\Omega}_X^p \longrightarrow \mathbf{R}f_*\Omega_{\widetilde{X}}^p \xrightarrow{\varphi} \mathbf{R}f_*\Omega_E^p \xrightarrow{+1}.$$

Since $X$ is a cone, we have that $\widetilde{X}$ has an $\mathbf{A}^1$-bundle structure over $Y$, denoted

$$\pi\colon \widetilde{X} \to Y.$$

Thus objects on $\widetilde{X}$ can be both pulled back and restricted to $X$, hence the map $\varphi$ in the triangle has a splitting given by $\pi^*$. Passing to cohomology, this implies that there are short exact sequences

$$0 \to \mathcal{H}^i \underline{\Omega}_X^p \to R^i f_* \Omega_{\widetilde{X}}^p \to R^i f_* \Omega_E^p \to 0 \tag{2.5.1}$$

except for

$$0 \to \mathcal{H}^0 \underline{\Omega}_X^0 \to f_* \mathscr{O}_{\widetilde{X}} \oplus \mathscr{O}_{\{0\}} \to f_* \mathscr{O}_E \to 0. \tag{2.5.2}$$

Let's first deal with (2.5.1). Since $X$ is affine, it follows that

$$\Gamma(X, \mathcal{H}^i \underline{\Omega}_X^p) \simeq \ker \big(\pi_* \colon H^i(Y, \pi_* \Omega_{\widetilde{X}}^p) \to H^i(Y, \Omega_Y^p)\big).$$

Hence our task is to describe the kernel on the right hand side. We will show in fact that there exists a split short exact sequence

$$0 \to \bigoplus_{m \geq 0} \Omega_Y^p \otimes L^{\otimes m} \to \pi_* \Omega_{\widetilde{X}}^p \to \bigoplus_{m \geq 1} \Omega_Y^{p-1} \otimes L^{\otimes m} \to 0, \tag{2.5.3}$$

which gives the result by passing to cohomology.

To this end, recall that we have the short exact sequence

$$0 \to \pi^* \Omega_Y^1 \to \Omega_{\widetilde{X}}^1 \to \Omega_{\widetilde{X}/Y}^1 \to 0,$$

and via the affine bundle structure we have $\Omega_{\widetilde{X}/Y}^1 \simeq \pi^* L$. By a standard argument, for any $p \geq 1$ this gives rise to the short exact sequence

$$0 \to \pi^* \Omega_Y^p \to \Omega_{\widetilde{X}}^p \to \pi^*(\Omega_Y^{p-1} \otimes L) \to 0.$$

Recall now that we have the formula

$$\pi_* \mathcal{O}_{\widetilde{X}} \simeq \bigoplus_{m \geq 0} L^{\otimes m}$$

given by the definition of $X$, hence pushing forward via $\pi$ and applying the projection formula we obtain a sequence as in (2.5.3). The splitting is given by

$$\bigoplus_{m \geq 1} \Omega_Y^{p-1} \otimes L^{\otimes m} \xrightarrow{d} \pi_* \Omega_{\widetilde{X}}^p \longrightarrow \bigoplus_{m \geq 1} \Omega_Y^{p-1} \otimes L^{\otimes m},$$

where $d$ is given by differentiation; a simple calculation shows that the composition is equal to multiplication by $m$ on the $m$-th summand.

We now deal with (2.5.2). Since $X$ is affine, taking global sections we get a short exact sequence

$$0 \to \Gamma(X, \mathcal{H}^0 \underline{\Omega}_X^0) \to H^0(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \oplus \mathbf{C} \to H^0(Y, \mathscr{O}_Y) \to 0.$$

The rest of the argument goes exactly as above, by noting that the first map in the composition

$$\mathscr{O}_Y \to \pi_* \mathscr{O}_{\widetilde{X}} \simeq \bigoplus_{m \geq 1} L^{\otimes m} \to \mathscr{O}_Y$$

is the inclusion of the summand corresponding to $m = 0$. ∎

# CHAPTER 3

# Du Bois singularities

## 3.1. Definition and first examples

In this section we always consider a (possible reducible) complex algebraic variety of dimension $n$.

Definition 3.1.1. We say that $X$ has *Du Bois* singularities if the canonical morphism $\varphi\colon \mathscr{O}_X \to \underline{\Omega}^0_X$ is an isomorphism in $\mathbf{D}^b_{\mathrm{coh}}(X)$.

Recall that in §1.3 we gave a preliminary definition in terms of resolution of singularities. Let's establish that the two definitions coincide (so consequently the preliminary definition is independent of the choice of resolution).

Let's embed $X$ (locally) in a smooth variety $Y$, and consider a strong log resolution $f\colon \widetilde{Y} \to Y$ of the pair $(Y, X)$, with $E = f^{-1}(X)_{\mathrm{red}}$. We have seen in §2.4(12) that there is an exact triangle

$$\mathscr{O}_Y \to \mathbf{R}f_*\mathscr{O}_{\widetilde{Y}} \oplus \underline{\Omega}^0_X \to \mathbf{R}f_*\underline{\Omega}^0_E \xrightarrow{+1} .$$

Since $Y$ is smooth, the natural map $\mathscr{O}_Y \to \mathbf{R}f_*\mathscr{O}_{\widetilde{Y}}$ is an isomorphism, while by §2.4(7). we have $\underline{\Omega}^0_E \simeq \mathscr{O}_E$. It follows that

$$\underline{\Omega}^0_X \simeq \mathbf{R}f_*\mathscr{O}_E. \tag{3.1.1}$$

This shows that the tow definitions agree, and provides a nice interpretation of $\underline{\Omega}^0_X$ in terms of a single (embedded) resolution of singularities. A strengthening of this interpretation that does not necessarily assume $f$ to be a log resolution was obtained by Schwede [**Sch**].

Example 3.1.2. Simple normal crossing divisors have Du Bois singularities by §2.4(7).

Example 3.1.3. We've seen in Example 2.2.7 that a nodal curve $C$ satisfies $\underline{\Omega}^0_C \simeq \mathscr{O}_C$, hence it has Du Bois singularities. (Locally it is of course a simple normal crossing divisor.) We've also seen in Example 2.2.6 that cuspidal curve satisfies $\underline{\Omega}^0_C \simeq f_*\mathscr{O}_{\widetilde{C}} \not\simeq \mathscr{O}_C$, where $\widetilde{C}$ is the normalization, hence it does not have Du Bois singularities.

Example 3.1.4. Let $X = C(Y) \subset \mathbf{C}^{n+1}$ be the cone over a smooth projective hypersurface $Y \subset \mathbf{P}^n$ of degree $d$. Then Exercise 1.3.2, using the alternative definition, says that $X$ has Du Bois singularities if and only if $d \le n+1$ (i.e. $Y$ is Fano or Calabi-Yau).

Remark 3.1.5. This also gives us the first examples of a varieties where $\underline{\Omega}^0_X$ has nontrivial higher cohomology. Indeed, if $Y$ has degree $d > n+1$, then $X$ is not Du Bois, but we have seen that $\mathcal{H}^0\underline{\Omega}^0_X \simeq \mathscr{O}_X$ (or see the discussion on seminormalization below).

This means that there must be another $i > 0$ such that $\mathcal{H}^i\underline{\Omega}^0_X \neq 0$. (In fact it turns out that the only such $i$ is $i = n-1$; can you see this?)

EXAMPLE 3.1.6. More generally, take $X = C(Y, L)$ as in §2.5, where $Y \subset \mathbf{P}^n$ is a smooth projective variety, and $L$ is an ample line bundle on $Y$. We have seen in Proposition 2.5.3 that $\mathcal{H}^0\underline{\Omega}^0_X \simeq \mathscr{O}_X$, while

$$\Gamma(X, \mathcal{H}^i\underline{\Omega}^0_X) \simeq \bigoplus_{m\geq 1} H^i(X, L^{\otimes m}) \quad \text{for all} \quad i > 0.$$

We conclude that

$$X \text{ is Du Bois} \iff H^i(X, L^{\otimes m}) = 0 \quad \text{for all} \quad i, m > 0.$$

Again, this holds automatically for any $L$ when $X$ is Fano or Calabi-Yau, or more generally with nef anticanonical bundle, by Kodaira Vanishing.

EXAMPLE 3.1.7. We have seen in §2.5 that if $X$ has quotient singularities we have $\underline{\Omega}^0_X \simeq \Omega^{[0]}_X = \mathscr{O}_X$, hence $X$ has Du Bois singularities.

## 3.2. Du Bois singularities are seminormal

We've seen in the previous example that Du Bois singularities are not necessarily normal, since they can appear in codimension one; take for instance a node in the plane, or the union of any two smooth divisors with transverse intersection. In this section we will see that we are however still not very far from normality. Let's introduce first some definitions in commutative algebra, in the restricted setting that is of interest to us here. For the material in this section and further references, see [?].

We fix a commutative ring $R$ which is a reduced finitely generated algebra over a field $k$.

DEFINITION 3.2.1. A ring extension $i\colon R \hookrightarrow S$, with $S$ a reduced $R$-algebra which is finitely generated as a module over $R$, is called *subintegral* if

- The induced $i\colon \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a bijection.
- For each $\mathfrak{p} \in \mathrm{Spec}(S)$, the induced morphism of residue fields $k(i^{-1}(\mathfrak{p})) \to k(\mathfrak{p})$ is an isomorphism.

Note that when $k$ is algebraically closed (like here, where we work over $\mathbf{C}$), the second condition is superfluous, so the important point is the bijection on spectra.

DEFINITION 3.2.2. Let $R \subseteq S$ be an extension of rings. The *seminormalization* of $R$ in $S$ is the (unique) largest subextension

$$R \subseteq R^{\mathrm{sn},S} \subseteq S$$

which is subintegral over $R$.

In particular, if $S = \bar{R}$, the integral closure of $R$ in its ring of fractions, we simply denote this by $R^{\mathrm{sn}}$ and call it the seminormalization of $R$. We say that $R$ is *seminormal* if $R = R^{\mathrm{sn}}$. If $X = \mathrm{Spec}(R)$ etc., we have morphisms

$$X^{\mathrm{n}} \to X^{\mathrm{sn}} \to X,$$

where $X^{\mathrm{n}}$ denotes the normalization of $X$, which are both isomorphisms on the residue fields (when $k = \bar{k}$), but in addition $X^{\mathrm{sn}} \to X$ is a bijection on points.

REMARK 3.2.3. One can show the following:

(i) $R$ is seminormal if and only if every subintegral extension $R \subseteq S$ is an isomorphism.

(ii) $R$ is seminormal if and only if $R_{\mathfrak{p}}$ is seminormal, for every prime ideal $\mathfrak{p}$ (or equivalently just every maximal ideal) in $R$.

How to check subintegrality or seminormality? Here is an important result of Swan and many others:

THEOREM 3.2.4. *(i) An ring $R$ is seminormal in an extension $R \subseteq S$ if and only if for every $x \in S$ such that $x^2, x^3 \in R$, we have $x \in R$.*

*(ii) $R$ is seminormal if and only if for every $x, y \in R$ such that $y^2 = x^3$, there exists a unique $z \in R$ such that $x = z^2$ and $y = z^3$.*

EXAMPLE 3.2.5. Let $C = (y^2 - x^3 = 0) \subseteq \mathbf{A}^2$ be a cusp. Its normalization corresponds to the morphism of rings

$$k[X,Y]/(Y^2 - X^3) \to k[T], \quad \bar{X} \mapsto T^2, \quad \bar{Y} \mapsto T^3.$$

In other words, the extension is

$$k[T^2, T^3] \subseteq k[T],$$

which is a subintegral extension. It follows that the seminormalization of the cusp is equal to its normalization; is pretty clear geometrically, since we know we have a bijection on points.

EXAMPLE 3.2.6. Let $C = (y^2 - x^2 - x^3 = 0) \subseteq \mathbf{A}^2$ be a node. For the same geometric reason, it should be clear that in this case the normalization does not coincide with the seminormalization. Indeed, the normalization corresponds to the morphism of rings

$$k[X,Y]/(Y^2 - X^2 - X^3) \to k[T], \quad \bar{X} \mapsto T^2 - 1, \quad \bar{Y} \mapsto T^3 - T.$$

Equivalently, the extension is

$$k[T^2 - 1, T^3 - T] \subseteq k[T].$$

and it is not hard to check that $k[T^2 - 1, T^3 - T]$ is subintegrally closed in $k[T]$. (Exercise!) Therefore the node is seminormal. This can be easily generalized to normal crossing divisors in arbitrary dimension.

EXERCISE 3.2.7. The seminormalization of a tacnode is a node.

Note that what this exercise suggests, and it is generally true, is that the seminormalization corresponds to making the self-intersections as transverse as possible. However, in order to achieve this, one may need to go to an ambient space of higher dimension. For instance:

EXERCISE 3.2.8. Show that the union of three concurrent lines in $\mathbf{A}^2$ is not seminormal, and its seminormalization is the union of the three coordinate axes in $\mathbf{A}^3$.

Here is another typical example, that shows a new phenomenon showing up: sometimes more complicated singularities are "limits" of normal crossing singularities.

EXAMPLE 3.2.9 (**Whitney's umbrella**). Let $X = (x^2 - y^2 z = 0) \subset \mathbf{C}^3$. This hypersurface is not normal, since it is singular in codimension 1. It has normal crossing singularities along the $z$ axis, everywhere except at the origin, where we get a so-called *pinch point*. We have a morphism of rings morphism of rings

$$k[X,Y]/(X^2 - Y^2 Z) \to k[U,V], \quad \bar{X} \mapsto UV, \quad \bar{Y} \mapsto U, \quad \bar{Z} \mapsto V^2,$$

or equivalently the extension

$$R = k[U, UV, V^2] \subseteq k[U,V] = S.$$

EXERCISE 3.2.10. Show that $R$ is seminormal.

The exercise shows that the Whitney umbrella is seminormal, and its normalization is smooth. An interesting point in this example is that to get a resolution of singularities, one needs to blow up the entire $z$-axis. Bowing up the origin will not yield any improvement; please see [**Kov3**, Example 6.8] for a nice explanation.

In fact we have an even better statement, at least once we take into account Proposition 3.2.13 below.

EXERCISE 3.2.11. The singularities of Whitney's umbrella are Du Bois.

We say that a variety $X$ is *seminormal* if all its local rings are seminormal; based on the property in Remark 3.2.3(ii), this is equivalent to requiring that there is an open cover of $X$ with affine open sets with seminormal coordinate rings.

We record a useful technical point:

LEMMA 3.2.12. *Let $X$ be a seminormal variety. Then for every open subset $U \subseteq X$, the ring $\mathscr{O}_X(U)$ is seminormal.*

PROOF. It suffices to assume $U = X$. Pick an affine open cover $X = \cup U_i$, where each $\mathscr{O}_X(U_i)$ is seminormal. Consider now $x, y \in \mathscr{O}_X(U)$ such that $y^2 = x^3$. Their images $x_i$ and $y_i$ in $\mathscr{O}_X(U_i)$ still satisfy $y_i^2 = x_i^3$, so by Theorem 3.2.4 there exist unique $z_i \in \mathscr{O}_X(U_i)$ such that $x_i = z_i^2$ and $y_i = z_i^3$. By uniqueness, the $z_i$ glue to a global section $z \in \mathscr{O}_X(U)$, satisfying $x = z^2$ and $y = z^3$. This section is unique since $\mathscr{O}_X(U)$ is reduced, hence again by Theorem 3.2.4, the ring $\mathscr{O}_X(U)$ is seminormal. ∎

We now come to the connection with Du Bois singularities. Here is the main result of the section, due to Saito and Schwede:

PROPOSITION 3.2.13. *For any complex variety $X$, we have*

$$\mathcal{H}^0 \underline{\Omega}_X^0 \simeq \mathscr{O}_{X^{\mathrm{sn}}}.$$

*In particular, Du Bois singularities are seminormal.*

Proof. The problem is local, so we may assume that $X$ is affine, embedded in a smooth variety $Y$. Consider a strong log resolution $f\colon \widetilde{Y} \to Y$ of the pair $(Y, X)$, with $E = f^{-1}(X)_{\mathrm{red}}$. We know that $\mathcal{H}^0\underline{\Omega}^0_X \simeq f_*\mathscr{O}_E$.

