MULTIVARIABLE CALCULUS

MATH S-21A

Unit 10: Linearization

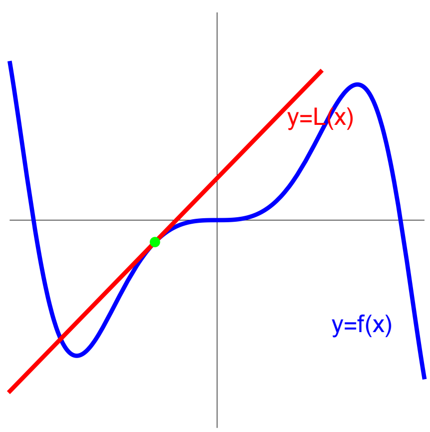
Lecture

10.1. In single variable calculus we have seen how to approximate functions by linear functions:

Definition: The **linear approximation** of $f(x)$ at a is the affine function

$$L(x) = f(a) + f'(a)(x - a) \ .$$

10.2. If you remember **Taylor series**, this is the part of the series $f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k/k!$, where only the $k = 0$ and $k = 1$ term are considered. We think about the linear approximation L as a function and not as a graph because we also will look at linear approximations for functions of three variables, where we can not draw graphs.



10.3. The graph of the function L is close to the graph of f at a. What about higher dimensions?

Definition: The **linear approximation** of $f(x, y)$ at (a, b) is the affine function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \ .$$

The **linear approximation** of $f(x, y, z)$ at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) \ .$$

10.4. Using the **gradient**

$$\nabla f(x,y) = [f_x, f_y], \qquad \nabla f(x,y,z) = [f_x, f_y, f_z] \, ,$$

the linearization can be written more compactly as

$$L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) \, .$$

10.5. How do we justify the linearization? If the second variable $y = b$ is fixed, we have a one-dimensional situation, where the only variable is x. Now $f(x,b) = f(a,b) + f_x(a,b)(x-a)$ is the linear approximation. Similarly, if $x = x_0$ is fixed y is the single variable, then $f(x_0,y) = f(x_0,y_0) + f_y(x_0,y_0)(y-y_0)$. Knowing the linear approximations in both the x and y variables, we can get the general linear approximation by $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

Examples

10.6. What is the linear approximation of the function $f(x,y) = \sin(\pi x y^2)$ at the point $(1,1)$? Answer: We have $[f_x(x,y), f_y(x,y)] = [\pi y^2 \cos(\pi x y^2), 2xy\pi \cos(\pi x y^2)]$ which is at the point $(1,1)$ equal to $\nabla f(1,1) = [\pi \cos(\pi), 2\pi \cos(\pi)] = [-\pi, -2\pi]$. The function is $L(x,y) = 0 + (-\pi)(x-1) - 2\pi(y-1)$.

10.7. Linearization can be used to estimate functions near a point. In the previous example,

$$f(1+0.01, 1+0.01) = -0.095$$

$$L(1+0.01, 1+0.01) = -\pi 0.01 - 2\pi 0.01 = -3\pi/100 = -0.0942 \, .$$

10.8. Here is an example in three dimensions: find the linear approximation to $f(x,y,z) = xy+yz+zx$ at the point $(1,1,1)$. Since $f(1,1,1) = 3$, and $\nabla f(x,y,z) = (y+z, x+z, y+x), \nabla f(1,1,1) = [2,2,2]$. we have $L(x,y,z) = f(1,1,1) + [2,2,2] \cdot [x-1, y-1, z-1] = 3 + 2(x-1) + 2(y-1) + 2(z-1) = 2x + 2y + 2z - 3$.

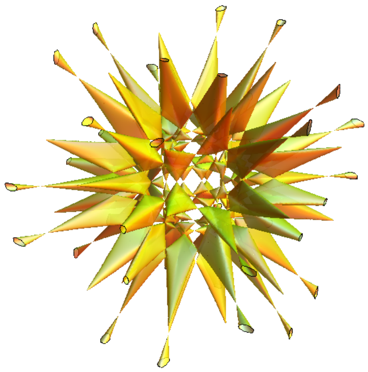
10.9. Estimate $f(0.01, 24.8, 1.02)$ for $f(x,y,z) = e^x \sqrt{y} z$.
Solution: take $(x_0,y_0,z_0) = (0,25,1)$, where $f(x_0,y_0,z_0) = 5$. The gradient is $\nabla f(x,y,z) = (e^x \sqrt{y} z, e^x z/(2\sqrt{y}), e^x \sqrt{y})$. At the point $(x_0,y_0,z_0) = (0,25,1)$ the gradient is the vector $(5, 1/10, 5)$. The linear approximation is $L(x,y,z) = f(x_0,y_0,z_0) + \nabla f(x_0,y_0,z_0)(x-x_0, y-y_0, z-z_0) = 5 + (5,1/10,5)(x-0, y-25, z-1) = 5x + y/10 + 5z - 2.5$. We can approximate $f(0.01, 24.8, 1.02)$ by $5 + (5, 1/10, 5) \cdot (0.01, -0.2, 0.02) = 5 + 0.05 - 0.02 + 0.10 = 5.13$. The actual value is $f(0.01, 24.8, 1.02) = 5.1306$, very close to the estimate.

