MULTIVARIABLE CALCULUS

MATH S-21A

Unit 4: Lines and Planes

LECTURE

3.1. A point $P = (p, q, r)$ and a vector $\vec{v} = [a, b, c]$ define the **line**

$$L = \{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \; t \in \mathbb{R} \} .$$

The line consists of all points obtained by adding a multiple of the vector $\vec{v} = [a, b, c]$ to the vector $\vec{OP} = [p, q, r]$. It contains the point P as well as a copy of $\vec{v} = \vec{PQ}$ attached to P. Every vector contained in the line is necessarily parallel to $\vec{v}$. We think about the parameter t as "time". At $t = 0$, we are at the end point P of $\vec{OP}$ and at $t = 1$, we are at the end point Q of $\vec{OQ} = \vec{OP} + \vec{v}$.

3.2. If t is restricted to values in a **parameter interval** $[t_1, t_2]$, then $L = \{[p, q, r] + t[a, b, c], \; t_1 \leq t \leq t_2 \}$ is a **line segment** which connects $\vec{r}(t_1)$ with $\vec{r}(t_2)$. For example, to get the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$, form the vector $\vec{v} = \vec{PQ} = [1, 3, 4]$ and get $L = \{[x, y, z] = [1, 1, 2] + t[1, 3, 4]; \}$. This can be written also as $\vec{r}(t) = [1+t, 1+3t, 2+4t]$. If we write $[x, y, z] = [1, 1, 2]+t[1, 3, 4]$ as a collection of equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ and solve the first equation for t:

$$L = \{(x, y, z) \mid (x - 1)/2 = (y - 1)/3 = (z - 2)/4 \} .$$

3.3. The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

The reason is that each of these expressions is equal to t. These symmetric equations have to be modified a bit one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$. The interpretation is that the line is written as an intersection of two planes.

3.4. A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.
An example is $\Sigma = \{[x, y, z] = [1, 1, 2] + t[2, 4, 6] + s[1, 0, -1] \; \}$. This is called the **parametric description** of a plane.

3.5. If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector $\vec{n} = \vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = [a, b, c]$, this means $ax + by + cz = ax_0 + by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$. We have shown:

Theorem: The equation for a plane containing $\vec{v}$ and $\vec{w}$ and a point P is $ax + by + cz = d$, where $[a, b, c] = \vec{v} \times \vec{w}$ and where d is obtained by plugging in P.

3.6. Problem: Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = \vec{PQ} = [1, 2, 0]$ and $\vec{w} = \vec{PR} = [2, 2, 2]$. The normal vector $\vec{n} = \vec{v} \times \vec{w} = [4, -2, -2]$ leads to the equation $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of one of the points. In our case, it is $4x - 2y - 2z = -4$.

3.7. Problem: Find the angle between the planes $x + y = -1$ and $x + y + z = 2$, The **angle between the two planes** $ax + by + cz = d$ and $ex + fy + gz = h$ is defined as the angle between the two normal vectors $\vec{n} = [a, b, c]$ and $\vec{m} = [e, f, g]$.
Answer: find the angle between $\vec{n} = [1, 1, 0]$ and $\vec{m} = [1, 1, 1]$. It is $\arccos(2/\sqrt{6})$.

Examples

3.8. To practice, we look at **distance formulas**.

1) If P is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point Q, then

$$d(P, \Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between P and the plane. Proof: use the angle formula in the denominator. For example, to find the distance from $P = (7, 1, 4)$ to $\Sigma : 2x + 4y + 5z = 9$, we find first a a point $Q = (0, 1, 1)$ on the plane. Then compute

$$d(P, \Sigma) = \frac{|[-7, 0, -3] \cdot [2, 4, 5]|}{|[2, 4, 5]|} = \frac{29}{\sqrt{45}} .$$

2) If P is a point in space and L is the line $\vec{r}(t) = Q + t\vec{u}$, then

$$d(P, L) = \frac{|(\vec{PQ}) \times \vec{u}|}{|\vec{u}|}$$

is the distance between P and the line L. Proof: the area divided by base length is height of parallelogram. For example, to compute the distance from $P = (2, 3, 1)$ to the line $\vec{r}(t) = (1,1,2) + t(5,0,1)$, compute

$$d(P,L) = \frac{|[-1,-2,1] \times [5,0,1]|}{[5,0,1]} = \frac{|[-2,6,10]|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}} .$$

3) If L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$ which is normal to both lines. For example, to compute the distance between $\vec{r}(t) = (2,1,4) + t(-1,1,0)$ and M is the line $\vec{s}(t) = (-1,0,2) + t(5,1,2)$ form the cross product of $[-1,1,0]$ and $[5,1,2]$ is $[2,2,-6]$. The distance between these two lines is

$$d(L,M) = \frac{|(3,1,2) \cdot (2,2,-6)|}{|[2,2,-6]|} = \frac{4}{\sqrt{44}} .$$

4) To get the distance between two planes $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$, then their distance is

$$d(\Sigma,\Pi) = \frac{|e-d|}{|\vec{n}|}$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane. For example, $5x + 4y + 3z = 8$ and $10x + 8y + 6z = 2$ have the distance

$$\frac{|8-1|}{|[5,4,3]|} = \frac{7}{\sqrt{50}} .$$

FIGURE 1. The **global positioning system** GPS uses the fact that a receiver can get the difference of distances to two satellites.

Homework

This homework is due on Tuesday, 7/02/2024.

Problem 4.1: Given the three points $P = (11, 5, 6)$ and $Q = (2, 4, 10)$ and $R = (5, 3, 11)$, find the parametric equation for the line perpendicular to the triangle PQR passing through its center of mass $(P + Q + R)/3$.

Problem 4.2: The regular tetrahedron from Problem 1.1 had vertices at $A = (3, 4, 5), B = (3, 2, 3), C = (1, 4, 3), D = (1, 2, 5)$. What is the distance between two non-intersecting edges AB and CD? Discuss whether this distance is smaller or larger than the distance between A and CD.

Problem 4.3: Find a parametric equation for the line through the point $P = (3, 1, 2)$ that is perpendicular to the line $L : x = 1+4t, y = 1-4t, z = 8t$ and intersects this line in a point Q.

Problem 4.4: Given three spheres of radius 9 centered at $A = (1, 2, 0), B = (4, 5, 0), C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches all of three spheres from the same side.

Problem 4.5: a) Find the distance between the point $P = (3, 3, 4)$ and the line $2x = 2y = 2z$.
b) Parametrize the line $\vec{r}(t) = [x(t), y(t), z(t)]$ in a) and find the minimum of the function $f(t) = d(P, \vec{r}(t))^2$. Verify this agrees with a).

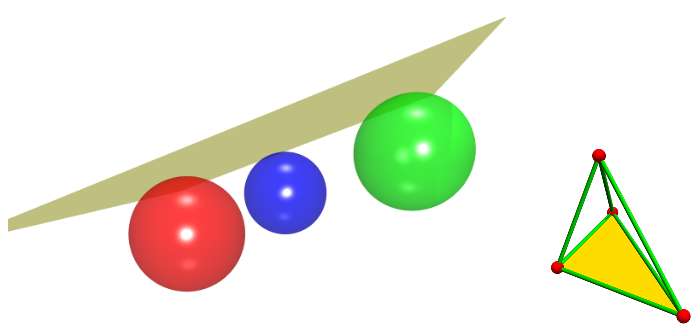

Figure 2. The sphere problem and the tetrahedron.

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