Since $E$ is an SNC divisor, it is seminormal. By Lemma 3.2.12, it follows that $f_*\mathscr{O}_E$ is a sheaf of seminormal rings. We now define $X' := \mathrm{Spec}(f_*\mathscr{O}_E)$, so that we have morphisms

$$E \xrightarrow{g} X' \xrightarrow{s} X$$

whose composition is $f$. (See [**Ha**, Ch.II, Ex.5.17].) Since $f$ is proper, it follows that $g$ is proper (see [**Ha**, Ch.II, 4.8]), and therefore it is surjective (since it is in any case dominant by construction). Moreover, $s$ is both affine and proper, hence finite; at the same time, since $f$ has connected fibers, $s$ has connected fibers as well. It follows that $s$ is a bijection on points, hence automatically the extension $\mathscr{O}_X(X) \to \mathscr{O}'_X(X')$ is subintegral, since we are working over $\mathbf{C}$. But $X'$ is seminormal, hence it is the seminormalization of $X$ (via $s$). We conclude that $f_*\mathscr{O}_E \simeq \mathscr{O}_{X^{\mathrm{sn}}}$. □

## 3.3. Connection with other types of singularities

We've seen in Theorem 1.2.21 that klt singularities are rational. The coarsest class of singularities appearing in the minimal model program are log canonical singularities. Kollár conjectured that these are Du Bois; this is now a well-known theorem due to Kollár and Kovács.

Theorem 3.3.1 ([**KK**, Theorem 1.4]). *If $(X, \Delta)$ is a log canonical pair, then $X$ has Du Bois singularities.*

Proving the full theorem is outside the scope of these lectures, but we will see a proof under the assumption that $X$ has Cohen-Macaulay singularities. While log canonical singularities suffice for running the minimal model program, Kollár and Kovács proved this for an even larger class of singularities, called semi-log canonical; these are the singularities of the varieties corresponding to points in the boundary of the moduli space of varieties of general type. Therefore the following observation of Du Bois-Jarreaud regarding families of varieties with Du Bois singularities is useful for moduli theory.

Proposition 3.3.2. *Let $f\colon X \to S$ be a flat projective morphism of complex varieties. If for all $s \in S$ the fiber $X_s$ has Du Bois singularities, then the higher direct images $R^i f_*\mathscr{O}_X$ are locally free for all $i$, and compatible with base change.*

The crucial point for this result is Hodge-theoretic: by the degeneration of the Hodge-to-de Rham type spectral sequence, if $Y$ is a projective variety, we have surjective maps

$$H^i(Y, \mathbf{C}) \to \mathbb{H}^i(Y, \underline{\Omega}^0_Y)$$

for all $i$. If in addition $Y$ has Du Bois singularities, then this is equivalent to the natural maps

$$H^i(Y, \mathbf{C}) \to H^i(Y, \mathscr{O}_Y)$$

being surjective.

Proof of Proposition 3.3.2. The inclusion $\mathbf{C}_X \to \mathscr{O}_X$ induces for each $s \in S$ a commutative diagram

$$\begin{array}{ccc} R^i f_* \mathbf{C}_X \otimes_{\mathbf{C}} \mathbf{C}(s) & \longrightarrow & R^i f_* \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathbf{C}(s) \\ \downarrow & & \downarrow \\ H^i(X_s, \mathbf{C}) & \longrightarrow & H^i(X_s, \mathscr{O}_{X_s}) \end{array}$$

By the remark preceding the proof, for every $s \in S$ the bottom horizontal map is surjective. Moreover, by the topological proper base change theorem, the left vertical map is an isomorphism. This implies that the right vertical map

$$\varphi_s \colon R^i f_* \mathscr{O}_X \otimes \mathbf{C}(s) \to H^i(X_s, \mathscr{O}_{X_s})$$

is surjective for all $s \in S$. Since this holds for all $i$, by the Cohomology and Base Change theorem (see [**Ha**, Theorem 12.11]) we obtain that $\varphi_s$ is an isomorphism for all $s$, hence $R^i f_* \mathscr{O}_X$ is locally free and compatible with base change. □

Since klt implies rational, but also implies log canonical which implies Du Bois, it is natural to ask whether rational implies Du Bois. This was in fact conjectured by Steenbrink, and is now a fact, shown by Kovács [**Kov1**] and Saito [**Sa3**].

Theorem 3.3.3. *If $X$ has rational singularities, then $X$ has Du Bois singularities.*

We will prove this result using a simplification of the original argument of Kovács, observed by Kovács-Schwede. It relies on the following technical injectivity theorem they proved, [**KoS1**, Theorem 3.3], which turns out to be important for many applications and generalizations.

Theorem 3.3.4. *For a variety $X$, the morphism*

$$\psi \colon \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) \to \mathbf{R}\mathcal{H}om(\mathscr{O}_X, \omega_X^\bullet) = \omega_X^\bullet,$$

*induced by dualizing the canonical morphism $\varphi \colon \mathscr{O}_X \to \underline{\Omega}_X^0$, is injective on cohomology.*

Let's assume this for the moment, and proceed towards the proof of Theorem 3.3.3. First, the theorem implies a more flexible criterion for detecting Du Bois singularities:

Corollary 3.3.5. *If the canonical morphism $\mathscr{O}_X \to \underline{\Omega}_X^0$ has a left inverse in $\mathbf{D}^b_{\mathrm{coh}}(X)$, then $X$ has Du Bois singularities.*

Proof. The hypothesis means that there exists a morphism $\nu \colon \underline{\Omega}_X^0 \to \mathscr{O}_X$ such that the composition

$$\mathscr{O}_X \xrightarrow{\varphi} \underline{\Omega}_X^0 \xrightarrow{\nu} \mathscr{O}_X$$

is an isomorphism. Dualizing this composition, we obtain an isomoprhism

$$\omega_X^\bullet \longrightarrow \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) \xrightarrow{\psi} \omega_X^\bullet.$$

This implies that $\psi$ is surjective on cohomology; it is however also injective on cohomology thanks to Theorem 3.3.4. Therefore $\psi$ is an isomorphism, and applying duality one more time we obtain the conclusion. □

This leads in turn to a criterion analogous to Theorem 1.2.14 for rational singularities.

COROLLARY 3.3.6. *If $f\colon Y \to X$ is a morphism such that $Y$ has Du Bois singularities and the induced morphism $\mathscr{O}_X \to \mathbf{R}f_*\mathscr{O}_Y$ has a left inverse, then $X$ has Du Bois singularities.*

PROOF. Apply the previous Corollary to the composition

$$\mathscr{O}_X \to \underline{\Omega}^0_X \to \mathbf{R}f_*\underline{\Omega}^0_Y \simeq \mathbf{R}f_*\mathscr{O}_Y \to \mathscr{O}_X.$$

The second morphism is given by functoriality, and the last morphism is the left inverse given by the hypothesis. We conclude that $\mathscr{O}_X \to \underline{\Omega}^0_X$ has a left inverse as well. □

We are now ready to show that rational singularities are Du Bois.

PROOF OF THEOREM 3.3.3. Let $f\colon \widetilde{X} \to X$ be a resolution of singularities. By hypothesis, the induced morphism

$$\mathscr{O}_X \to \mathbf{R}f_*\mathscr{O}_{\widetilde{X}}$$

is an isomorphism, hence we conclude by Corollary 3.3.6. □

We are left with proving the injectivity result, Theorem 3.3.4. First we need to study a technical point, namely the behavior of the 0-th Du Bois complex under cyclic covering constructions.

Assume that $X$ is projective, and consider a semiample line bundle $L$ on $X$. For some positive integer $m$, and divisor $D \in |L^{\otimes m}|$, we can construct the $m$-fold cyclic cover of $X$ branched along $D$, denoted

$$\pi\colon Y \to X.$$

See for instance [**EV**, §3]. One of the key properties of such a cover is that

$$\pi_*\mathscr{O}_Y \simeq \bigoplus_{i=0}^{m-1} L^{\otimes -i}.$$

If $X$ and $D$ are smooth, then $Y$ is also smooth; see e.g. [**La**, §4.1.B]. The key point for us is the following:

LEMMA 3.3.7. *With the notation above, if $D$ is a general divisor in $|L^{\otimes m}|$, we have*

$$\pi_*\underline{\Omega}^0_Y \simeq \underline{\Omega}^0_X \otimes \pi_*\mathscr{O}_Y \simeq \underline{\Omega}^0_X \otimes \Big(\bigoplus_{i=0}^{m-1} L^{\otimes -i}\Big).$$

PROOF. Let $\varepsilon_\bullet\colon X_\bullet \to X$ be a hyperresolution. For each $\varepsilon_i\colon X_i \to X$, set $L_i := \varepsilon_i^* L$, and $D_i := \varepsilon_i^* D$, which is still a general member of a basepoint-free linear system. Since $X_i$ is smooth, by Bertini $D_i$ is smooth as well, hence the $m$-fold cyclic cover $\pi_i\colon Y_i \to X_i$ branched along $D_i$ is smooth as well.

By construction, for each $i$ we obtain a commutative diagram

$$\begin{array}{ccc} Y_i & \xrightarrow{\pi_i} & X_i \\ {\scriptstyle \varepsilon_i^Y}\downarrow & & \downarrow{\scriptstyle \varepsilon_i} \\ Y & \xrightarrow{\pi} & X \end{array}$$

and it is not hard to check that the resulting $\varepsilon_\bullet^Y : Y_\bullet \to Y$ is a hyperresolution (exercise!). Therefore we have

$$\mathbf{R}\pi_*\underline{\Omega}_Y^0 \simeq \mathbf{R}\pi_*\mathbf{R}\varepsilon^Y_{\bullet\,*}\mathscr{O}_{Y_\bullet} \simeq \mathbf{R}\varepsilon_{\bullet*}\mathbf{R}\pi_{\bullet*}\mathscr{O}_{Y_\bullet} \simeq$$

$$\mathbf{R}\varepsilon_{\bullet*}\big(\mathscr{O}_{X_\bullet} \otimes \big(\bigoplus_{i=0}^{m-1} \varepsilon_\bullet^* L^{\otimes -i}\big)\big) \simeq \mathbf{R}\varepsilon_{\bullet*}\mathscr{O}_{X_\bullet} \otimes \big(\bigoplus_{i=0}^{m-1} L^{\otimes -i}\big) \simeq \underline{\Omega}_X^0 \otimes \big(\bigoplus_{i=0}^{m-1} L^{\otimes -i}\big).$$

□

Remark 3.3.8. It is straightforward to see that the induced morphism $\pi_*\mathscr{O}_Y \to \pi_*\underline{\Omega}_Y^0$ is compatible with the direct sum decompositions described above.

Corollary 3.3.9. *With the notation above, the natural morphisms*

$$H^j(X, L^{\otimes -i}) \to \mathbb{H}^j(X, \underline{\Omega}_X^0 \otimes L^{\otimes -i}),$$

*induced from the canonical $\mathscr{O}_X \to \underline{\Omega}_X^0$ by tensoring with $L^{\otimes -i}$ and passing to cohomology, are surjective for all $i, j \geq 0$.*

Proof. Fix $i \geq 0$. Let $m \gg 0$ such that $L^{\otimes}m$ is globally generated, and assume that $m > i$. Pick a general divisor $D \in |L^{\otimes m}|$, and consider the corresponding cyclic cover $\pi\colon Y \to X$ discussed above.

We now apply Hodge theory on $Y$, namely the fact that the composition

$$H^i(Y, \mathbf{C}) \to H^i(Y, \mathscr{O}_Y) \to \mathbb{H}^i(Y, \underline{\Omega}_Y^0)$$

is surjective. Therefore so is the morphism on the right. But Lemma 3.3.7 says that this is the same as the morphism

$$\bigoplus_{\ell=0}^{m-1} H^j(X, L^{\otimes -\ell}) \to \bigoplus_{\ell=0}^{m-1} \mathbb{H}^j(X, \underline{\Omega}_X^0 \otimes L^{\otimes -\ell}),$$

induced on each component by the morphism described in the statement. Since $i$ is among the $\ell$, we obtain the conclusion. □

Proof of Theorem 3.3.4. The statement is local, so we may first assume that $X$ is affine, and then by compactifying it we may assume it is projective. Take $L$ to be an ample line bundle on $X$. By Corollary 3.3.9, for every $i, j \geq 0$ the morphisms

$$H^j(X, L^{\otimes -i}) \to \mathbb{H}^j(X, \underline{\Omega}_X^0 \otimes L^{\otimes -i})$$

are surjective. Using the notation $\mathbf{D}(\underline{\Omega}_X^0) = \mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet)$ for simplicity, applying Gothendieck-Serre duality we deduce that the morphisms

$$\mathbb{H}^j(X, \mathbf{D}(\underline{\Omega}_X^0) \otimes L^{\otimes i}) \to \mathbb{H}^j(X, \omega_X^\bullet \otimes L^{\otimes i})$$

are injective, for all $i, j \geq 0$. Now for any object $A^\bullet$ in $\mathbf{D}^b_{\mathrm{coh}}(X)$, we have a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^j A^\bullet \otimes L^{\otimes i}) \implies \mathbb{H}^{p+q}(X, A^\bullet \otimes L^{\otimes i}),$$

and if $i \gg 0$, by Serre vanishing we deduce that

$$\mathbb{H}^j(X, A^\bullet \otimes L^{\otimes i}) \simeq H^0(X, \mathcal{H}^j A^\bullet \otimes L^{\otimes i}).$$

Thus for sufficiently large $i$ we deduce the existence of injective morphisms

$$H^0(X, \mathcal{H}^j \mathbf{D}(\underline{\Omega}^0_X) \otimes L^{\otimes i}) \to H^0(X, \mathcal{H}^j \omega^\bullet_X \otimes L^{\otimes i}).$$

By the same token, $i$ can be chosen sufficiently large so that the sheaves whose global sections we are taking are globally generated. This implies that we already have an inclusion at the level of sheaves, and then tensoring with $L^{\otimes -i}$ we obtain inclusions

$$\mathcal{H}^j \mathbf{D}(\underline{\Omega}^0_X) \hookrightarrow \mathcal{H}^j \omega^\bullet_X \quad \text{for all } j.$$

□

Theorem 3.3.4 has many other interesting applications. Let's first draw some obvious conclusions:

Corollary 3.3.10. *If $X$ is a Cohen-Macaulay variety of dimension $n$, then*

$$\mathcal{E}xt^j(\underline{\Omega}^0_X, \omega^\bullet_X) = 0 \quad \text{for} \quad j \neq -n.$$

Proof. This is clear from the injectivity theorem, since in the Cohen-Macaulay case $\omega^\bullet_X$ is a sheaf supported in degree $-n$. □

Reinterpreted in birational terms, this is a sort of Grauert-Riemenschneider type criterion for embedded resolutions of Cohen-Macaulay varieties.