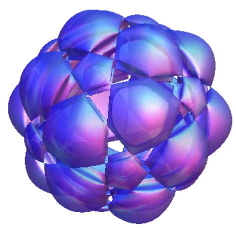
10.10. Find the tangent line to the graph of the function $g(x) = x^2$ at the point $(2,4)$.
Solution: the level curve $f(x,y) = y - x^2 = 0$ is the graph of a function $g(x) = x^2$ and the tangent at a point $(2, g(2)) = (2,4)$ is obtained by computing the gradient $[a,b] = \nabla f(2,4) = [-g'(2), 1] = [-4, 1]$ and forming $-4x + y = d$, where $d = -4 \cdot 2 + 1 \cdot 4 = -4$. The answer is $\boxed{-4x + y = -4}$ which is the line $y = 4x - 4$ of slope 4.

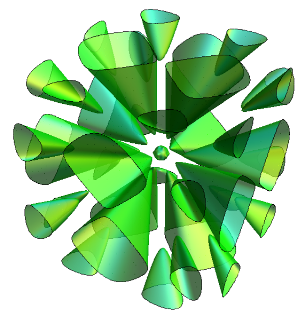
10.11. The **Barth surface** is defined as the level surface $f = 0$ of

$$f(x,y,z) = (3+5t)(-1+x^2+y^2+z^2)^2(-2+t+x^2+y^2+z^2)^2$$
$$+ 8(x^2-t^4y^2)(-(t^4x^2)+z^2)(y^2-t^4z^2)(x^4-2x^2y^2+y^4-2x^2z^2-2y^2z^2+z^4) ,$$

where $t = (\sqrt{5}+1)/2$ is a constant called the **golden ratio**. If we replace t with $1/t = (\sqrt{5}-1)/2$ we see the surface to the middle. For $t = 1$, we see to the right the surface $f(x,y,z) = 8$. Find the tangent plane of the later surface at the point $(1,1,0)$.
Answer: We have $\nabla f(1,1,0) = [64,64,0]$. The surface is $x+y = d$ for some constant d. By plugging in $(1,1,0)$ we see that $x+y = 2$.



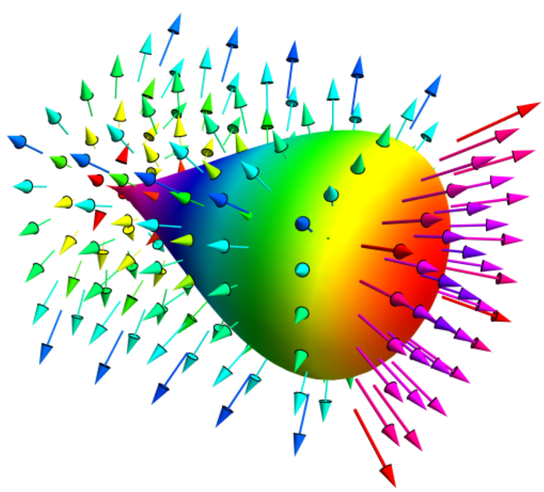




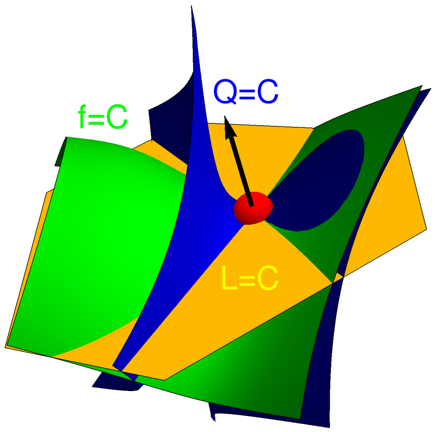
The quartic surface

$$f(x,y,z) = x^4 - x^3 + y^2 + z^2 = 0$$

is called the **piriform**. What is the equation for the tangent plane at the point $P = (2,2,2)$ of this pair shaped surface? We get $[a,b,c] = [20,4,4]$ and so the equation of the plane $20x+4y+4z = 56$, where we have obtained the constant to the right by plugging in the point $(x,y,z) = (2,2,2)$.



10.12. Linearization is just the first step for more accurate approximations. One could do **quadratic approximations** for example. In one dimension, one has $Q(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!}$. In two dimensions, this becomes $Q(x,y) = L(x,y) + H(a,b)[x-a, y-b] \cdot [x-a, y-b]/2$, where H is the **Hessian matrix** $H(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$. We will see this matrix next week, when we maximize or minimize functions.



Homework

This homework is due on Tuesday, 7/12/2022.

Problem 10.1: Estimate $400'000'000^{1/9}$ using linear approximation of $f(x) = x^{1/9}$ near $x_0 = 9^9$.

Problem 10.2: Given $f(x,y) = \frac{6yx}{\pi} - \cos(x)$. Estimate $f(\pi + 0.01, \pi - 0.03)$ using linearization

Problem 10.3: Estimate $f(0.003, 0.9999)$ for $f(x,y) = \cos(\pi y) + \sin(x + \pi y)$ using linearization.

Problem 10.4: Find the linear approximation $L(x,y)$ of the function

$$f(x,y) = \sqrt{10 - x^2 - 5y^2}$$

at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.

Problem 10.5: Estimate $(99^3 * 101^2)$ by linearising the function $f(x,y) = x^3y^2$ at $(100,100)$. What is the difference between $L(100,100)$ and $f(100,100)$?