Chebotarev Questions

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In my lecture I asked a question of a Chebotarev-sort about knots in S^3, but I set up the question incorrectly; it needs correction... or, at least, clarification.

So, this gives me a reason to return to the subject of knots in Chebotarev arrangements and

- to review some of the relevant literature about what one might call *Chebotarev Questions* in various contexts, and
- to remind people of ways of 'reverse engineering' number-theoretic results, such as the Chebotarev Theorem to get analogous questions about knots in S^3.

For a beautiful overview of this subject, see Morishita's [3].

1 Ordered sequences of disjoint knots as 'Chebotarev arrangements'

In [1] I described a "thought-experiment" that was not so much a problem to be resolved but rather " a somewhat casual way of appreciating *visually* how vastly *entangled* the collection of all primes are."

Imagine choosing one knot in every commensurable equivalence class of hyperbolic knots, fixing an orientation on each of these knots, and then arranging these (oriented) knots (up to equivalence) in S^3 so that they form a mutually disjoint ensemble:

$$\mathcal{C} := \sqcup_i K_i \ \subset S^3$$

where we have ordered them compatibly with their hyperbolic volume.

By an **admissible Galois cover of S^3 (relative to $\mathcal{C}$)** let us mean a finite connected cover

$$\begin{array}{ccccc} Y & \xrightarrow{=} & M^3 - f^{-1}\Sigma & \xrightarrow{\subset} & M^3 \\ \downarrow f & & \downarrow f & & \downarrow f \\ X & \xrightarrow{=} & S^3 - \Sigma & \xrightarrow{\subset} & S^3 \end{array}$$

Galois and ramified over at worst a finite subcollection of knots $\Sigma = K^{(1)} \sqcup K^{(2)} \sqcup \ldots \sqcup K^{(n)}$ of $\mathcal{C}$ in the natural sense[1]. So f restricted to $Y := M^3 - f^{-1}\Sigma$ the pullback of $S^3 - \Sigma$ is a locally trivial covering space of $X := S^3 - \Sigma$ with free action of a finite group G on M^3 (the "Galois group "of the cover) such that $Y/G = X$. The knots in $\mathcal{C}$ that are ramified in $M^3 \to S^3$ are contained in the finite set of knots $\Sigma \subset \mathcal{C}$.

One has a surjection

$$\pi_1(X) \twoheadrightarrow G$$

well-defined up to conjugacy, and for any (oriented) unramified knot $K \in \mathcal{C} \setminus \Sigma$ we have homomorphisms,

$$\mathbf{Z} \simeq \pi_1(K) \twoheadrightarrow \pi_1(X) \to G, \tag{1}$$

also well-defined up to conjugacy. Define, then, $\{Frob_K(M^3/S^3)\} \subset G$, the **Frobenius conjugacy class associated to** K in $G = \mathrm{Gal}(M^3/S^3)$ to be the conjugacy class containing the image of $1 \in \mathbf{Z}$ under the composition of the homomorphisms of Equation 1 above.

Thus, for all knots not in Σ—hence for all but finitely many knots in $\mathcal{C}$—we have a well-defined conjugacy class

$$\{Frob_K(M^3/S^3)\} \subset G.$$

Let us say that the collection $\mathcal{C}$ is a **Chebotarev Arrangement** if for all finite subsets $\Sigma \subset \mathcal{C}$ and for all finite Galois covers $M^3 \to S^3$ unramified outside Σ and every conjugacy class

$$\{c\} \subset G = \mathrm{Gal}(M^3/S^3)$$

the following statistical rule holds:

$$\lim_{k\to\infty} \frac{1}{k} \#\{K_i\, i \le k\ ,\ K_i \notin \Sigma\ ,\ \{Frob_{K_i}(M^3/S^3)\} = \{c\}\} \quad = \quad \frac{|\{c\}|}{|G|},$$

where the limit here is compiled by ordering the knots compatibly with their hyperbolic volume.

In effect, one is asking that—with these conventions—the Frobenius conjugacy classes are uniformly distributed in fundamental groups.

How many 'natural' (and interesting) Chebotarev arrangements are there? Given such, the rate of growth of the volumes of the K_i's—being analogous to the rate of growth of primes numbers—might be interesting to study.

[1] We call a knot in $\mathcal{C}$ *ramified* in this cover if it is branched in $f : M^3 \to S^3$; and if it isn't we say it is *unramified in the cover.*

2 Geodesics, closed orbits, and primes

In Peter Sarnak's PhD Thesis [6]*Prime Geodesic Theorems* (1980) he considers (among other things) (finite) Galois unramified coverings of Riemann surfaces of genus > 1, and says:

> By abuse of language, we will call an *oriented primitive closed geodesic* on any of the surfaces (relative to the Poincaré metric) a **prime**.

Sarnak then goes on to prove analogues of all the basic theorems regarding splitting of primes, and the Chebotarev theorem.

See also Sunada's Chebotarev-type theorem (Proposition II-2-12) in [8] which is framed in the context of Galois coverings of compact Riemannian manifolds of negative curvature.

Slightly later, William Parry and Mark Policott proved a Chebotarev-type theorem for closed orbits in the context of (Galois coverings of) Axiom A flows in Riemann surfaces (Axiom A as in Smale's [7]). See [4], [5]—and the bibliography there.

In three slightly different formats, McMullen's paper *Knots which behave like the prime numbers* [2] establish theorems of a Chebotarev type.

1. The knots $K_i \subset S^3$ arising from the periodic cycles of monodromy around the figure-eight knot, ordered by their lengths in a generic metric, obey the Chebotarev law. McMullen notes that "The same construction works for any fibered hyperbolic knot in S^3".

   More generally:

2. The closed orbits $\{K_i\}$ (ordered by length) of any topologically mixing pseudo-Anosov flow on any closed 3-manifold M obey an analogous Chebotarev law.
3. And for X is a closed surface of constant negative curvature, if $\{K_i\}$ (ordered by length) are the closed orbits of the geodesic flow in its tangent bundle, then these $\{K_i\}$ obey an analogous Chebotarev law.

Proof: Corollary 1.3, and Theorems 1.1 and 1.2 (respectively) of [2].

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References

[1] B. Mazur, Primes, knots and Po, For the conference "Geometry, Topology and Group Theory" in honor of the 80th birthday of Valentin Poenaru held in Autrans, July (2012) https://people.math.harvard.edu/~mazur/papers/Po8.pdf

[2] C. McMullen Knots which behave like the prime numbers (2012) Compos. Math., **149** (2013) 1235-1244 (`https://people.math.harvard.edu/~ctm/papers/home/text/papers/cheb/cheb.pdf`)

[3] M. Morishita, Knots and Primes: An Introduction to Arithmetic Topology (2012) Springer

[4] W. Parry, M. Pollicott, The Chebotarov theorem for Galois coverings of Axiom A flows, Ergod. Th. & D ynam. Sys. (1986) **6** 133-148

[5] W. Parry, M. Pollicott. Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics. Astérisque, (1990) 187-188

[6] P. Sarnak, Prime Geodesic Theorems, PhD. Stanford University (1980)

[7] S. Smale, Differentiable Dynamical Systems, Bull. Amer. Math. Soc. (1967) **73** 747-817

[8] T. Sunada, Geodesic Flows and Geodesic Random Walks, Advanced Studies in Pure Mathematics, *Geometry of Geodesics and Related Topics* (1984) **3** 47-85