Corollary 3.3.11. *Let $X$ be a Cohen-Macaulay variety, embedded with codimension $r$ in a smooth variety $Y$, and let $f \colon \widetilde{Y} \to Y$ be a strong log resolution of $Y$ with $E = f^{-1}(X)_{\mathrm{red}}$. Then*

$$R^i f_* \omega_E = 0 \quad \text{for} \quad j \neq r - 1.$$

Proof. Recall by (3.1.1) that $\underline{\Omega}^0_X \simeq \mathbf{R} f_* \mathscr{O}_E$. Applying Grothendieck duality to this isomorphism yields

$$\mathbf{R}\mathcal{H}om(\underline{\Omega}^0_X, \omega^\bullet_X) \simeq \mathbf{R} f_* \omega_E[\dim E].$$

The statement then follows from Corollary 3.3.10. □

Remark 3.3.12. Note that by the injectivity theorem we have an inclusion

$$R^{r-1} f_* \omega_E \hookrightarrow \omega_X \simeq \mathcal{H}^{-n} \omega^\bullet_X,$$

where $\omega_X$ is the dualizing sheaf of $X$. If $X$ is Gorenstein, this is a line bundle, hence we obtain an isomorphism

$$R^{r-1} f_* \omega_E \hookrightarrow \omega_X \otimes \mathcal{J}$$

where $\mathcal{J}$ is an ideal sheaf corresponding to a closed subscheme of $X$ supported on the non-Du Bois locus of $X$.

Here is another application to the vanishing of various cohomologies of $\underline{\Omega}_X^0$; this is [**PSV**, Corollary B].

COROLLARY 3.3.13. *Assume that a variety $X$ is Du Bois away from a closed subset of dimension $s$. Then*

$$\mathcal{H}^i\underline{\Omega}_X^0 = 0 \quad \text{for all} \;\; 0 < i < \operatorname{depth}(\mathscr{O}_X) - s - 1.$$

EXAMPLE 3.3.14. If $X$ is Cohen-Macaulay of dimension $n$, then $\operatorname{depth}(\mathscr{O}_X) = n$; if in addition it is Du Bois away from finitely many points (e..g. if it has isolated singularities), then we obtain

$$\mathcal{H}^i\underline{\Omega}_X^0 = 0 \quad \text{for all} \;\; 0 < i < n - 1.$$

Given what we already know about $\mathcal{H}^i\underline{\Omega}_X^0$, it follows that besides $i = 0$, the only other possible non-zero cohomology is for $i = n - 1$. The extends the phenomenon we have observed for cones over smooth hypersurfaces in $\mathbf{P}^n$.

PROOF OF COROLLARY 3.3.13. Dualizing the canonical map $\mathscr{O}_X \to \underline{\Omega}_X^0$, we obtain an exact triangle

$$\mathbf{R}\mathcal{H}om(\underline{\Omega}_X^0, \omega_X^\bullet) \to \omega_X^\bullet \to A^\bullet[n] \xrightarrow{+1},$$

where $n = \dim X$, and $A^\bullet[n]$ is simply convenient notation for the cone of the dual morphism.

By Theorem 3.3.4, we obtain short exact sequences

$$0 \to \mathcal{E}xt^i(\underline{\Omega}_X^0, \omega_X^\bullet) \to \mathcal{H}^i\omega_X^\bullet \to \mathcal{H}^{i+n}A^\bullet \to 0.$$

By general commutative algebra we know that we can have $\mathcal{H}^i\omega_X^\bullet \neq 0$ only in the interval $[-n, -d]$, where $d = \operatorname{depth}(\mathscr{O}_X)$, hence $\mathcal{H}^k A^\bullet$ can be $\neq 0$ only for $k \in [0, n - d]$.

Dualizing the triangle back, we get an exact triangle

$$\mathbf{R}\mathcal{H}om(A^\bullet[n], \omega_X^\bullet) \to \mathscr{O}_X \to \omega_X^\bullet \xrightarrow{+1},$$

so we have

$$\mathcal{H}^i\underline{\Omega}_X^0 \simeq \mathcal{E}xt^{i+1-n}(A^\bullet, \omega_X^\bullet), \quad \text{for all} \;\; i > 0. \tag{3.3.1}$$

Now there is a standard spectral sequence converging to $\mathcal{E}xt^{i+1-n}(A^\bullet, \omega_X^\bullet)$, whose $E_2$ terms are

$$E_2^{i+1-n+k,k} := \mathcal{E}xt^{i+1-n+k}(\mathcal{H}^k A^\bullet, \omega_X^\bullet).$$

We claim that for $i < d - s - 1$, all these terms are 0, which gives us the conclusion we want.

To see this, note first that by general commutative algebra, for any coherent sheaf $\mathscr{F}$ on $X$, we have

$$\mathcal{E}xt^j(\mathscr{F}, \omega_X^\bullet) = 0, \quad \text{for all} \;\; j < -\dim \operatorname{Supp}(\mathscr{F}).$$

Since all $\mathcal{H}^k A^\bullet$ are supported in dimension at most $s$, this means that

$$E_2^{i+1-n+k,k} = 0 \quad \text{for} \;\; i < n - k - s - 1.$$

On the other hand, we know that $\mathcal{H}^k A^\bullet$ can only contribute for $k \in [0, n-d]$. Combining these two facts, it follows that all the $E_2$ terms are 0 if $i < d - s - 1$, which implies what we want thanks to (3.3.1). □

Here is an amusing consequence of this last Corollary.

Corollary 3.3.15. *Let $X$ be a projective seminormal Cohen-Macaulay variety of dimension $n$, with isolated singularities, or more generally Du Bois away from a finite set of points. If $H^n(X, \mathscr{O}_X) = 0$, then $X$ is Du Bois.*

*More precisely, we have $h^n(X, \mathscr{O}_X) \geq h^n(X, \underline{\Omega}^0_X)$, and $X$ is Du Bois $\iff$ $h^n(X, \mathscr{O}_X) = h^n(X, \underline{\Omega}^0_X)$ $\iff$ the natural map $H^n(X, \mathscr{O}_X) \to \mathbb{H}^n(X, \underline{\Omega}^0_X)$ is injective (hence an isomorphism).*

Proof. We consider the cone

$$\mathscr{O}_X \to \underline{\Omega}^0_X \to C^\bullet \xrightarrow{+1} .$$

Note that $C^\bullet$ is supported on a finite set. We clearly have $\mathcal{H}^i C^\bullet = 0$ for $i \leq 0$ (since the seminormality condition is equivalent to $\mathscr{O}_X \simeq \mathcal{H}^0 \Omega^0_X$), while $\mathcal{H}^i C^\bullet \simeq \mathcal{H}^i \underline{\Omega}^0_X$ for $i \geq 1$. In particular, by (5) in §2.3 and Proposition 2.4.2 we have $\mathcal{H}^i C^\bullet = 0$ for $i > n-1$. Moreover, since $X$ is Cohen-Macaulay, by Corollary 3.3.13 we have $\mathcal{H}^i C^\bullet = 0$ for $i < n - 1$.

Note now that we have a short exact sequence

$$0 \to \mathbb{H}^{n-1}(X, C^\bullet) \to H^n(X, \mathscr{O}_X) \to \mathbb{H}^n(X, \underline{\Omega}^0_X) \to 0.$$

The last map is surjective thanks to the degeneration of the Hodge-to-de Rham spectral sequence, as in §2.3 (4), since it sits in the surjective composition

$$H^n(X, \mathbf{C}) \to H^n(X, \mathscr{O}_X) \to \mathbb{H}^n(X, \underline{\Omega}^0_X).$$

The hypercohomology group $\mathbb{H}^{n-1}(X, C^\bullet)$ is computed by a spectral sequence whose $E_2$-terms are

$$E_2^{p,q} = H^p(X, \mathcal{H}^q C^\bullet), \quad \text{with} \ \ p + q = n - 1.$$

Since $\mathcal{H}^i C^\bullet = 0$ for $i \neq n - 1$, this gives

$$H^n(X, \mathscr{O}_X) \simeq \mathbb{H}^n(X, \underline{\Omega}^0_X) \iff H^0(X, \mathcal{H}^{n-1} C^\bullet) = 0.$$

As the support of $C^\bullet$ is finite, this last condition is equivalent to $\mathcal{H}^{n-1} C^\bullet = 0$, hence to $C^\bullet = 0$, i.e. to $X$ being Du Bois. □

For instance this applies to any low degree normal complete intersection with isolated singularities in $\mathbf{P}^N$.

## 3.4. Behavior with respect to hypersurfaces

Let $H$ be an effective Cartier divisor in a variety $X$, meaning that locally $H$ is defined by the vanishing of one equation. A hyperresolution $\varepsilon_\bullet \colon X_\bullet \to X$ can be put in a diagram

$$\begin{array}{ccccc} H_\bullet & \longrightarrow & H'_\bullet := X_\bullet \times_X H & \xrightarrow{\ j\ } & X_\bullet \\ & \searrow^{\varepsilon_\bullet} & \downarrow{\scriptstyle \varepsilon'_\bullet} & & \downarrow{\scriptstyle \varepsilon_\bullet} \\ & & H & \xrightarrow{\ \ j\ \ } & X \end{array}$$

where $H'_\bullet$ is given by the component-wise fiber products $H \times_X X_i$, and $H_\bullet$ is a hyperresolution of $H$ factoring through $H'_\bullet$, which can be constructed by the general technique of [**GNPP**]. We then have a sequence of morphisms

$$\underline{\Omega}^\bullet_X|_H = \mathbf{L}j^*\mathbf{R}\varepsilon_{\bullet *}\Omega^\bullet_{X_\bullet} \to \mathbf{R}\varepsilon'_{\bullet *}\mathbf{L}j^*\Omega^\bullet_{X_\bullet} \to \mathbf{R}\varepsilon'_{\bullet *}\Omega^\bullet_{H'_\bullet} \to \mathbf{R}\varepsilon_{\bullet *}\Omega^\bullet_{H_\bullet} = \underline{\Omega}^\bullet_H,$$

where the first morphism follows from the the push-pull formula, while the others are given by functoriality. In other words, we have an induced factorization

$$\underline{\Omega}^\bullet_X|_H \to \mathbf{R}\varepsilon'_{\bullet *}\Omega^\bullet_{H'_\bullet} \to \underline{\Omega}^\bullet_H. \tag{3.4.1}$$

Proposition 3.4.1. *If $X$ is a quasi-projective variety with Du Bois singularities, and $H$ is a general hyperplane section of $X$, then $H$ has Du Bois singularities.*

Proof. If $H$ is a general member of a basepoint-free linear system, then each $H'_i$ is a smooth divisor in $X_i$, and $H'_\bullet$ is already a hyperresolution of $H$ (exercise!). Therefore we can take $H_\bullet = H'_\bullet$, and since $j^*\mathscr{O}_{X_i} \simeq \mathscr{O}_{H_i}$, it is easy to deduce from the chain of morphisms above that we actually have

$$\underline{\Omega}^0_H \simeq \underline{\Omega}^0_X|_H.$$

Since $\underline{\Omega}^0_X \simeq \mathscr{O}_X$, it follows immediately that $\underline{\Omega}^0_H \simeq \mathscr{O}_H$. □

Exercise 3.4.2. If $X$ is a quasi-projective variety with rational singularities, and $H$ is a general hyperplane section of $X$, then $H$ has rational singularities.

Can we go backwards? If phrased appropriately, the answer is yes; this type of result usually goes under the name “inversion of adjunction”. Inspired by Elkik’s proof of a similar result in the case of rational singularities, Kovács and Schwede [**KoS1**] showed the following:

Theorem 3.4.3. *Let $H$ be an effective Cartier divisor with Du Bois singularities, on a variety $X$. Then $X$ has Du Bois singularities in a neighborhood of $H$.*

Proof. We use the discussion at the beginning of the section. Denoting $C^\bullet := \mathbf{R}\varepsilon'_{\bullet *}\Omega^\bullet_{H'_\bullet}$ for simplicity, by (3.4.1) we have a factorization

$$\underline{\Omega}^0_X|_H \to C^\bullet \to \underline{\Omega}^0_H.$$

In particular, applying the functor $\mathbf{D}(\,\cdot\,) = \mathbf{R}\mathcal{H}om(\,\cdot\,, \omega^\bullet_X)$, we obtain a morphism $\mathbf{D}(\underline{\Omega}^0_H) \to \mathbf{D}(C^\bullet)$.

The problem is local, so we may assume that $X$ is affine, and $H$ is given by the vanishing of a function $f$. Since the action of $f$ can be lifted to the hyperresolution $X_\bullet$, we have an induced commutative diagram of exact triangles

$$\begin{array}{ccccccc}
\mathscr{O}_X & \xrightarrow{\cdot f} & \mathscr{O}_X & \longrightarrow & \mathscr{O}_H & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\underline{\Omega}_X^0 & \xrightarrow{\cdot f} & \underline{\Omega}_X^0 & \longrightarrow & C^\bullet & \xrightarrow{+1} &
\end{array}$$

Dualizing this diagram, we obtain exact triangles

$$\begin{array}{ccccccc}
\mathbf{D}(C^\bullet) & \longrightarrow & \mathbf{D}(\underline{\Omega}_X^0) & \xrightarrow{\cdot f} & \mathbf{D}(\underline{\Omega}_X^0) & \xrightarrow{+1} & \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} & & \\
\mathbf{D}(\mathscr{O}_H) & \longrightarrow & \omega_X^\bullet & \xrightarrow{\cdot f} & \omega_X^\bullet & \xrightarrow{+1} &
\end{array}$$

Now the main ingredient is Theorem 3.3.4, which says that $\psi$ is injective on cohomology. On the other hand, we have a composition of morphisms

$$\mathbf{D}(\underline{\Omega}_H^0) \to \mathbf{D}(C^\bullet) \to \mathbf{D}(\mathscr{O}_H),$$

where the first morphism arises from the discussion in the first paragraph of the proof, while the second is $\nu$. It is an easy check that the composition is the canonical morphism induced by $\mathscr{O}_H \to \underline{\Omega}_H^0$; since this is an isomorphism by hypothesis, it follows that $\nu$ is surjective on cohomology.

Putting all this together, we obtained that $\psi$ is an isomorphism as a consequence of the general Lemma below. $\square$

LEMMA 3.4.4. *Let $R$ be a reduced Noetherian local ring, and $f \in R$ a non-invertible element. Consider finitely generated $R$-modules $A_i, A_i', B_i, B_i'$, with $i \in I$ a finite set, such that there is a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc}
A_i & \xrightarrow{\beta_i} & B_i & \xrightarrow{\cdot f} & B_i & \xrightarrow{\alpha_i} & A_{i+1} & \xrightarrow{\beta_{i+1}} & \cdots \\
\downarrow{\scriptstyle \nu_i} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \nu_{i+1}} & & \\
A_i' & \xrightarrow{\beta_i'} & B_i' & \xrightarrow{\cdot f} & B_i' & \xrightarrow{\alpha_i'} & A_{i+1}' & \xrightarrow{\beta_{i+1}'} & \cdots
\end{array}$$

*with $\nu_i$ surjective, and $\varphi_i$ and $\psi_i$ injective, for all $i$. Then $\psi_i$ is an isomorphism for all $i$.*

PROOF. Let $z \in B_i'$. By assumption, there exists $w \in A_{i+1}$ such that

$$\nu_{i+1}(w) = \alpha_i'(z).$$

Since $\varphi_{i+1}$ is injective, we have $\beta_{i+1}(w) = 0$, so there exists $t \in B_i$ such that $w = \alpha_i(t)$. Therefore

$$\alpha_i'(\psi_i(t)) = \nu_{i+1}(\alpha_i(t)) = \alpha_i'(z)$$

and consequently

$$z - \psi_i(t) \in \ker(\alpha_i') = f \cdot B_i'.$$

Denoting $K_i = \operatorname{coker}(\psi_i)$, this implies that $K_i = f \cdot K_i$. Thanks to Nakayama's Lemma, we deduce that $K_i = 0$. □

A quick consequence of Theorem 3.4.3 is the fact that Du Bois singularities are preserved under deformation.

