7/25/2019 SECOND HOURLY PRACTICE I Maths 21a, O.Knill, Summer 2019

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- As usual, all functions are assumed to be differentiable unless stated otherwise.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

1) [**T**] [F] The gradient of $f(x,y) = x^2 + y^2$ at $(1,2)$ is perpendicular to the tangent line to $x^2 + y^2 = 5$ at $(1,2)$.

Solution:

It is indeed. This is the fundamental theorem of gradients.

2) [T] [**F**] The surface area of a surface parametrized by $\vec{r}(u,v)$ over a domain R in the uv-plane is given by the integral $\int\int_R |\vec{r}_u \times \vec{r}_v|^2 \, dudv$.

Solution:

NOOO!!! It is not the square. That would later hunt many in the surface area problem. It is the area of the parallelogram, not the square of the area!

3) [**T**] [F] If the partial derivative $f_x(x,y)$ is zero for all x, then f only depends on y.

Solution:

Yes, this is like in single variable calculus

4) [**T**] [F] The surface area of a cylinder of height 2 and radius 1 is the same than the surface area of a sphere of radius 1

Solution:

This was mentioned in class. A discovery of Archimedes

5) [T] [**F**] The point $(1,1)$ is a critical point of $f(x,y) = x^5 - y^5$.

Solution:

There is only one critical point $(0,0)$

6) T F

The function $f(x, y) = 4x^2 - 5y^2$ has a global minimum under the constraint $y = 0$.

Solution:

It is $4x^2$.

7) T F

Assume $(1, 1)$ is a local maximum and $(0, 0)$ is a local minimum of a function $f(x, y)$, then $\nabla f(0, 0) = \nabla f(1, 1)$.

Solution:

They are both zero.

8) T F

If $(0, 0)$ is a minimum for f with $D > 0$, then the second derivative $D_{\vec{v}} f(0, 0)$ is negative for all unit vectors $\vec{v}$.

Solution:

It is zero. The second derivative is negative for all v.

9) T F

If $(0, 0)$ is a local maximum for f, then $f_{xx}(0, 0) \leq 0$.

Solution:

Yes, it can be negative or zero. The later happens for $1 - x^4 - y^4$. If it were positive, then it would not be a local maximum.

10) T F

For $u = [1, 0]^T$ and $v = [0, 1]^T$ we have $D_v D_u f = D_u D_v f$.

Solution:

This is Clairaut.

11) T F

If $\vec{r}(t)$ is a curve on the surface $f(x, y, z) = 1$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ for all t.

Solution:

Yes, this is the key insight for the Gradient theorem

12) T **F** The function $f(x, y) = x^2 - y^2$ solves the partial differential equation $f_x^2 + f_y^2 = 4f$.

Solution:

Just differentiate.

13) **T** F If $f_x(x, y) = f_y(x, y)$ then $f(x, y) = f(y, x)$.

Solution:

Yes, then f is a solution of the transport equation and of the form $g(x + y)$

14) T **F** The identity $\int_0^2 \int_0^{x^2} 1 \, dydx = \int_0^4 \int_0^{y^2} 1 dxdy$ holds.

Solution:

The change of the order is different

15) T **F** We have $\int_{-\infty}^{\infty} e^{x^2} \, dx = \pi$.

Solution:

This was from the movie clip gifted where the girl was given the wrong sign problem

16) **T** F Using linearization we can estimate $(1.005)^2(1.0002)^3 \approx 1 + 2 \cdot 0.005 + 3 \cdot 0.0002$.

Solution:

Yes, this is the linearization.

17) | T | F | If $f(x,t)$ solves the wave equation, then $f(x,-t)$ solves the wave equation.

Solution:

Both the sign of f_{tt} and f_{xx} change sign.

18) | T | F | If the value $D_{\vec{u}}f(0,0)$ is independent of the choice of unit vector $\vec{u}$, then $(0,0)$ is a critical point of $f(x,y)$.

Solution:

If the gradient is not zero, there will be a direction where the directional derivative is positive (the direction of the gradient) and directions, where it is negative (like the direction opposite to the gradient.

19) | T | F | $f(x,y) = e^{x^2+y}\cos(x-y)$ satisfies the partial differential equation $f_{xxyy} = f_{yyxx}$.

