7/25/2019 SECOND HOURLY PRACTICE IV Maths 21a, O.Knill, Summer 2019

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points). No justifications needed.

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) [✓ T] F If $x^2 + 2y^2 + 3z^2 = 1$ describes the surface of an ellipsoid, then the gradient ∇f points outwards.

Solution:

The gradient is $[2x, 4y, 6z]^T$.

2) T [✓ F] The chain rule tells that $\frac{d}{dt}\vec{r}(f(t)) = \nabla \vec{r}(f(t)) \cdot \vec{f}\ '(t)$ for any function $f(x, y, z)$ and any curve $\vec{r}(t)$.

Solution:

Nothing makes sense.

3) T [✓ F] The function $f(x, y) = x^4 - y^4$ has infinitely many critical points.

Solution:

There is only one critical point

4) [✓ T] F The point $(0, 1)$ is a maximum of $f(x, y) = y$ under the constraint $g(x, y) = x^2 + y^2 = 1$.

Solution:

The gradients are parallel.

5) [✓ T] F Every linear function $u(x, y) = ax + by + c$ solves the partial differential equation $u_{xx}(x, y) = u_{yy}(x, y)$.

Solution:

Just differentiate.

6) [✓ T] F Let $f(x,y) = x^3y$. At every point (x,y) there is a direction v for which $D_vf(x,y) = 0$.

Solution:

It is a general fact for directional derivatives as $D_vf(x,y) = -D_vf(x,y)$ implies that if the directional derivative is positive in one direction, it is negative in the opposite direction.

7) T [✓ F] If $f_{xy} = f_{yx}$ then $f(x,y) = xy$.

Solution:

Clairaut is always true even if $f(x,y)$ is not xy.

8) T [✓ F] $g(x,y) = \int_0^x \int_0^y f(s,t)\, dsdt$ satisfies $g_{xy} = f(x,y)$.

Solution:

By the fundamental theorem of calculus we have $g_{xy} = f(y,x)$ which is not the same.

9) [✓ T] F If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$.

Solution:

Because the vector r_v is tangent to the surface.

10) [✓ T] F The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $|D_{\vec{v}}f(0,0)|$ among all unit vectors $\vec{v}$.

Solution:

By the cos-formula.

11) [✓ T] F Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x,y)$ we have $\frac{d}{dt}f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t = 0$.

Solution:

This is the chain rule. The assumption $t = 0$ was necessary as otherwise $2t$ is a different point.

12) [✓ T] F $\int_0^{2\pi} \int_0^2 r \; dr d\theta = 4\pi$.

Solution:

It is the area.

13) [✓ T] F If $u(t, x)$ solves both the heat and wave equation, then $u_t = c \; u_{tt}$ for some constant c.

Solution:

Equal to u_{xx}.

14) T [✓ F] If the Lagrange multiplier λ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.

Solution:

The sign of λ has nothing to do with the nature of the critical point.

15) [✓ T] F The equation $f_{xy} f_{xx} f_{yy} = 1$ is an example of a partial differential equation.

Solution:

Yes, it is an equation for a function f involving partial derivatives.

16) T [✓ F] If the discriminant D of $f(x, y)$ is positive at $(0, 0)$ then $|\nabla f(0, 0)| > 0$.

Solution:

Take a minimum of $f(x, y) = x^2 + y^2$.

17) T [✓ F] If $f(x,y)$ is a continuous function then $\int_7^9 \int_5^7 f(x,y) \quad dxdy = \int_5^7 \int_7^9 f(y,x) \, dxdy$.

Solution:

Take a simple example like $f(x,y) = x$, then $f(y,x) = y$, which gives an other result.

18) T [✓ F] The point $(-3,1)$ is a critical point of $f(x,y) = x^2 + 3y^2$.

Solution:

The function f has only 0 as a critical point.

19) T [✓ F] If f has the critical point $(0,0)$, then f_x has the critical point $(0,0)$.

Solution:

Take an example $xy + y^2$

20) T [✓ F] $\int\int_R \sqrt{f_x^2 + f_y^2} \, dxdy$ is the surface area of the graph $z = f(x,y) = x^3 + y^3$ defined over the region R.