COROLLARY 3.4.5. *Let $f\colon X \to C$ be a proper flat morphism, with $C$ a smooth curve. If the fiber $X_t$ has Du Bois singularities for some $t \in C$, then $X_s$ has Du Bois singularities for all $s$ in a neighborhood of $t$.*

PROOF. We have that $X_t$ is a Cartier divisor, so by Theorem 3.4.3, there exists an open set $X_t \subset U$ in $X$ that has Du Bois singularities. If $Z = X \smallsetminus U$, then $f(Z)$ is a closed subset of $C$ which is proper (since it does not contain $t$). Throwing $Z$ and its preimage away, we may then assume from the beginning that $X$ has Du Bois singularities. Since the fibers of $f$ form a basepoint-free linear system, the general one then has Du Bois singularities by Proposition 3.4.1. □

CHAPTER 4

# Higher Du Bois and rational singularities

## 4.1. First definitions and examples

**Higher Du Bois singularities.** We introduce the following generalization of the notion of Du Bois singularities. This should be seen as a temporary definition only, as it will need to be modified in the case of varieties that are not local complete intersections (LCI).

Definition 4.1.1. Let $X$ be a complex variety. For an integer $m \geq 0$, we say that $X$ has *$m$-Du Bois singularities* if the canonical morphisms

$$\Omega_X^p \to \underline{\Omega}_X^p$$

are isomorphisms for all $p \leq m$.

Sometimes it is convenient to refer to this condition for $m < 0$, when it simply means that we impose no conditions.

Since the issues in the non-LCI case will arise from the behavior of the 0-th cohomology, we also introduce the following concept:

Definition 4.1.2. The variety $X$ has *pre-$m$-Du Bois singularities* if the canonical morphisms

$$\mathcal{H}^0\underline{\Omega}_X^p \to \underline{\Omega}_X^p$$

are isomorphisms for all $p \leq m$. Equivalently, this says that $\mathcal{H}^i\underline{\Omega}_X^p = 0$ for all $i > 0$ and $p \leq m$.

Example 4.1.3. Clearly 0-Du Bois is the same as Du Bois. Moreover, by Proposition 3.2.13, a variety $X$ is Du Bois if and only if it is pre-0-Du Bois and seminormal.

A cuspidal curve is pre-0-Du Bois, but it is not Du Bois, since it is not seminormal.

Example 4.1.4 (**Quotient singularities**). From Section 2.5, we know that quotient singularities are pre-$m$-Du Bois for all $m$.

Example 4.1.5 (**Toric varieties**). By ???, toric varieties are pre-$m$-Du Bois for all $m$.

Example 4.1.6 (**Cones**). Proposition 2.5.3 implies that an abstract cone $X = C(Y, L)$, with $Y \subset \mathbf{P}^N$ a smooth projective variety and $L$ an ample line bundle on $Y$, is pre-$m$-Du Bois if and only if

$$H^i(Y, \Omega_Y^p \otimes L^{\otimes \ell}) = 0, \quad \text{for all} \;\; i > 0, \ell > 0, p \leq m.$$

The question of when this happens is interesting and quite deep. Note first that by Nakano vanishing we know that it holds whenever $p > n - i$, where $n = \dim Y$; however here we are interested in small values of $p$.

As an example, when $X$ is a surface and we are interested in the pre-1-Du Bois condition, given the remark above, the only thing we need to worry about is the vanishing of cohomology groups of the form

$$H^1(Y, \Omega^1_Y \otimes A)$$

where $A$ is an ample line bundle on $Y$. We will see in the next example that this may or may not happen, even when we stay within a fixed class, like that of $K3$ surfaces. First we recall the following:

Definition 4.1.7. A smooth projective variety *satisfies Bott vanishing* if for every ample line bundle $A$ on $Y$ we have

$$H^i(Y, \Omega^p_Y \otimes A) = 0 \quad \text{for all} \quad i > 0, p \geq 0.$$

Corollary 4.1.8. *If $Y$ satsifies Bott vanishing, then $X = C(Y, L)$ is pre-$m$-Du Bois for all $m$, for any ample line bundle $L$ on $Y$.*

Example 4.1.9. Here are some examples of varieties satisfying Bott vanishing. The list is certainly not complete.

Perhaps the most obvious class of varieties that satisfies Bott vanishing is that of abelian varieties, when all $\Omega^p_Y$ are trivial vector bundles. The projective space $\mathbf{P}^n$ satisfies Bott vanishing; this was first shown by Bott [?], and is the reason for the terminology. More generally, toric varieties satisfy Bott vanishing.

In recent years, Totaro has studied this question extensively. He showed that Del Pezzo surfaces of degre at least 5 satisfy Bott vanishing [?], and so do 37 types of Fano threefolds [?]. He also showed that $K3$ surfaces with Picard number 1 satisfy Bott vanishing iff their degree is 20 or $\geq 24$.

It is much harder to specify when a variety is $m$-Du Bois, even when we know that it is pre-$m$-Du Bois. The extra condition that needs to be satisfied is that the natural maps

$$\Omega^p_X \to \mathcal{H}^0 \underline{\Omega}^p_X$$

are isomorphisms for all $p \leq m$. By (9) in Section 2.4, an immediate obstruction is that $\Omega^p_X$ should be a torsion-free sheaf for all $p \leq m$, and even reflexive when $X$ has rational singularities.

For repeated use, we make the following:

Definition 4.1.10. Let $\mathscr{F}$ be a coherent sheaf on $X$, and consider the natural morphism $\varphi \colon \mathscr{F} \to \mathscr{F}^{\vee\vee}$. Then the torsion and cotorsion sheaf of $\mathscr{F}$ are, respectively,

$$\mathrm{tor}(\mathscr{F}) := \ker(\varphi) \quad \text{and} \quad \mathrm{cotor}(\mathscr{F}) := \mathrm{coker}(\varphi).$$

EXAMPLE 4.1.11. When $X$ is reducible, the forms supported on $X_{\text{sing}}$ are torsion. For example, when $X = E$ is an SNC divisor, we have seen in Exercise 2.4.7 that

$$\underline{\Omega}_E^p \simeq \Omega_E^p/\text{tor}(\Omega_E^p) \quad \text{for all } p,$$

and $\text{tor}(\Omega_E^p) \neq 0$ for all $p \geq 1$, so $E$ is not $m$-Du Bois for any $m \geq 1$.

EXAMPLE 4.1.12 (**Hypersurfaces**). Let $X$ be an $n$-dimensional hypersurface in a smooth variety $Y$, and say $c = \operatorname{codim} X_{\text{sing}}$. Buliding on work of Vetter and Greuel, Graf proved the following precise result regarding the torsion and cotorsion of the sheaves of Kähler differentials on $X$.

THEOREM 4.1.13 ([**Gr**, Theorem 1.11]). *With the notation above, we have*

$$\text{tor}(\Omega_X^p) = 0 \iff p \leq c - 1$$

*and*

$$\text{cotor}(\Omega_X^p) = 0 \iff p \leq c - 2.[1]$$

In other words, the first part of the theorem says that $\Omega_X^p$ is torsion-free if and only if $p \leq c-1$. For instance, if $X$ is normal, then we are only guaranteed the torsion-freeness of $\Omega_X^1$. If $X$ is $m$-Du Bois, the theorem implies the obstruction $c \geq m+1$. In fact much more is true; one can show with more sophisticated methods that $c \geq 2m+1$.

EXAMPLE 4.1.14. Let $X = (x_1^2 + x_2^2 + x_3^2 = 0) \subset \mathbf{C}^3$ be the cone over a smooth plane conic. In this case $c = 2$, so $\Omega_X^1$ is torsion-free. Moreover, $X$ is pre-1-Du Bois by Corollary 4.1.8. On the other, $X$ is not 1-Du Bois, since $\Omega_X^1$ is not reflexive; we have seen in Example 2.2.8 that $\underline{\Omega}_X^1 \simeq f_*\Omega_{\widetilde{X}}^1$, where $f\colon \widetilde{X} \to X$ is a resolution of singularities, and this can be shown to be reflexive by direct calculation. As a high-brow alternative, $X$ has rational singularities, and therefore (10) in Section 2.4 says that $\mathcal{H}^0\underline{\Omega}_X^1$ has to be reflexive.

It turns out that higher Du Bois hypersurface singularities can be characterized numerically. I will only state the main result for now, but will try to address at least some of it later; the full result uses a lot of technology that we have not discussed in this class, like Bernstein-Sato polynomials and mixed Hodge module theory.

THEOREM 4.1.15. *Let $X$ be a reduced hypersurface in a smooth variety $Y$. The following are equivalent:*

*(1) $X$ is $m$-Du Bois*
*(2) $\widetilde{\alpha}(X) \geq m+1$, where $\widetilde{\alpha}(X)$ is the minimal exponent of $X$ (i.e. the negative of the largest root of the reduced Bernstein-Sato polynomial of $X$).*
*(3) $I_p(X) = \mathscr{O}_Y$ for all $p \leq m$, where $I_p(X)$ is the $p$-th Hodge ideal of $X$.*
*(4) $I_m(X) = \mathscr{O}_X$.*

The proof of this theorem spans many papers. Initially, the equivalence between (3) and (4) was shown in [**MP1**, Proposition 13.1], and their equivalence with (2) was shown in [**Sa4**, Corollary 1]. More recently, and most importantly for the discussion here,

[1]The result is in fact more precise; see *loc. cit.*

the implication (2) $\Longrightarrow$ (1) was shown in [**MOPW**, Theorem A], while the converse (1) $\Longrightarrow$ (2) was subsequently shown in [**JKSY**] (which also introduced the terminology *$m$-Du Bois*!).

The strength of this restul stems from the fact that the minimal exponent can sometimes be computed explicitly.

EXAMPLE 4.1.16. When $X$ is the zero locus of a weighted homogeneous polynomial in $X_1, \ldots, X_n$, with weights $w_1, \ldots, w_n$ such that the total degree is 1, and with isolated singularities, then by [?] we have

$$\widetilde{\alpha}(X) = w_1 + \cdots + w_n.$$

For instance, if $X$ is the diagonal hypersurface $X_1^{a_1} + \cdots + X_n^{a_n} = 0$, with all $a_i \geq 2$, then

$$\widetilde{\alpha}(X) = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

Thus for the cone over a quadric, when $a_1 = \cdots = a_n = 2$, for $n = 3$ the minimal exponent is $3/2$, hence as we discussed in earlier example $X$ is not 1-Du Bois, but as soon as $n \geq 4$, $X$ does become 1-Du Bois.

EXAMPLE 4.1.17. When $X$ is an ordinary singular point (meaning its projectivized tangent cone is smooth) of multiplicity $m$, for instance the cone over a smooth projective hypersurface of degree $m$ in $\mathbf{P}^{n-1}$, then $\widetilde{\alpha}(X) = n/m$; see [?].

**Duality morphism.** If $Y$ is a smooth variety, for each $p \geq 0$ we have an obvious isomorphism

$$\Omega_Y^p \simeq \mathcal{H}om(\Omega_Y^{n-p}, \omega_Y) \simeq \mathbf{R}\mathcal{H}om(\Omega_Y^{n-p}, \omega_Y).$$

We cannot hope for something this strong in the case of Du Bois complexes, but at the very least we have canonical morphisms arising from duality.

PROPOSITION 4.1.18. *For every irreducible variety $X$ of dimension $n$ and every $p \geq 0$, there is a canonical morphism*

$$\psi_p \colon \underline{\Omega}_X^p \longrightarrow \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(\underline{\Omega}_X^{n-p}, \omega_X^\bullet[-n]\big)$$

*in the bounded derived category of coherent sheaves on $X$.*

PROOF. Let $f \colon \widetilde{X} \to X$ be any resolution of singularities. The functoriality of the Du Bois complex gives a morphism

$$\alpha_p \colon \underline{\Omega}_X^p \longrightarrow \mathbf{R}f_*\Omega_{\widetilde{X}}^p$$

for each $p$. On the other hand, on $\widetilde{X}$ we have the isomorphism

$$\tau_p^{\widetilde{X}} \colon \Omega_{\widetilde{X}}^p \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\widetilde{X}}}\big(\Omega_{\widetilde{X}}^{n-p}, \omega_{\widetilde{X}}\big)$$

and we get an isomorphism $\beta_p$ on $X$ as the composition

$$\mathbf{R}f_*\Omega_{\widetilde{X}}^p \xrightarrow{\simeq} \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathscr{O}_{\widetilde{X}}}\big(\Omega_{\widetilde{X}}^{n-p}, \omega_{\widetilde{X}}\big) \xrightarrow{\simeq} \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}\big(\mathbf{R}f_*\Omega_{\widetilde{X}}^{n-p}, \omega_X^\bullet[-n]\big),$$

where the first isomorphism is $\mathbf{R}f_*(\tau_p^{\widetilde{X}})$ and the second isomorphism is provided by relative duality for $f$. Finally, taking the Grothendieck dual of $\alpha_{n-p}$ provides a morphism

$$\gamma_p\colon \mathbf{R}\mathcal{H}om\big(\mathbf{R}f_*\Omega_{\widetilde{X}}^{n-p}, \omega_X^\bullet[-n]\big) \longrightarrow \mathbf{R}\mathcal{H}om\big(\underline{\Omega}_X^{n-p}, \omega_X^\bullet[-n]\big).$$

We define

$$\psi_p := \gamma_p \circ \beta_p \circ \alpha_p.$$

We need to show that this is independent of the choice of resolution. Since any two resolutions are dominated by a third one, it is enough to show that if $f$ is as above and $g\colon W \to \widetilde{X}$ is a proper birational morphism, with $W$ smooth, then the morphisms $\psi_p$ corresponding to $f$ and $h = f \circ g$ coincide. This follows easily from the definitions once we know that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{R}f_*\Omega_{\widetilde{X}}^p & \longrightarrow & \mathbf{R}\mathcal{H}om\big(\mathbf{R}f_*\Omega_{\widetilde{X}}^{n-p}, \omega_X^\bullet[-n]\big) \\ \downarrow & & \uparrow \\ \mathbf{R}h_*\Omega_W^p & \longrightarrow & \mathbf{R}\mathcal{H}om\big(\mathbf{R}h_*\Omega_W^{n-p}, \omega_X^\bullet[-n]\big), \end{array}$$

in which the top horizontal map is the $\beta_p$ with respect to $f$ and the bottom map is $\beta_p$ with respect to $h$. This commutativity follows from the functoriality of relative duality and its compatibility with composition of proper morphisms, together with the commutativity of the diagram

$$\begin{array}{ccc} \Omega_{\widetilde{X}}^p & \xrightarrow{\tau_p^{\widetilde{X}}} & \mathbf{R}\mathcal{H}om_{\mathscr{O}_{\widetilde{X}}}\big(\Omega_{\widetilde{X}}^{n-p}, \omega_{\widetilde{X}}\big) \\ \downarrow & & \uparrow \\ \mathbf{R}g_*\Omega_W^p & \xrightarrow{\mathbf{R}g_*(\tau_p^W)} & \mathbf{R}g_*\mathbf{R}\mathcal{H}om_{\mathscr{O}_W}\big(\Omega_W^{n-p}, \omega_W\big). \end{array}$$

The latter follows in turn from the fact that it trivially holds over any open subset over which $g$ is an isomorphism (note that we are comparing two morphisms between vector bundles). This completes the proof of the proposition. ☐

Remark 4.1.19. It's quite clear that having $\psi_p$ be an isomorphism for some $p$ requires assumptions on the singularities. For instance, when $p = 0$, even when $X$ is Cohen-Macaulay and Du Bois, $\psi_0$ being an isomorphism is equivalent to the condition $f_*\omega_{\widetilde{X}} \simeq \omega_X$, where $f\colon \widetilde{X} \to X$ is a resolution of singularities; in other words, it is equivalent to $Z$ having rational singularities.