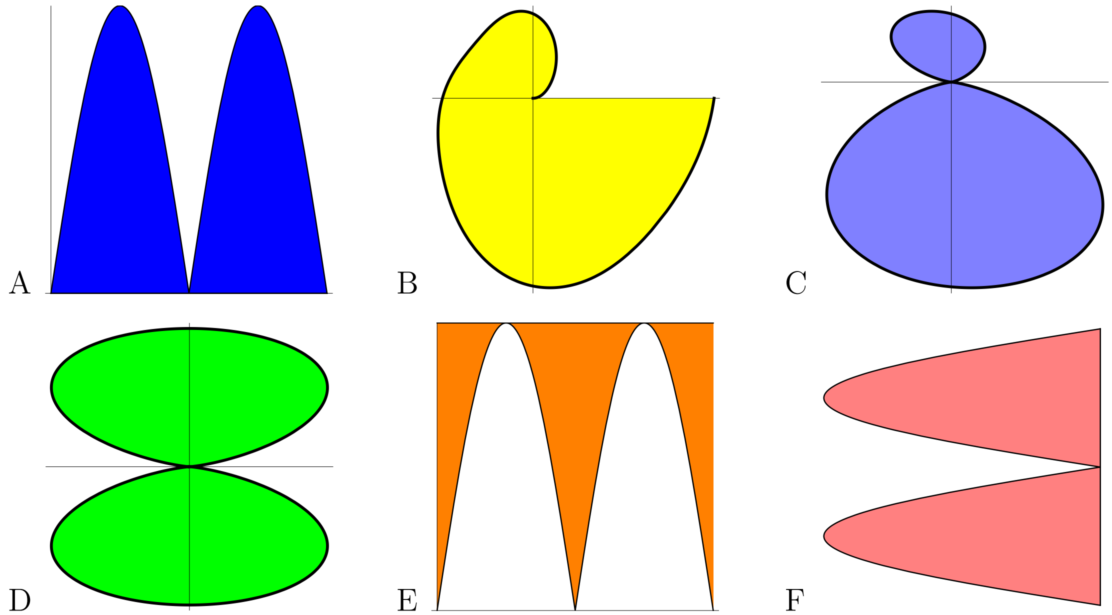
20) | T | F | If $f_{xx}f_{yy} = f_{xy}^2$ for all x, y, then f is constant.

Solution:

Take x^4 for example.

Problem 2) (10 points) No justifications are needed

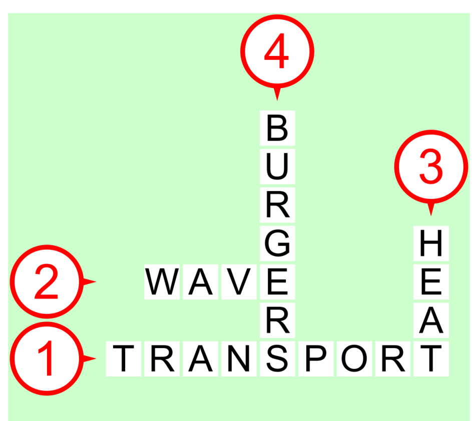
a) (6 points) Match the following regions with their area computation. There should be an exact match meaning that every entry A-F as well as the entry O (for no match) all appear exactly once.



Enter A-F or O	Area Integral
	$\int_0^{2\pi} \int_0^{\lvert \sin(x) \rvert} 1 \; dydx$
	$\int_0^{2\pi} \int_{1-\lvert \sin(y) \rvert}^{1} 1 \; dxdy$
	$\int_0^{2\pi} \int_0^{\lvert \sin(y) \rvert} 1 \; dxdy$
	$\int_0^{2\pi} \int_{\lvert \sin(x) \rvert}^{1} 1 \; dydx$
	$\int_0^{2\pi} \int_0^{\sin^4(\theta)} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{\theta \sin^4(\theta)} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{1-\sin^4(\theta)} r \; drd\theta$

b) (4 points) We design a crossword puzzle. Identify the PDEs. There will be two entries O and all the entries 1-4 appear exactly once.