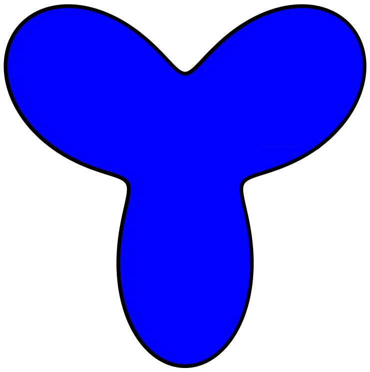
Solution:

It is not the formula. If you make the computation, you get $\sqrt{1 + f_x^2 + f_y^2}$.

Problem 2) (10 points) No justifications are needed

a) (6 points) Match the following regions with their area computation.

A



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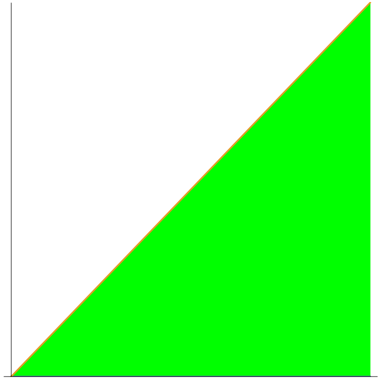

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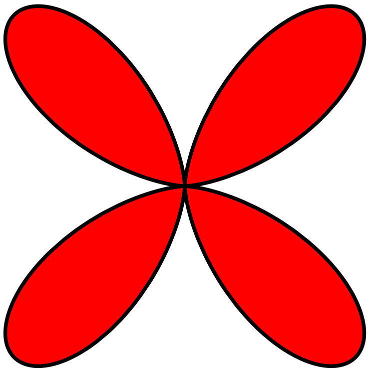

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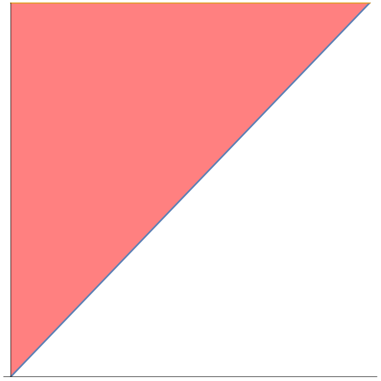

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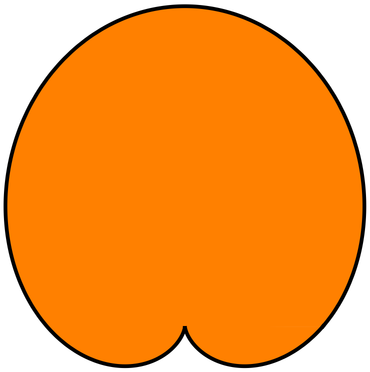


F

Enter A-F	Area Integral
	$\int_0^1 \int_y^1 1\ dxdy$
	$\int_0^{2\pi} \int_0^{\sin^2(\theta)\cos^2(\theta)} 1\ rdrd\theta$
	$\int_0^{2\pi} \int_0^{2+\sin(3\theta)} 1\ rdrd\theta$
	$\int_0^1 \int_0^{1-x} 1\ dydx$
	$\int_0^{2\pi} \int_0^{1+\sin(\theta)} 1\ rdrd\theta$
	$\int_0^1 \int_0^y 1\ dxdy$

b) (4 points) Match the following partial differential equations:

Fill in 1-4	Name
	Black Scholes
	Laplace
	Burgers
	Heat

Equation Number	PDE
1	$f_t = f_{xx}$
2	$f_t = f - xf_x - x^2 f_{xx}$
3	$f_t = ff_x - f_{xx}$
4	$f_{tt} + f_{xx} = 0$

Solution:

a) BCAFED
b) 2431

Problem 3) (10 points) (No justifications are needed.)

a) (5 points) Each answer is a number:

i) If $|\nabla f| = 1$ and $\vec{v} = \nabla f$, then $D_{\vec{v}} f =$ ☐

ii) The z component of the gradient ∇g of $g(x, y, z) = f(x, y) - z$ is ☐.