**Higher rational singularities.** By analogy with the Du Bois case, we temporarily introduce the following:

Definition 4.1.20. Let $X$ be an irreducible complex variety of dimension $n$. For an integer $m \geq 0$, we say that $X$ has *m-rational singularities* if the morphisms $\Omega_X^p \to \mathbf{D}(\underline{\Omega}_X^{n-p})$ obtained as the canonical compositions

$$\Omega_X^p \to \underline{\Omega}_X^p \to \mathbf{D}(\underline{\Omega}_X^{n-p})$$

are isomorphisms for all $p \leq m$.

Sometimes it is convenient to refer to this condition for $m < 0$, when it simply means that we impose no conditions.

Again, this condition will need to be modified when $X$ is not LCI.

DEFINITION 4.1.21. The variety $X$ has *pre-$m$-rational singularities* if the canonical morphisms

$$\mathcal{H}^0\mathbf{D}(\underline{\Omega}_X^{n-p}) \to \underline{\Omega}_X^{n-p}$$

are isomorphisms for all $p \leq m$. Equivalently, this says that $\mathcal{H}^i\mathbf{D}(\underline{\Omega}_X^{n-p}) = 0$ for all $i > 0$ and $p \leq m$.

EXAMPLE 4.1.22. Since we have

$$\underline{\Omega}_X^n \simeq f_*\omega_{\widetilde{X}} \simeq \mathbf{R}f_*\omega_{\widetilde{X}}$$

for any resolution of singularities $f\colon \widetilde{X} \to X$, by Grothendieck duality we have

$$\mathbf{D}(\underline{\Omega}_X^n) \simeq \mathbf{R}f_*\mathscr{O}_{\widetilde{X}}.$$

Therefore $X$ has pre-0-rational singularities if and only if $R^if_*\mathscr{O}_{\widetilde{X}} = 0$ for all $i > 0$.

Moreover, $X$ has rational singularities if and only if $X$ has 0-rational singularities, if and only if $X$ has pre-0-rational singularities and is normal.

EXAMPLE 4.1.23. We will see soon that quotient singularities are pre-$m$-rational for all $m$. Therefore so are simplicial toric varieties. However, non-simplicial toric varieties are not even pre-1-rational; see [**SVV1**, §6.2] and [**SVV2**].

Theorem 4.1.15 has an analogue for higher rational singularities; I only state the part in terms of the minimal exponent. This is shown in [**MP3**, Theorem E] and [**Sa5**, Theorem A.1].

THEOREM 4.1.24. *Let $X$ be a reduced hypersurface in a smooth variety $Y$. The following are equivalent:*

*(1) $X$ is $m$-rational*
*(2) $\widetilde{\alpha}(X) > m + 1$.*

Combining the two theorems implies the following extension of "rational implies Du Bois" in the case of hypersurface singularities:

COROLLARY 4.1.25. *If a hypersurface $X$ has $m$-rational singularities, then it has $m$-Du Bois singularities.*

## 4.2. Basic results

We have seen that rational singularities are Du Bois. The main focus of this section is the "higher" analogue of this statement, namely the fact that $m$-rational singularities are $m$-Du Bois, but we will establish various key properties along the way.

First we aim to understand the behavior with respect to general hyperplane sections:

PROPOSITION 4.2.1. *Let $X$ be a quasi-projective variety, and $H$ a general hyperplane section. Then for each $p \geq 0$, there is an exact triangle*

$$\underline{\Omega}_H^{p-1} \otimes \mathscr{O}_H(-H) \to \underline{\Omega}_X^p|_H \to \underline{\Omega}_H^p \xrightarrow{+1}$$

THEOREM 4.2.2. *If $X$ is a quasi-projective variety that is pre-$m$-Du Bois (resp. pre-$m$-rational), and $H$ is a general hyperplane section of $X$, then $H$ is pre-$m$-Du Bois (resp. pre-$m$-rational).*

PROOF. Assume first that $X$ is pre-$m$-Du Bois. The case $m = 0$ is clear; see the proof of Proposition 3.4.1. We assume $m > 0$, and use induction on $m$. For any $p \leq m$, we consider the following commutative diagram, where the vertical maps are the canonical morphisms:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{H}^0\underline{\Omega}_H^{p-1}(-H) & \longrightarrow & \mathcal{H}^0(\underline{\Omega}_X^p|_H) & \longrightarrow & \mathcal{H}^0\underline{\Omega}_H^p & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \underline{\Omega}_H^{p-1}(-H) & \longrightarrow & \underline{\Omega}_X^p|_H & \longrightarrow & \underline{\Omega}_H^p & \xrightarrow{+1} & \end{array}$$

The bottom exact triangle is given by Proposition 4.2.1. Since $X$ is also pre-$(m-1)$-Du Bois, by induction so is $H$; therefore the left vertical map is an isomorphism, and the top row is a short exact sequence. Moreover, since $H$ is general, we have

$$(\mathcal{H}^i\underline{\Omega}_X^p)|_H \simeq \mathcal{H}^i(\underline{\Omega}_X^p|_H),$$

and consequently the middle vertical map is also an isomorphism. We conclude that the right vertical map is an isomorphism as well.

Assume now that $X$ is pre-$m$-rational. We proceed again by induction on $m$, the case $m = 0$ being clear. For any $p \leq m$, consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{H}^0\mathbf{D}_H(\underline{\Omega}_H^{n-p}) & \longrightarrow & \mathcal{H}^0\mathbf{D}_H(\underline{\Omega}_X^{n-p}|_H) & \longrightarrow & \mathcal{H}^0\mathbf{D}_H(\underline{\Omega}_H^{n-p-1})(H) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbf{D}_H(\underline{\Omega}_H^{n-p}) & \longrightarrow & \mathbf{D}_H(\underline{\Omega}_X^{n-p}|_H) & \longrightarrow & \mathbf{D}_H(\underline{\Omega}_H^{n-p-1})(H) & \xrightarrow{+1} & \end{array}$$

Here the bottom triangle is obtained by dualizing the triangle in Proposition 4.2.1, with $p$ replaced by $n - p$.

Denote by $i\colon H \hookrightarrow X$ the inclusion of $H$ in $X$. Recall that in our hypersurface case, the functor

$$i^! = \mathbf{D}_H \circ \mathbf{L}i^* \circ \mathbf{D}_X$$

acts as $\bullet \otimes \mathscr{O}_H(H)[-1]$; see [Sta, Tag 0AU3, 4(b) and 7]. Therefore we have

$$\mathbf{D}_H(\underline{\Omega}_X^{n-p}|_H) \simeq i^!\mathbf{D}_X(\underline{\Omega}_X^{n-p}[1] \simeq \mathbf{D}_X(\underline{\Omega}_X^{n-p})|_H(H),$$

hence the middle vertical map in the diagram above is an isomorphism. Furthermore, $X$ is also pre-$(m-1)$-rational, hence by induction so is $H$. Therefore the left vertical map is an isomorphism, and the top row is a short exact sequence. We conclude again that the right vertical map is an isomorphism. $\square$

Combining this result with basic properties of Kähler differentials, we get:

Corollary 4.2.3. *The statement of Theorem 4.2.2 also holds for $m$-Du Bois and $m$-rational singularities.*

The main result we are after is the following:

Theorem 4.2.4. *If $X$ is a normal variety with pre-$m$-rational (resp. $m$-rational[2]) singularities, then $X$ has pre-$m$-Du Bois (resp. $m$-Du Bois) singularities.*

Recall that for $m = 0$, the proof of this result made essential use of the injectivity result, Theorem 3.3.4. Along, these lines, the following is proposed in [**PSV**]:

Conjecture 4.2.5. *Let $X$ be a variety with pre-$(m-1)$-Du Bois singularities. Then the map*

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}_X^m, \omega_X^\bullet) \to \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\mathcal{H}^0\underline{\Omega}_X^m, \omega_X^\bullet)$$

*in $\mathbf{D}^b_{\mathrm{coh}}(X)$, obtained by dualizing the canonical morphism $\mathcal{H}^0\underline{\Omega}_X^m \to \underline{\Omega}_X^m$, is injective on cohomology.*

There are two main results known in this direction. The first is a proof of the conjecture when $X$ has isolated singularities, given in [**PSV**, Theorem D].

Theorem 4.2.6. *Conjecture 4.2.5 holds when $X$ has isolated singularities.*

The second, which in fact predates the conjecture, is a proof of a strong version of this result in the case of local complete intersections, assuming the stronger $(m-1)$-Du Bois condition; this is [**MP3**, Theorem A].

Theorem 4.2.7. *If $X$ is local complete intersection with $(m-1)$-Du Bois singularities, then the morphism*

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}_X^m, \omega_X) \to \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\Omega_X^m, \omega_X)$$

*in $\mathbf{D}^b_{\mathrm{coh}}(X)$, obtained by dualizing the canonical morphism $\Omega_X^m \to \underline{\Omega}_X^m$, is injective on cohomology.*

Remark 4.2.8. Note that a conclusion as in Theorem 4.2.7 implies a conclusion as in Conjecture 4.2.5, because of the factorization

$$\Omega_X^m \to \mathcal{H}^0\underline{\Omega}_X^m \to \underline{\Omega}_X^m.$$

Note however that it is also shown in [**MP2**], [**MP3**] that, at least for $m \geq 2$, under the $(m-1)$-Du Bois assumption we have the isomorphisms

$$\Omega_X^m \simeq \Omega_X^{[m]} \simeq \mathcal{H}^0\underline{\Omega}_X^m.$$

We can prove Theorem 4.2.6 without much trouble using the methods developed up to now, so let's first do this.

**Proof of Theorem 4.2.6.** The statement of the theorem is local, hence we may assume first that $X$ is quasi-projective. Since the singular locus $S$ of $X$ is a finite set, using resolution of singularities we may choose a compactification $\bar{X}$ of $X$ such that the singular

[2] Note that in this case normality is automatic.

locus of $\bar{X}$ is still $S$, and prove the statement for $\bar{X}$. Hence it suffices to assume that $X$ is projective to begin with. Under this assumption, we prove a stronger statement:

THEOREM 4.2.9. *Let $X$ be a projective variety which is pre-$(m-1)$-Du Bois, and pre-$m$-Du Bois away from a finite set. Then the morphism*

$$\mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}_X^m, \omega_X^\bullet) \to \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\mathcal{H}^0\underline{\Omega}_X^m, \omega_X^\bullet)$$

*in $\mathbf{D}^b_{\mathrm{coh}}(X)$, obtained by dualizing the canonical map $\mathcal{H}^0\underline{\Omega}_X^m \to \underline{\Omega}_X^m$, is injective on cohomology.*

The key point in the proof is the following:

PROPOSITION 4.2.10. *Let $X$ be a projective variety with pre-$(m-1)$-Du Bois singularities. Then for each $i$, the natural map*

$$H^i(X, \mathcal{H}^0\underline{\Omega}_X^m) \to \mathbb{H}^i(X, \underline{\Omega}_X^m),$$

*obtained by applying cohomology to $\mathcal{H}^0\underline{\Omega}_X^m \to \underline{\Omega}_X^m$, is surjective.*

PROOF. For each $p \geq 0$, we denote

$$\underline{\Omega}_X^{\leq p} := \underline{\Omega}_X^\bullet / F^{p+1}\underline{\Omega}_X^\bullet.$$

So we have an exact triangle

$$\underline{\Omega}_X^p[-p] \longrightarrow \underline{\Omega}_X^{\leq p} \longrightarrow \underline{\Omega}_X^{\leq p-1} \xrightarrow{+1} . \tag{4.2.1}$$

We also denote by $\Omega_{X,h}^{\leq p}$ the object in the derived category of differential complexes on $X$,[3] represented by the complex

$$[\mathcal{H}^0\underline{\Omega}_X^0 \xrightarrow{d} \mathcal{H}^0\underline{\Omega}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}^0\underline{\Omega}_X^p],$$

placed in cohomological degrees $0, \ldots, p$. This is not to be confused with $\mathcal{H}^0(\underline{\Omega}_X^{\leq p})$. Here we have an exact triangle

$$\mathcal{H}^0\underline{\Omega}_X^p[-p] \longrightarrow \Omega_{X,h}^{\leq p} \longrightarrow \Omega_{X,h}^{\leq p-1} \xrightarrow{+1} . \tag{4.2.2}$$

Using the triangles (4.2.1) and (4.2.2), it is immediate to see by induction on $p$ that there exist natural morphisms

$$\Omega_{X,h}^{\leq p} \longrightarrow \underline{\Omega}_X^{\leq p}.$$

Since $X$ is projective, the $E_1$-degeneration of the Hodge-to-de Rham spectral sequence for the filtered de Rham complex of $X$ implies that the induced composition

$$H^i(X, \mathbf{C}) \to \mathbb{H}^i(X, \Omega_{X,h}^{\leq p}) \to \mathbb{H}^i(X, \underline{\Omega}_X^{\leq p})$$

is surjective for each $i$, hence so is the second map.

[3] The notation is motivated by the fact that $\mathcal{H}^0\underline{\Omega}_X^k$ agrees with the $h$-differentials $\Omega_{X,h}^k$ studied in [?HJ].

Let's now consider the integer $m$ in the statement. The map $\Omega^{\leq m}_{X,h} \to \underline{\Omega}^{\leq m}_X$ and its analogue for $m-1$, combined with the two exact triangles described above, give rise to a morphism of exact triangles

$$\begin{array}{ccccccc}
\mathcal{H}^0\underline{\Omega}^m_X[-m] & \longrightarrow & \Omega^{\leq m}_{X,h} & \longrightarrow & \Omega^{\leq m-1}_{X,h} & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
\underline{\Omega}^m_X[-m] & \longrightarrow & \underline{\Omega}^{\leq m}_X & \longrightarrow & \underline{\Omega}^{\leq m-1}_X & \xrightarrow{+1} & .
\end{array}$$

Since $X$ is pre-$(m-1)$-Du Bois, the right-most vertical map is an isomorphism. Passing to hypercohomology, we obtain a morphism of long exact sequences

$$\begin{array}{ccccccc}
\mathbb{H}^i(X, \Omega^{\leq m-1}_{X,h}) & \longrightarrow & H^iX, (\mathcal{H}^0\underline{\Omega}^m_X[-m]) & \longrightarrow & \mathbb{H}^i(X, \Omega^{\leq m}_{X,h}) & \longrightarrow & \mathbb{H}^i(X, \Omega^{\leq m-1}_{X,h}) \\
\downarrow\simeq & & \downarrow & & \downarrow & & \downarrow\simeq \\
\mathbb{H}^i(X, \underline{\Omega}^{\leq m-1}_X) & \longrightarrow & \mathbb{H}^i(X, \underline{\Omega}^m_X[-m]) & \longrightarrow & \mathbb{H}^i(X, \underline{\Omega}^{\leq m}_X) & \longrightarrow & \mathbb{H}^i(X, \underline{\Omega}^{\leq m-1}_X)
\end{array}$$

Since the third vertical map is surjective for all $i$, basic homological algebra shows that so is the second. ☐

We now consider the exact triangle

$$\mathcal{H}^0\underline{\Omega}^m_X \to \underline{\Omega}^m_X \to C \xrightarrow{+1} .$$

By definition $X$ is pre-$m$-Du Bois away from a finite set of points if and only if $C$ is supported on a finite set. After dualizing, we obtain an exact triangle

$$K \to \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}^m_X, \omega^\bullet_X) \to \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\mathcal{H}^0\underline{\Omega}^m_X, \omega^\bullet_X) \xrightarrow{+1},$$

where again $K$ is supported on a finite set. Applying Grothendieck-Serre duality to the surjections in Proposition 4.2.10, we obtain that the induced morphisms

$$\mathbb{H}^i\big(X, \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}^m_X, \omega^\bullet_X)\big) \to \mathbb{H}^i\big(X, \mathbf{R}\mathcal{H}om_{\mathscr{O}_X}(\mathcal{H}^0\underline{\Omega}^m_X, \omega^\bullet_X)\big)$$

are injective for all integers $i$.

Theorem 4.2.9 is then a consequence of the following general result:

Lemma 4.2.11. *Let $X$ be a projective variety, and let*

$$K \to F \to G \xrightarrow{+1}$$

*be an exact triangle in $\mathbf{D}^b_{\mathrm{coh}}(X)$. Suppose that $K$ has zero-dimensional support, and that the induced maps on hypercohomology*

$$\mathbb{H}^i(X, F) \to \mathbb{H}^i(X, G)$$

*are injective for all $i$. Then the induced maps on cohomology*

$$\mathcal{H}^iF \to \mathcal{H}^iG$$

*are injective for all $i$.*

Proof. The injectivity on hypercohomology implies that for each $i$ we have short exact sequences:

$$0 \to \mathbb{H}^i(X, F) \to \mathbb{H}^i(X, G) \to \mathbb{H}^{i+1}(X, K) \to 0.$$

Now the hypercohomology of $G$ is computed by a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q G) \implies \mathbb{H}^{p+q}(X, G),$$

while the similar spectral sequence for $K$ shows that

$$\mathbb{H}^{i+1}(X, K) \simeq H^0(X, \mathcal{H}^{i+1} K),$$

because of the assumption that $K$ is supported in dimension zero. Passing to the first associated graded term of the filtration on the total object in each of these two cases leads to a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^i(X, G) & \longrightarrow & \mathbb{H}^{i+1}(X, K) \\ \downarrow & & \downarrow \\ E_\infty^{0,i} & \longrightarrow & H^0(X, \mathcal{H}^{i+1} K) \end{array}$$

and by the observations above, it follows that the bottom horizontal map is surjective. On the other hand, note that in fact this map has a factorization

$$E_\infty^{0,i} \hookrightarrow E_2^{0,i} = H^0(X, \mathcal{H}^i G) \xrightarrow{\varphi} H^0(X, \mathcal{H}^{i+1} K),$$

where $\varphi$ comes from the connecting homomorphism $H^i G \to \mathcal{H}^{i+1} K$ induced by the original triangle. Since the support of $\mathcal{H}^{i+1} K$ is zero-dimensional, it follows immediately that this connecting homomorphism is surjective for each $i$, which is equivalent to our assertion. □

**Proof of Theorem 4.2.4.** As before, we want to use injectivity in order to address this question. First however, we need to understand a different, and perhaps more natural, interpretation of higher rational singularities.

Lemma 4.2.12. *Let $X$ be an irreducible variety, and $f \colon \widetilde{X} \to X$ a strong log resolution, with reduced exceptional divisor $E$. Then:*

*(1) There is a natural morphism*

$$\mathbf{D}(\underline{\Omega}_X^{n-p}) \to \mathbf{R} f_* \Omega_{\widetilde{X}}^p(\log E),$$

*which is an isomorphism for $p <$ $\operatorname{codim}_X X_{\mathrm{sing}}$.*

*(2) If $X$ is normal, the induced map*

$$\mathcal{H}^0 \mathbf{D}(\underline{\Omega}_X^{n-p}) \to f_* \Omega_{\widetilde{X}}^p(\log E)$$

*is an isomorphism.*[4]

[4] For $p = 0$, this is the familiar $\mathcal{H}^0 \mathbf{D}(\underline{\Omega}_X^n) \simeq f_* \mathscr{O}_{\widetilde{X}}$.

Proof. Recall that if $Z = X_{\mathrm{sing}}$, we have Steenbrink's triangle

$$\mathbf{R}f_*\Omega^p_{\widetilde{X}}(\log E)(-E) \longrightarrow \underline{\Omega}^{n-p}_X \longrightarrow \underline{\Omega}^{n-p}_Z \xrightarrow{+1} .$$

If $n-p > \dim Z$, then $\underline{\Omega}^{n-p}_Z = 0$, and we get (1) by dualizing and applying Grothendieck duality.

For (2), since $X$ is normal, we have that $c = \operatorname{codim}_X X_{\mathrm{sing}} \geq 2$. Dualizing the triangle above, we have

$$\mathbf{D}(\underline{\Omega}^{n-p}_Z) \longrightarrow \mathbf{D}(\underline{\Omega}^{n-p}_X) \longrightarrow \mathbf{R}f_*\Omega^{n-p}_{\widetilde{X}}(\log E) \xrightarrow{+1} .$$

Passing to cohomology, we obtain an exact sequence

$$0 \to \mathcal{H}^0\mathbf{D}_X(\underline{\Omega}^{n-p}_Z) \longrightarrow \mathcal{H}^0\mathbf{D}(\underline{\Omega}^{n-p}_X) \longrightarrow f_*\Omega^{n-p}_{\widetilde{X}}(\log E) \to \mathcal{H}^1\mathbf{D}_X(\underline{\Omega}^{n-p}_Z).$$

Note now that by Grothendieck duality we have $\mathbf{D}_X(\cdot) = \mathbf{D}_Z(\cdot)[-c]$, hence for all $i$ we have

$$\mathcal{H}^i\mathbf{D}_X(\underline{\Omega}^{n-p}_Z) \simeq \mathcal{H}^{i-c}\mathbf{D}_Z(\underline{\Omega}^{n-p}_Z).$$

Since $\mathbf{D}_Z(\underline{\Omega}^{n-p}_Z)$ is supported in non-negative degrees, and $c \geq 2$, it follows that these cohomologies are 0 for $i = 0, 1$, hence the exact sequence above implies what we want. ☐

Assume now that $X$ is normal. In this case we've seen that pre-0-rational is equivalent to rational singularities. In this case, by (10) in Section 2.4, for all $p$ we have

$$\mathcal{H}^0\underline{\Omega}^p_X \simeq \Omega^{[p]}_X \simeq f_*\Omega^p_{\widetilde{X}} \simeq f_*\Omega^p_{\widetilde{X}}(\log E) \simeq \mathcal{H}^0\mathbf{D}(\Omega^{n-p}_X),$$

where for the last isomorphism we use Lemma 4.2.12.

We can now proceed with deducing the statement of Theorem 4.2.4 from the injectivity result/conjecture. Let's assume therefore that $X$ is normal and pre-$m$-rational for some $m \geq 0$. According to the definition and the discussion above, this means that the composition

$$\mathcal{H}^0\underline{\Omega}^m_X \to \underline{\Omega}^m_X \to \mathbf{D}(\underline{\Omega}^{n-m}_X)$$

is an isomorphism. Dualizing, we obtain that the composition

$$\underline{\Omega}^{n-m}_X \to \mathbf{D}(\underline{\Omega}^m_X) \to \mathbf{D}\mathcal{H}^0\underline{\Omega}^m_X)$$

is an isomorphism, and therefore the morphism on the right is injective on cohomology. Since it is also injective on cohomology by Theorem 4.2.6, it is an isomorphism, hence dualizing again we get that $X$ is pre-$m$-Du Bois.

Since Conjecture 4.2.5 is not yet known in general, this argument of course works only when $X$ has isolated singularities, or when $X$ is a local complete intersection with $m$-rational singularities, if we use Theorem 4.2.7 in a similar way. However, one can directly modify the arguments in the proof of Theorem 4.2.9 in order to obtain the statement of Theorem 4.2.4, even without knowing injectivity. Here is a sketch, following [**SVV1**].

First, the problem is local, so we may assume $X$ is quasi-projective. One proceeds by induction on both $m$ and the dimension of $X$. The statement is clear when $\dim X = 0$ or when $m = -1$. Assume now that $n = \dim X > 0$, and $X$ has pre-$m$-rational singularities, with $m \geq 0$. By induction, $X$ is pre-$(m-1)$-Du Bois, hence all we need to show is

$$\mathcal{H}^i \underline{\Omega}_X^m = 0 \quad \text{for} \quad i > 0.$$

Denote by $\Sigma$ the locus where this statement does not hold, and let $H$ be a general hyperplane section of $X$. According to Proposition 4.2.1, we have an exact triangle

$$\underline{\Omega}_H^{m-1}(-H) \to \underline{\Omega}_X^m|_H \to \underline{\Omega}_H^m \xrightarrow{+1}$$

By Theorem 4.2.2, $H$ has pre-$m$-rational singularities, hence by induction it also has pre-$m$-Du Bois singularities. The long exact sequence on cohomology associated to the triagle above then yields

$$(\mathcal{H}^i \underline{\Omega}_X^m)|_H \simeq \mathcal{H}^i(\underline{\Omega}_X^m|_H) = 0 \quad \text{for all} \quad i > 0,$$

hence $X$ is pre-$m$-Du Bois in a neighborhood of $H$. Hence $\Sigma \cap H = \emptyset$, and since $H$ is general, it follows that $\Sigma = \emptyset$ (in which case we're done), or $\dim \Sigma = 0$.

Using induction, we have therefore reduced to the case where $X$ is pre-$(m-1)$-Du Bois, and pre-$m$-Du Bois away from a zero-dimensional subset. If $X$ is projective, we are done by Theorem 4.2.9. Otherwise one needs to perform a few more (nontrivial) auxiliary steps in order to reduce to this case.

## 4.3. The Hodge-Du Bois diamond

Let $X$ be an irreducible projective variety of dimension $n$.

Definition 4.3.1 (**Hodge-Du Bois numbers**). The *Hodge-Du Bois spaces* of $X$ are defined as

$$\underline{H}^{p,q}(X) := \mathbb{H}^q(X, \underline{\Omega}_X^p).$$

As discussed in earlier lectures, these are the associated graded spaces with respect to Deligne's Hodge filtration, part of the mixed Hodge structure on the singular cohomology $H^{p+q}(X, \mathbb{Q})$. The *Hodge-Du Bois numbers* of $X$ are defined as

$$\underline{h}^{p,q}(X) := \dim_{\mathbf{C}} H^{p,q}(X).$$

The name is given in order to emphasize the singular setting, and the difference with the Hodge-Deligne numbers of the pure Hodge structures on the associated graded pieces with respect to the weight filtration. As in the smooth setting, these numbers are organized into a *Hodge-Du Bois diamond*:

$$\begin{array}{ccccccc}
 & & & \underline{h}^{n,n} & & & \\
 & & \underline{h}^{n,n-1} & & \underline{h}^{n-1,n} & & \\
 & \iddots & & \vdots & & \ddots & \\
\underline{h}^{n,0} & \underline{h}^{n-1,1} & & \cdots & & \underline{h}^{1,n-1} & h^{0,n} \\
 & \ddots & & \vdots & & \iddots & \\
 & & \underline{h}^{1,0} & & \underline{h}^{0,1} & & \\
 & & & \underline{h}^{0,0} & & &
\end{array}$$

What do we know about these numbers? There are very few things one can say in general; the main thing is that, by the degeneration of the generalized Hodge-to-de Rham spectral sequence, one has

$$b_k(X) = \sum_{p+q=k} \underline{h}^{p,q}(X) \tag{4.3.1}$$

for all $k$, where $b_k(X)$ are the Betti numbers of $X$. In particular, we have

$$\underline{h}^{0,0}(X) = 1 \quad \text{and} \quad \underline{h}^{n,n}(X) = 1. \tag{4.3.2}$$

Pretty much all of the other symmetries of the Hodge diamond of a smooth projective variety, given by conjugation, Poincaré duality, or Serre duality, see (1.1.1), fail in general.

Example 4.3.2 (**Curves**). Let $C$ be an irreducible projective curve. In this case, we only need to compare $\underline{h}^{1,0}(C)$ and $\underline{h}^{0,1}(C)$. Recall that we know what the Du Bois complexes look like, namely

$$\underline{\Omega}^0_C \simeq \mathscr{O}_{C^{\mathrm{sn}}} \quad \text{and} \quad \underline{\Omega}^1_X \simeq f_*\omega_{\widetilde{C}},$$

where $C^{\mathrm{sn}}$ is the semi-normalization of $C$, and $f\colon \widetilde{C} \to C$ is the normalization. We deduce that

$$\underline{h}^{1,0}(C) = g(\widetilde{C}) = p_g(C),$$

the geometric genus of $C$, while

$$\underline{h}^{0,1}(C) = p_a(C^{\mathrm{sn}}),$$

the arithmetic genus of the seminormalization.

For example, when $C$ is a nodal curve, it is already seminormal, $\underline{h}^{0,1}(C) = p_a(C)$. The two can of course be different; for example, if $C$ is rational, then $p_g(C) = 0$, while $p_a(C) = s$, the number of nodes (e.g. $s = 1$ if $C$ is a singular plane cubic).

It is not hard to see that for any $C$ we have

$$\underline{h}^{1,0}(C) \leq \underline{h}^{0,1}(C),$$

usually with strict inequality.

EXAMPLE 4.3.3 (**Surface cones**). Let $X = C(B)$ be a projectivized cone over a smooth plane curve $B$ of degree $d$. Since the singularity is isolated, $X$ is a normal surface. We will see later that this implies that $H^1(X,\mathbb{Q})$ is a pure Hodge structure, hence

$$\underline{h}^{1,0}(X) = \underline{h}^{0,1}(X).$$

We also have that $H^3(X,\mathbb{Q})$ is a pure Hodge structure, see Theorem 4.3.4, hence

$$\underline{h}^{2,1}(X) = \underline{h}^{1,2}(X).$$

But do we have Poincaré duality $H^1(X,\mathbb{Q}) \simeq H^3(X,\mathbb{Q})$, or equivalently, $\underline{h}^{0,1}(X) = \underline{h}^{2,1}(X)$?