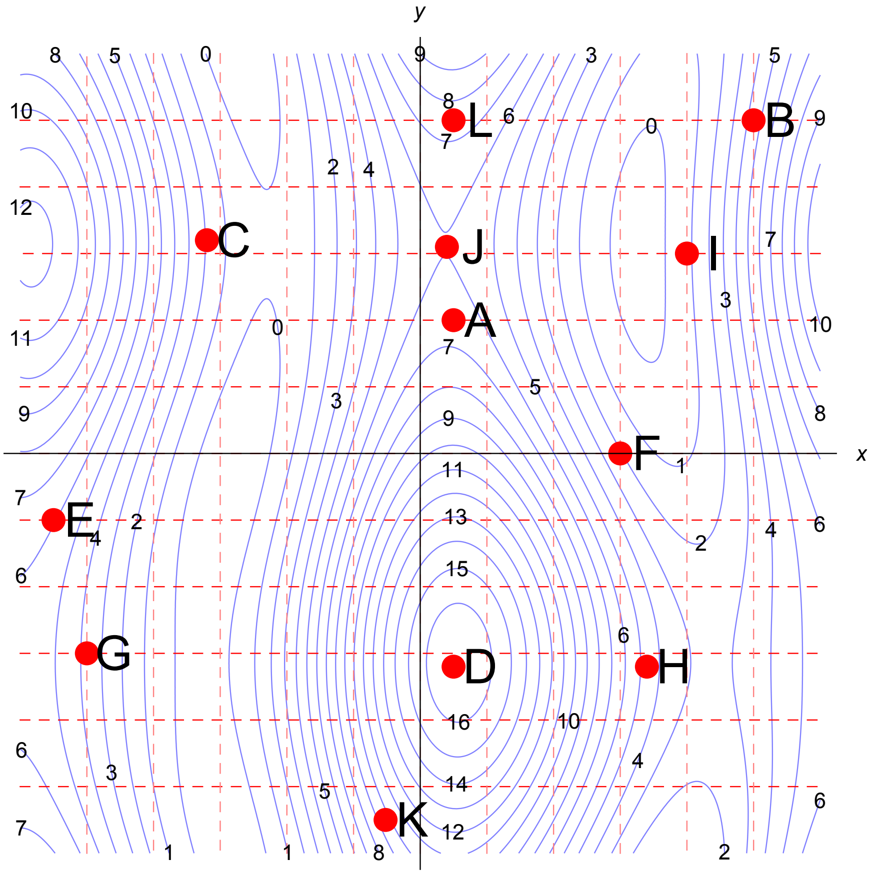
(iii) The directional derivative in a direction parallel to the level set is ☐.

(iv) If $\vec{r}(t)$ is a curve on $\{f = 1\}$, then $\frac{d}{dt} f(\vec{r}(t))$ is ☐

(v) The angle between the normal vector $\vec{n}$ to $L(x, y) = z$ at a point (x_0, y_0) and the gradient of f at (x_0, y_0) is ☐

b) (5 points) In each part, pick the correct point in $A - L$.

	Choose one A-H
A saddle point	
A local maximum	
A point where $\|\nabla f\|$ maximal	
A point where $f_x = 0$ and $f_y > 0$	
A point where $f_y = 0$ and $f_x > 0$	



Solution:

a) $1, -1, 0, 0, 0$
b) JDKLI

Problem 4) (10 points)

Between Harvard and MIT, there is a building with octagonal cross section. The living area is $f(a,b) = a^2 + 2b^2$, while the illuminated window area is $g(a,b) = 4ab$. Solve with Lagrange:

a) (5 points) Find (a,b) minimizing f under the constraint $g = 4$.

b) (5 points) Find (a,b) maximizing g under the constraint $f = \sqrt{8}$.

Of course, you should get the same solution $a > 0$ and $b > 0$ in both problems.