Let's focus on the Du Bois case, i.e. $d \leq 3$. In this case

$$\underline{h}^{0,1}(X) = h^1(X, \underline{\Omega}^0_X) = h^1(X, \mathscr{O}_X).$$

Moreover

$$\underline{h}^{2,1}(X) = h^1(X, \underline{\Omega}^2_X) = h^1(\widetilde{X}, \omega_{\widetilde{X}}) = h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}),$$

where the last equality follows from Serre duality on $\widetilde{X}$. Using the Leray spectral sequence, it is not hard to check that the equality $h^1(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = h^1(X, \mathscr{O}_X)$ holds if and only if $B \simeq \mathbf{P}^1$, i.e. $d = 2$, which is also equivalent to $X$ having rational singularities. When $d = 3$, i.e. when $B$ is an elliptic curve, we do not have equality, hence we also have $b_1(X) \neq b_3(X)$.

A couple more general facts about the Hodge-Du Bois numbers follow from the general theory of mixed Hodge structures. First, the top part of the Hodge diamond is well behaved, in a range dictated by the size of the singular locus; see e.g. [**PS**, Theorem 6.32].

THEOREM 4.3.4. *If $X$ is a projective variety of dimension $n$, and $s = \dim X_{\mathrm{sing}}$, then the Hodge structure on $H^i(X,\mathbb{Q})$ is pure for $i > n+s$.*

Thus in this region we have horizontal symmetry; in fact we even have isomorphisms $H^i(X,\mathbb{Q}) \simeq IH^i(X,\mathbb{Q})$ with the intersection cohomology spaces.

We also have some universal inequalities satisfies by the Hodge-Du Bois numbers, as a consequence of the existence and properties of the mixed Hodge structure on singular cohomology.

LEMMA 4.3.5 ([?FL22, Lemma 3.23]). *Let $X$ be a complex projective variety of dimension $n$. For all $0 \leq p \leq i \leq n$ we have*

$$\sum_{a=0}^{p} \underline{h}^{i-a,a}(X) \leq \sum_{a=0}^{p} \underline{h}^{a,i-a}(X).$$

*Moreover, equality holds for all $0 \leq p \leq k \iff \underline{h}^{i-p,p}(X) = \underline{h}^{p,i-p}(X)$ for all $0 \leq p \leq k$ $\iff \mathrm{gr}^p_F W_{i-1} H^i(X,\mathbf{C}) = 0$ for all $p \leq k$.*

This contains the well-known fact that the Hodge structure on $H^i(X,\mathbb{Q})$ is pure if and only if $\underline{h}^{i-p,p}(X) = \underline{h}^{p,i-p}(X)$ for all $0 \leq p \leq i$, which will be used repeatedly throughout the paper.

To gain a better understanding of the symmetries of the Hodge-Du Bois diamond, and more generally of the topological and Hodge-theoretic properties of singular varieties, it is important to focus on the following on the following invariants, coming from rather different areas of study:

(i) the local cohomological defect of $X$.

(ii) the higher rational singularities level of $X$.

(iii) the defect of $\mathbb{Q}$-factoriality of $X$.

**Local cohomological defect.** Let $X$ be an equidimensional variety of dimension $n$. If $Y$ is a smooth variety containing $X$ (locally), the local cohomological dimension of $X$ in $Y$ is

$$\mathrm{lcd}(X,Y) := \max\, \{q \mid \mathcal{H}^q_X \mathscr{O}_Y \neq 0\}.$$

Here $\mathcal{H}^q_X \mathscr{O}_Y$ is the $q$-th local cohomology sheaf of $\mathscr{O}_Y$ along $X$. These sheaves also satisfy the property

$$\mathrm{codim}_Y X := \min\, \{q \mid \mathcal{H}^q_X \mathscr{O}_Y \neq 0\}.$$

A discussion of this circle of ideas can be found for instance in [**MP2**, Section 2.2], where it is explained in particular that

$$\mathrm{lcd}(X,Y) \le r(X,Y), \tag{4.3.3}$$

where $r(X,Y)$ is the minimal number of defining equations for $X$ in $Y$.

We define the *local cohomological defect* $\mathrm{lcdef}(X)$ of $X$ as

$$\mathrm{lcdef}(X) := \mathrm{lcd}(X,Y) - \mathrm{codim}_Y X.$$

The topological characterization of $\mathrm{lcd}(X,Y)$ in [**Og**], or its holomorphic characterization in [**MP2**], imply that $\mathrm{lcdef}(X)$ depends only on $X$, and not on the embedding, and that $0 \le \mathrm{lcdef}(X) \le n$.

For instance, as explained in [**PSh**, Section 2], a reinterpretation of the characterization of local cohomological dimension in [**MP2**, Theorem E], that makes the relationship with Hodge theory apparent, is stated as follows:

THEOREM 4.3.6. *We have the identity*

$$\mathrm{lcdef}(X) = n - \min_{p\ge 0}\, \{\mathrm{depth}\, \underline{\Omega}^p_X + p\}.$$

Here we will be more often concerned with the topological interpretation of $\mathrm{lcdef}(X)$, more precisely with its more modern interpretation via Riemann-Hilbert correspondence and the perverse $t$-structure on the bounded derived category $D^b_c(X,\mathbb{Q})$ of $\mathbb{Q}$-sheaves on $X$ with constructible cohomologies, as formulated in [**RSW**] and [**BBL**$^+$].

THEOREM 4.3.7 ([**Og**, Theorem 2.13], [**RSW**, Theorem 1], [**BBL**$^+$, Section 3.1]). *We have the identity*

$$\mathrm{lcdef}(X) = \max\, \{j \ge 0 \mid {}^p\mathcal{H}^{-j}(\mathbb{Q}_X[n]) \neq 0\}.[5]$$

[5] The shift is included for compatibility with the theory of perverse sheaves and Hodge modules.

Note that thanks to (4.3.3) we have

$$\operatorname{lcdef}(X) \leq s(X) := r(X,Y) - \operatorname{codim}_Y X. \tag{4.3.4}$$

The quantity on the right hand side measures the defect of being a local complete intersection. When $X$ is so, we obviously have $\operatorname{lcdef}(X) = 0$. We will see that in many respects the topology of $X$ is best behaved when this condition is satisfied, or in any case when $\operatorname{lcdef}(X)$ is small. Interestingly, this can happen for varieties that may be quite far from being local complete intersections.

EXAMPLE 4.3.8 (**Varieties with** $\operatorname{lcdef}(X) = 0$). Besides local complete intersections, the condition $\operatorname{lcdef}(X) = 0$ it is known to hold when $X$ has quotient singularities (see [**MP2**, Corollary 11.22], or use the known fact that $X$ is a rational homology manifold; see the example below), for affine varieties with Stanley-Reisner coordinate algebras that are Cohen-Macaulay [**MP2**, Corollary 11.26], and for arbitrary Cohen-Macaulay surfaces and threefolds in [**Og**, Remark p.338-339] and [**DT**, Corollary 2.8] respectively.

More generally, Dao-Takagi [**DT**, Theorem 1.2] show that for $k \leq 3$ one has

$$\operatorname{depth} \mathscr{O}_X \geq k \Longrightarrow \operatorname{lcdef}(X) \leq n - k$$

This does not continue to hold for $k \geq 4$, but for instance it implies that a Cohen-Macaulay fourfold satisfies $\operatorname{lcdef}(X) \leq 1$. Moreover, in this case we do have $\operatorname{lcdef}(X) = 0$ if the local analytic Picard groups are torsion [**DT**, Theorem 1.3].

Another general class of varieties that satisfy $\operatorname{lcdef}(X) = 0$ is provided by the example below.

EXAMPLE 4.3.9 (**Rational homology manifolds**). A *rational homology manifold (or RHM)* is classically defined by the fact that the homology of the link of each singularity of $X$ is the same as that of a sphere. It is however known (see [?BM83]) that this is equivalent to the fact that the map $\gamma_X \colon \mathbb{Q}_X^H[n] \to \mathrm{IC}_X^H$ is an isomorphism. Thanks to Theorem 4.3.7, we thus have the implication

$$X \text{ is an RHM } \Longrightarrow \operatorname{lcdef}(X) = 0.$$

This is usually not an equivalence. For instance, any normal surface has $\operatorname{lcdef}(X) = 0$; however, when $X$ is Du Bois, the RHM condition is equivalent to $X$ having rational singularities (see Exercise 4.3.20).

For the story we are developing here, one of the ways in which the local cohomological defect comes into play through the following version of the Lefschetz hyperplane theorem.

THEOREM 4.3.10. *Let $X$ be a projective variety of dimension $n$, $D$ an ample effective Cartier divisor on $X$, and $U = X \smallsetminus D$. Then the restriction map*

$$H^i(X,\mathbb{Q}) \to H^i(D,\mathbb{Q})$$

*is an isomorphism for $i \leq n - 2 - \operatorname{lcdef}(U)$ and injective for $i = n - 1 - \operatorname{lcdef}(U)$.*

The proof relies on a strong version of Artin vanishing, combined with Theorem 4.3.7. Here is the sketch of a proof more in line with the material discussed in this class, in the case $D$ is a general hyperplane section, and we use $\operatorname{lcdef}(X)$ instead of $\operatorname{lcdef}(U)$.

The key point is the following dual version of the Nakano-type vanishing theorem for Du Bois complexes.

THEOREM 4.3.11. *If $L$ is an ample line bundle on a projective variety $X$, then*

$$\mathbb{H}^q(X, \underline{\Omega}_X^p \otimes L^{-1}) = 0 \quad \text{for all} \;\; p + q < n - \mathrm{lcdef}(X).$$

The proof of this statement uses the theory of mixed Hodge modules. Equivalently, one needs to show that

$$\mathbb{H}^{n-q}(X, \mathbf{D}(\underline{\Omega}_X^p) \otimes L) = 0 \quad \text{for all} \;\; p + q < n - \mathrm{lcdef}(X).$$

The reason for this is that $\mathbf{D}(\underline{\Omega}_X^p)$ is an associated graded quotient of the de Rham complex of an object in the derived category of mixed Hodge modules, whose amplitude is precisely $\mathrm{lcdef}(X)$; for such objects there is a standard vanishing theorem, called Kodaira-Saito vanishing. For details, see [**PSh**, §5].

Assuming this vanishing result, the argument goes as follows. Since $D$ is a general hyperplane section of $X$, by Proposition 4.2.1 for each $p$ there is an exact triangle

$$\underline{\Omega}_D^{p-1}(-D) \to \underline{\Omega}_X^p|_D \to \underline{\Omega}_D^p \xrightarrow{+1} .$$

We of course also have an exact triangle

$$\underline{\Omega}_X^p(-D) \to \underline{\Omega}_X^p \to \underline{\Omega}_X^p|_D \xrightarrow{+1} .$$

Using these, and the octahedral axiom, it is not hard to see that we obtain an exact triangle

$$C_{X,D}^p \to \underline{\Omega}_X^p \to \underline{\Omega}_D^p \xrightarrow{+1}, \tag{4.3.5}$$

where in turn the object $C_{X,D}^p$ sits in an exact triangle

$$\underline{\Omega}_X^p(-D) \to C_{X,D}^p \to \underline{\Omega}_D^{p-1}(-D) \xrightarrow{+1} . \tag{4.3.6}$$

Since by generality we have $\mathrm{lcdef}(D) \le \mathrm{lcdef}(X)$, applying Theorem 4.3.11 to both extremes of (4.3.6) gives

$$\mathbb{H}^q(X, C_{X,D}^p) = 0 \quad \text{for all} \;\; p + q \le n - 1 - \mathrm{lcdef}(X),$$

which by (4.3.5) is equivalent to the conclusion of Theorem 4.3.10 (for each of the individual Hodge spaces $H^{p,q}(X)$ with $p + q = k$, hence for all of $H^k(X, \mathbf{C})$).

Theorem 4.3.10 has some interesting consequences regarding the purity of Hodge structures.

COROLLARY 4.3.12. *Let $X$ be an equidimensional projective variety with* $\mathrm{codim}\ X_{\mathrm{sing}} \ge k+1$. *Then the Hodge structure on $H^i(X, \mathbb{Q})$ is pure for $i \le k - \mathrm{lcdef}(X)$. Moreover, we have*

$$H^i(X, \mathbb{Q}) \simeq IH^i(X, \mathbb{Q}) \;\; \text{for} \;\; i \le k - 1 - \mathrm{lcdef}(X),$$

*so in particular $H^i(X, \mathbb{Q})$ is Poincaré dual to $H^{2n-i}(X, \mathbb{Q})$ for such $i$.*

Proof. Note that if $U$ is any open set in $X$, we have $\mathrm{lcdef}(U) \le \mathrm{lcdef}(X)$, and if $D$ is a general hyperplane section of $X$ we have $\mathrm{lcdef}(D) \le \mathrm{lcdef}(X)$. Thus by cutting with $(n-k)$ general hyperplane sections and applying the theorem repeatedly, we can reduce to the case when $D$ is smooth, when $H^i(D,\mathbb{Q})$ is a pure Hodge structure. Moreover, in this case we have $H^i(D,\mathbb{Q}) \simeq IH^i(D,\mathbb{Q})$, but we always have $IH^i(X,\mathbb{Q}) \simeq IH^i(D,\mathbb{Q})$ by the weak Lefschetz theorem for intersection cohomology.

Note that since the singular locus of $X$ has dimension at most $n-k-1$, we also have the isomorphism $H^i(X,\mathbb{Q}) \simeq IH^i(X,\mathbb{Q})$ for $i \ge 2n-k$, which implies the last assertion as intersection cohomology satisfies Poincaré duality. □

Example 4.3.13 (**Isolated singularities**). When $X$ has isolated singularities we can take $k = n-1$; if moreover $\mathrm{lcdef}(X) = 0$, we obtain that all $H^i(X,\mathbb{Q})$ carry pure Hodge structure, except perhaps $H^n(X,\mathbb{Q})$.

According to Example 4.3.8, this is the case for instance when $X$ is any Cohen-Macaulay threefold, or a Cohen-Macaulay fourfold which is locally analytically $\mathbb{Q}$-factorial, with isolated singularities.

If $\mathrm{depth}(\mathscr{O}_X) \ge 3$ in any dimension, then $\mathrm{lcdef}(X) \le n-3$ (again by Example 4.3.8), and therefore $H^2(X,\mathbb{Q})$ carries pure Hodge structure.

Example 4.3.14 (**Normal and rational singularities**). When $X$ is a projective normal variety, so that $\mathrm{depth}(\mathscr{O}_X) \ge 2$ and hence $\mathrm{lcdef}(X) \le n-2$, the iterative application of Theorem 4.3.10 implies the injection

$$H^1(X,\mathbb{Q}) \to H^1(C,\mathbb{Q})$$

where $C$ is a general complete intersection curve in $X$. Since $C$ is smooth by the normality of $X$, this recovers the well-known fact that $H^1(X,\mathbb{Q})$ is a pure Hodge structure of weight 1 (see e.g. [?Saito18]). As in the previous example, when $X$ has rational singularities and $\dim X \ge 3$, we have $\mathrm{lcdef}(X) \le n-3$, hence a similar procedure recovers the folklore fact that $H^2(X,\mathbb{Q})$ is a pure Hodge structure of weight 2, using the well-known statement that surfaces with rational singularities are rational homology manifolds.

**Higher rationality.** Recall that if $X$ has (pre-)$m$-rational singularities, then it has (pre-)$m$-Du Bois singularities, and satisfies the duality condition

$$\underline{\Omega}_X^p \simeq \mathbf{D}(\underline{\Omega}_X^{n-p}), \quad \text{for all} \quad p \le m.$$

Let's call this last condition $(D_m)$, for simplicity.