Solution:

a) The Lagrange equations are

$$\begin{aligned} 2a &= \lambda 4b \\ 4b &= \lambda 4a \\ ab &= 1 \end{aligned}$$

Cross multiplying the first two equations gives $8a^2 = 16b^2$ showing $a = \sqrt{2}b$. Plugging into the constraint gives $a = 2^{1/4}, b = 2^{-1/4}$.

b) The Lagrange equations are

$$\begin{aligned} 4b &= \lambda 2a \\ 4a &= \lambda 2b \\ ab &= 1 \end{aligned}$$

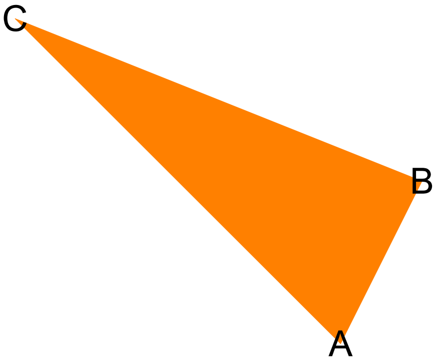
Cross multiplying the first two equations again gives $16b^2 = 8a^2$ showing $a = \sqrt{2}b$. Plugging into the constraint gives $a = 2^{1/4}, b = 2^{-1/4}$.

Problem 5) (10 points)

For every (x, y) we can define a triangle with vertices $(0, 0), (x, y), (y - 6x, 2x^2)$. Its area is

$$f(x, y) = 2x^3 - 6xy - y^2 \ .$$

Classify all critical points of f and in particular determine the values (x, y) for which the area is maximal.



Solution:

The gradient is $\nabla f(x, y) = [6x^2 + 6y, 6x - 2y]^T$. There are two critical points $(0, 0)$ and $(-3, 9)$. The first is a saddle point, the second is a maximum with value 27.

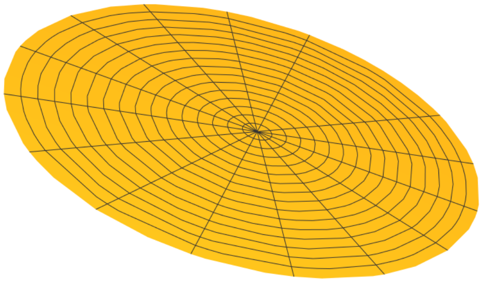
x	y	D	f_{xx}	Type	f
-3	9	36	-36	maximum	27
0	0	-36	0	saddle	0

Problem 6) (10 points)

Find the surface area of the surface parametrized by

$$\vec{r}(u, v) = [u + v, u - v, u + v]^T \ ,$$

where u, v is in the unit disc $u^2 + v^2 \leq 1$.



Solution:

The partial derivatives are

$$\vec{r}_u(u,v) = [1,1,1]^T ,$$
$$\vec{r}_v(u,v) = [1,-1,1]^T ,$$

The cross product is $[2,0,-2]^T$. It has length $\sqrt{8}$. The surface area is therefore $\sqrt{8}$ times the area of the disc which is $\int_0^{2\pi}\int_0^1 \sqrt{8}\; d\theta dr = \pi$. The answer is $\sqrt{8}\pi$.

Problem 7) (10 points)

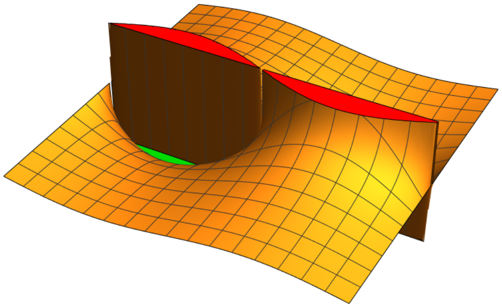
Estimate

$$\frac{\sin(\pi+0.001)}{(2.0001)} + 2.0001$$

by linearizing

$$f(x,y) = \frac{\sin(x)}{y} + y$$

at $(\pi, 2)$.



Solution:

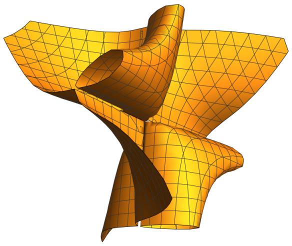
The gradient of f is $\nabla f(x,y) = [\cos(x)/y, 1-\sin(x)/y^2]^T$. At $(\pi,2)$, it is $[-1/2,1]^T$. Since $f(\pi,2)=2$, we estimate the number $2+(-1/2)\cdot 0.001 + 1\cdot 0.0001 = 1.9996$. The real value is only 2.510^{-8} off. It is 1.99960002508207.