Assume now that $X$ satisfies $(D_m)$ and is projective. Using Serre duality, we get the following symmetry:

$$\underline{H}^{p,q} = \mathbb{H}^q(X,\underline{\Omega}_X^p) \simeq \mathbb{H}^q(X,\mathbf{D}(\underline{\Omega}_X^{n-p})) \simeq \mathbb{H}^{n-q}(X,\underline{\Omega}_X^{n-p}) = \underline{H}^{n-p,n-q}(X).$$

In particular, we have

$$\underline{h}^{p,q}(X) = \underline{h}^{n-p,n-q}(X), \quad \text{for all} \quad p \le m \quad \text{and all} \quad q.$$

Friedman and Laza noted in [**FL1**] that there is in fact further symmetry under the $m$-rationality assumption, in the LCI setting. This was later extended in [**SVV1**] to arbitrary pre-$m$-rational singularities.

THEOREM 4.3.15 ([**FL1**], [**SVV1**]). *If $X$ is projective, with pre-$m$-rational singularities, then for all $q$ and all $0 \le p \le m$, we have:*

$$\underline{h}^{p,q}(X) = \underline{h}^{q,p}(X) = \underline{h}^{n-p,n-q}(X).$$

Let me exemplify with the case $m = 0$, i.e. that of rational singularities, by using a powerful tool. (The result can be shown with a more elementary method.) By the discussion above, in this case we only need to check that $\underline{h}^{0,q}(X) = \underline{h}^{q,0}(X)$ for all $q$. We have:

$$\underline{h}^{0,q}(X) = h^q(X, \underline{\Omega}^0_X) = h^q(X, \mathscr{O}_X) = h^q(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) = h^0(\widetilde{X}, \Omega^q_{\widetilde{X}}) = h^0(X, f_*\Omega^q_{\widetilde{X}}).$$

Here the second and third equality follow from the fact that $X$ has rational singularities, while the third follows from Hodge symmetry for smooth varieties. Also:

$$\underline{h}^{q,0}(X) = h^0(X, \underline{\Omega}^q_X) = h^0(X, \mathcal{H}^0\underline{\Omega}^q_X) = h^0(X, f_*\Omega^q_{\widetilde{X}}),$$

which establishes the equality we want. Here the second equality is due to the fact that $X$ is (pre-)$m$-Du Bois, while the last follows from the Kebekus-Schnell theorem, which says that if $X$ has rational singularities, then $\mathcal{H}^0\underline{\Omega}^q_X \simeq \Omega^{[q]}_X \simeq f_*\Omega^q_{\widetilde{X}}$, for all $q$.

More holds in fact, as the answer to the natural question of whether the Hodge number $\underline{h}^{n-q,n-p}$ can be included in this picture as well. This is shown in [**PP**], but requires more technical methods, based on Hodge modules and intersection cohomology.

THEOREM 4.3.16. *If $X$ is projective and satisfies condition $(D_m)$, then for all $q$ and all $0 \le p \le m$, we have:*

$$\underline{h}^{p,q}(X) = \underline{h}^{q,p}(X) = \underline{h}^{n-p,n-q}(X) = \underline{h}^{n-q,n-p}(X).$$

EXAMPLE 4.3.17 (**Rational singularities**). If $X$ has rational singularities, the theorem says that

$$\underline{h}^{0,q}(X) = \underline{h}^{q,0}(X) = \underline{h}^{n,n-q}(X) = \underline{h}^{n-q,n}(X).$$

for all $q$, i.e. the boundary of the Hodge-Du Bois diamond is symmetric.

EXAMPLE 4.3.18. If $X$ is a surface with rational singularities, we deduce that the Hodge-Du Bois diamond is fully symmetric. In this case more (a priori) is in fact known, from classical work of Mumford: $X$ is a rational homology manifold. In fact we have:

EXERCISE 4.3.19. Show that if $X$ is a projective normal surface, then $X$ is a rational homology manifold if and only if its Hodge-Du Bois diamond is fully symmetric.

Among all Du Bois surfaces (in particular the log canonical ones), the converse of the statement above also holds:

EXERCISE 4.3.20. Let $X$ be a surface with Du Bois singularities. Then $X$ is a rational homology manifold if and only if $X$ has rational singularities.

EXAMPLE 4.3.21. Threefolds and fourfolds with 1-rational singularities have fully symmetric Hodge-Du Bois diamonds. However this is a rather strong assumption. For instance, one can show that a threefold with 1-rational singularities is in fact smooth. For this reason, a finer analysis of symmetry is needed.

Let's focus on the case of threefolds, assumed for simplicity to have isolated singularities. This provides a clear illustration of how, by gradually imposing more and more singularity conditions, we have more and more Hodge symmetry, up to the case of rational homology manifolds. Let $X$ be one such threefold, and consider its Hodge-Du Bois diamond:

$$\begin{array}{ccccccc}
 & & & \underline{h}^{3,3} & & & \\
 & & \underline{h}^{3,2} & & \underline{h}^{2,3} & & \\
 & \underline{h}^{3,1} & & \underline{h}^{2,2} & & \underline{h}^{1,3} & \\
\underline{h}^{3,0} & & \underline{h}^{2,1} & & \underline{h}^{1,2} & & \underline{h}^{0,3} \\
 & \underline{h}^{2,0} & & \underline{h}^{1,1} & & \underline{h}^{0,2} & \\
 & & \underline{h}^{1,0} & & \underline{h}^{0,1} & & \\
 & & & \underline{h}^{0,0} & & &
\end{array}$$

First, the isolated singularities condition implies that the "top" part of the Hodge-Du Bois diamond, i.e. $H^k(X, \mathbb{Q})$ with $k \geq 4$, carries pure Hodge structure, and therefore we have the horizontal symmetry $\underline{h}^{p,q} = \underline{h}^{q,p}$ for $p + q \geq 4$.

If in addition $X$ is *normal*, then $H^1(X, \mathbb{Q})$ is pure, and therefore $\underline{h}^{1,0} = \underline{h}^{0,1}$; see Example 4.3.14.

Now for the new input: if $X$ is *Cohen-Macaulay*, then $\mathrm{lcdef}(X) = 0$ by Example 4.3.8, and therefore $H^1(X, \mathbb{Q}) \simeq IH^1(X, \mathbb{Q})$ and $H^2(X, \mathbb{Q})$ is pure, by Corollary 4.3.12. The latter implies that $\underline{h}^{2,0} = \underline{h}^{0,2}$ hence, except for $H^3(X, \mathbb{Q})$, we have horizontal symmetry. The former implies that $H^1(X, \mathbb{Q})$ is Poincaré dual to $H^5(X, \mathbb{Q})$, which gives

$$\underline{h}^{1,0} = \underline{h}^{0,1} = \underline{h}^{3,2} = \underline{h}^{2,3}.$$

We need more for further symmetries. If $X$ has *rational singularities*, then Example 4.3.17 implies

$$\underline{h}^{2,0} = \underline{h}^{0,2} = \underline{h}^{3,1} = \underline{h}^{1,3} \quad \text{and} \quad \underline{h}^{3,0} = \underline{h}^{0,3}.$$

(The hypothesis on rational singularities is necessary, for instance when $X$ is Du Bois with $\underline{h}^{0,2} = 0$.)

Hence under the rational singularities assumption, the only potential source of non-symmetry is the middle rhombus (even when the singularities are not isolated, in fact). The answer is provided by the following result from [**PP**].

THEOREM 4.3.22. *Let $X$ be a projective threefold with rational singularities. Then:*

*(1) $\underline{h}^{2,2}(X) \geq \underline{h}^{1,1}(X)$, and equality holds if and only if $X$ is $\mathbb{Q}$-factorial.*
*(2) $\underline{h}^{1,2}(X) \geq \underline{h}^{2,1}(X)$[6] and equality holds if and only if $X$ is analytically $\mathbb{Q}$-factorial.*

Being locally analytically $\mathbb{Q}$-factorial is stronger than being $\mathbb{Q}$-factorial, so finally this condition achieves full symmetry in the Hodge-Du Bois diamond of $X$. (It can in fact be shown that it is equivalent to $X$ being a rational homology manifold.)

### 4.4. Du Bois complexes in terms of Hodge modules

Let $X$ be a complex variety, and $i\colon X \hookrightarrow Y$ an embedding into a smooth variety. We also consider the inclusion $j\colon U = Y \smallsetminus X \hookrightarrow Y$. There is a canonical morphism $\mathbb{Q}_Y^H[n] \longrightarrow j_*\mathbb{Q}_U^H[n]$ in the derived category of mixed Hodge modules on $Y$, which we complete to an exact triangle

$$\mathrm{LC}_Y(X) \longrightarrow \mathbb{Q}_Y^H[n] \longrightarrow j_*\mathbb{Q}_U^H[n] \xrightarrow{+1} . \tag{4.4.1}$$

(In fact we have $\mathrm{LC}_Y(X) = i_*i^!\mathbb{Q}_X^H[n]$, as shown in [**Sa2**, Section 4].)

Here $\mathbf{Q}_Y^H[n]$ is the trivial (pure) Hodge module on $Y$; its underlying perverse sheaf ifs $\mathbb{Q}_Y[n]$, while its underlying $\mathscr{D}_Y$-module is $\mathscr{O}_Y$, with Hodge filtration is defined by $F_k\mathscr{O}_Y = \mathscr{O}_Y$ for $k \geq 0$, and $F_k\mathscr{O}_Y = 0$ for $k < 0$. Note that this implies that for each $p$ we have an isomorphism have an isomorphism

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(\mathbb{Q}_Y^H[n]\big) \simeq \Omega_Y^{n-p}[p].$$

On the other hand, the cohomologies of the extremal objects are mixed Hodge modules, whose underlying $\mathscr{D}_Y$-modules are

$$\mathcal{H}^q\mathrm{LC}_Y(X) = \mathcal{H}^q_Y\mathscr{O}_X \quad \text{and} \quad \mathcal{H}^q j_*\mathbb{Q}_U^H[n] = R^q j_*\mathscr{O}_U.$$

In particular, these $\mathscr{D}_Y$-modules carry canonical Hodge filtrations.

Moreover, the long exact sequence of cohomology corresponding to (4.4.1) gives an exact sequence of filtered $\mathscr{D}_Y$-modules

$$0 \to \mathscr{O}_Y \to j_*\mathscr{O}_U \to \mathcal{H}^1_X\mathscr{O}_Y \to 0 \tag{4.4.2}$$

and an isomorphism of filtered $\mathscr{D}_Y$-modules

$$R^q j_*\mathscr{O}_U \simeq \mathcal{H}^{q+1}_X\mathscr{O}_Y \quad \text{for all} \quad q \geq 1. \tag{4.4.3}$$

A key point for what follows is a formula for the graded pieces of the de Rham complex of $j_*\mathbb{Q}_U^H[n]$.

[6] This follows from Lemma 4.3.5, since we know that $\underline{h}^{3,0}(X) = \underline{h}^{0,3}(X)$ thanks to Theorem 4.3.16.

Lemma 4.4.1. *For every $p \in \mathbb{Z}$, we have an isomorphism in $\mathbf{D}^b_{\mathrm{coh}}(Y)$:*

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(j_*\mathbb{Q}^H_U[n]\big) \simeq \mathbf{R}f_*\Omega^{n-p}_{\widetilde{Y}}(\log E)[p].$$

Proof. We use the approach and notation in Section **??**. Recall that we have an isomorphism

$$j_*\mathbb{Q}^H_U[n] \simeq f_*j'_*\mathbb{Q}^H_V[n]. \tag{4.4.4}$$

The filtered resolution (**??**) gives an isomorphism

$$\mathrm{Gr}^F_{i-n}\mathrm{DR}_Y\big(j'_*\mathbb{Q}^H_V[n]\big) \simeq \Omega^{n-i}_Y(\log E)[i]$$

(see also [**MP1**, §6]). Using the isomorphism (4.4.4) and Saito's strictness-type result on the commutation of the direct image functor with the graded pieces of the de Rham complex (see e.g. [**Sa1**, Section 2.3.7]) we deduce that

$$\mathrm{Gr}^F_{i-n}\mathrm{DR}_X\big(j_*\mathbb{Q}^H_U[n]\big) \simeq \mathbf{R}f_*\mathrm{Gr}^F_{i-n}\mathrm{DR}_Y\big(j'_*\mathbb{Q}^H_V[n]\big) \simeq \mathbf{R}f_*\Omega^{n-i}_Y(\log E)[i].$$

□

The characterization of Du Bois complexes in terms of Hodge modules is provided by the following:

Proposition 4.4.2. *Let $X$ be a complex variety, and let $i\colon X \hookrightarrow Y$ be its inclusion map into a smooth, irreducible variety $Y$. Then, for every integer $p$, we have an isomorphism*

$$\underline{\Omega}^p_X \simeq \mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y(\mathrm{LC}_Y(X)), \omega_Y\big)[p]$$

*in $\mathbf{D}^b_{\mathrm{coh}}(X)$. In particular, if $X$ is a local complete intersection of pure codimension $r$, then*

$$\underline{\Omega}^p_X \simeq \mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\mathcal{H}^r_X(\mathscr{O}_Y), \omega_Y\big)[p+r].$$

Proof. Consider a strong log resolution $f\colon \widetilde{Y} \to Y$ of the pair $(Y, X)$, with $E = f^{-1}(X)_{\mathrm{red}}$. By Lemma 4.4.1 we have an isomorphism

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(j_*\mathbb{Q}^H_U[n]\big) \simeq \mathbf{R}f_*\Omega^{n-p}_{\widetilde{Y}}(\log E)[p].$$

We similarly have an isomorphism

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(\mathbb{Q}^H_Y[n]\big) \simeq \Omega^{n-p}_Y[p].$$

We apply the functor $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}(\cdot, \omega_Y)$; by Grothendieck duality we have

$$\mathbf{R}f_*\Omega^p_Y(\log E) \simeq \mathbf{R}f_*\big(\mathbf{R}\mathcal{H}om(\Omega^{n-p}_Y(\log E)(-E), \omega_Y)\big) \simeq$$

$$\simeq \mathbf{R}\mathcal{H}om\big(\mathbf{R}f_*\big(\Omega^{n-p}_Y(\log E)(-E)\big), \omega_X\big),$$

and therefore we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y(j_*\mathbb{Q}^H_U[n]), \omega_Y\big) & \longrightarrow & \mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\mathbb{Q}^H_Y[n], \omega_Y\big) \\
\downarrow & & \downarrow \\
\mathbf{R}f_*\big(\Omega^p_{\widetilde{Y}}(\log E)(-E)\big)[-p] & \longrightarrow & \Omega^p_Y[-p]
\end{array}$$

in which the vertical maps are isomorphisms. On the other hand, the exact triangle (4.4.1) in the derived category of Hodge modules induces the exact triangle

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y(\mathrm{LC}_Y(X)) \longrightarrow \mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\mathbb{Q}^H_Y[n] \longrightarrow \mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(j_*\mathbb{Q}^H_U[n]\big) \xrightarrow{+1} .$$

Applying $\mathbf{R}\mathcal{H}om_{\mathscr{O}_Y}(\,\cdot\,,\omega_Y)$ and using the above commutative diagram, as well as Steenbrink's triangle for Du Bois complexes in Section 2.4 (12), we obtain the first assertion in the proposition. The second one is an immediate consequence. □

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