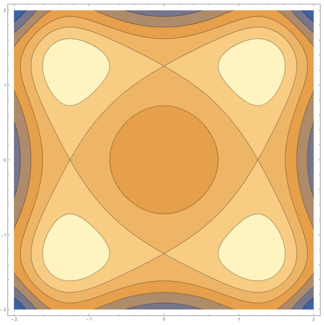
Problem 8) (10 points)

a) (5 points) Find the tangent plane to the surface

$$x^3y^3 + z^5 - 2xyz = 0$$

at the point $(1, 1, 1)$.





b) (5 points) Find the tangent line to the curve

$$\sin(x^2) + \sin(y^2) = c$$

which passes through the point $(1, 1)$.

Solution:

a) The gradient is $[1, 1, 3]^T$. The tangent plane is $x + y + 3z = 5$, where 5 was obtained by plugging in the point $(1, 1, 1)$.
b) The gradient is $[\cos(1), \cos(1)]^T$. The equation of the line is $x + y = 2$, where $d = 2$ was obtained by plugging in the point $(1, 1)$.

Problem 9) (10 points)

The following two integrals are called "Mad Max" integrals because they were written while watching that movie:

a) (5 points) Integrate

$$\int_0^1 \int_{\arcsin(y)}^{\pi/2} \frac{xy}{\sin(x)} \, dxdy \, .$$

b) (5 points) Integrate the double integral

$$\int \int_R \sin(x^2 + y^2) \, dxdy$$

where R is the disk of radius $\sqrt{\pi/2}$.

Solution:

a) Make a picture! Change the order of integration to get

$$\int_0^{\pi/2} \int_0^{\sin(x)} \frac{xy}{\sin(x)} \; dydx \; .$$

After solving the first integral, we get

$$\int_0^{\pi/2} x \sin(x) \; dx = 1/2 \; .$$

1/2.
b)

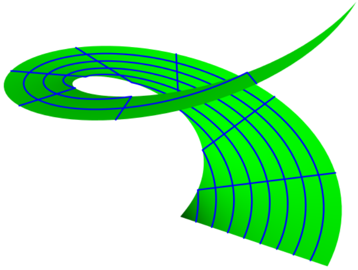
$$2\pi \int_0^{\sqrt{\pi/2}} \sin(r^2) r \; dr = -\pi \cos(r^2)|_0^{\sqrt{\pi}} = \pi \; .$$

Problem 10) (10 points)

We compute the **surface area** of the surface

$$\vec{r}(u,v) = [v\cos(u), v\sin(u), u]^T$$

over the region $R : 0 \leq u \leq 2\pi, u \leq v \leq 2\pi$.



a) (5 points) First verify that the integral is of the form

$$\iint_R \sqrt{1+v^2} \; dudv \; .$$

b) (5 points) Now compute the surface area integral.

Solution:

a) Take the cross product of $\vec{r}_u = [-v\sin(u), v\cos(u), 1]^T$ and $\vec{r}_v = [\cos(u), \sin(u), 0]^T$ to get $\vec{r}_u \times \vec{r}_u = [-\sin(u), \cos(u), -v]^T$ which has length $\sqrt{1+v^2}$.
b) The integral

$$\int_0^{2\pi}\int_u^{2\pi} \sqrt{1+v^2}\; dvdu$$

is not pleasant (doable as you have done in the homework). Better is to switch the order of integration. Draw the region which is a triangle and switch to a type II integral

$$\int_0^{2\pi}\int_0^{v} \sqrt{1+v^2}dudv = \int_0^{2\pi} v\sqrt{1+v^2}dv = (1/3)(1+v^2)^{3/2}|_0^{2\pi} = ((1+4\pi^2)^{3/2}-1)/3 \; .$$

The answer is $\boxed{(1+4\pi^2)^{3/2}/3 - 1/3}$.