Enter 1-4 or O if no match	
	$f_t = f_{xx}$
	$f_t = f_x$
	$f_{tt} = -f_{xx}$
	$f_{tt} = f_{xx}$
	$f_t + f f_x = f_{xx}$
	$f_x^2 + f_y^2 = 1$



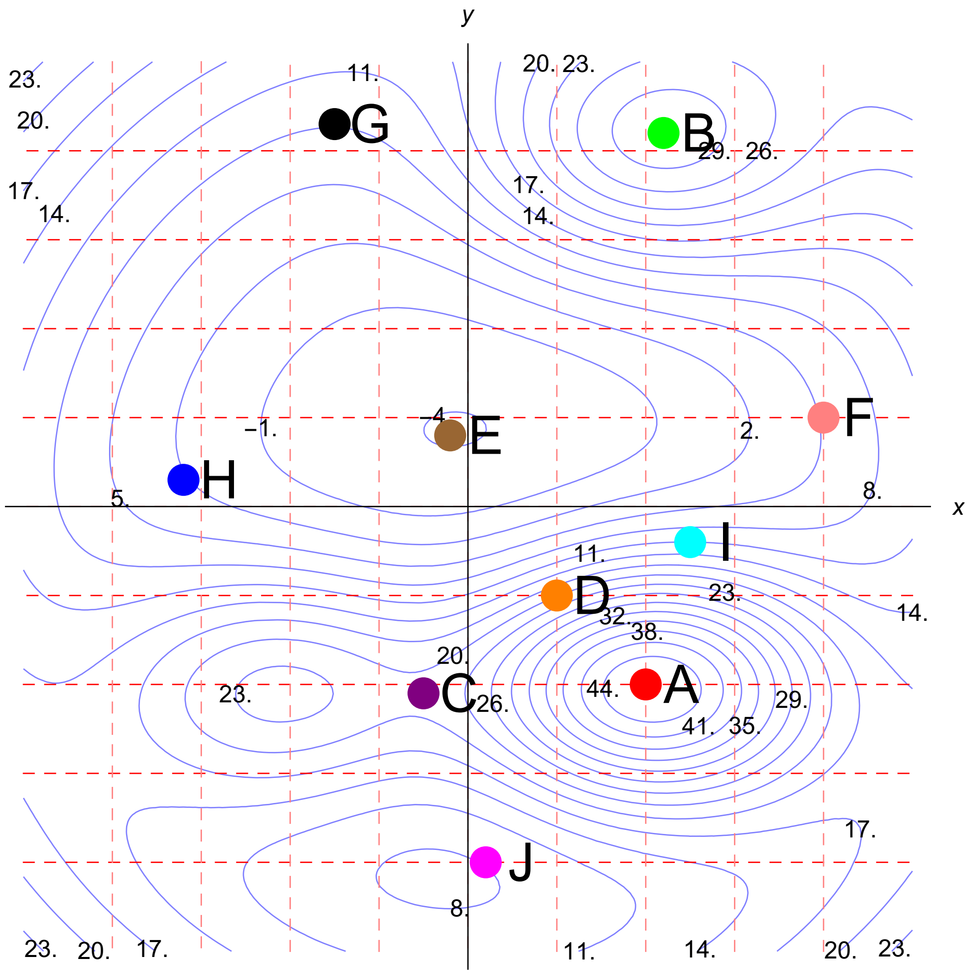
Solution:

a) AFOEDCB.
In the actual exam, it was stated $\int_0^{2\pi} \int_{|\sin(y)|}^1 1 \, dxdy$ so that $AOFEDCB$ or $AOOFEDCB$ were both also accepted.
b) The magic number is 310240.

Problem 3) (10 points) (No justifications are needed.)

There is an exact match here meaning that each of the points $A - J$ appears exactly once. Of course, every correct answer gives one point. We use below the unit vectors $v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $w = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

	Choose one A-J
At which point is $f_x = 0$ and $f_y > 0$?	
Which point is a local but not global maximum on the region?	
Which point is a saddle point?	
At which point is $f_x < 0$ and $f_y < 0$?	
At which point is $f_x > 0$ and $f_y = 0$?	
Which point is a global maximum on the region?	
At which point is the length of the gradient maximal among $A - J$?	
Which point is a global minimum in the region?	
At which point is $f_x = 0$ and $f_y < 0$?	
At which point is $D_v f > 0, D_w f = 0$?	



Solution:

GBCHFADEIJ

Problem 4) (10 points)

On the top of a MIT building there is a radar dome in the form of a spherical cap. Insiders call it the **"Death star" radar dome**. We know that with the height h and base radius a, we have volume and surface area given by $V = \pi r h^2 - \pi h^3/3$, $A = 2\pi r h = \pi$. This leads to the problem to extremize

$$f(x, y) = xy^2 - \frac{y^3}{3}$$

under the constraint

$$g(x, y) = 2xy = 1 \ .$$

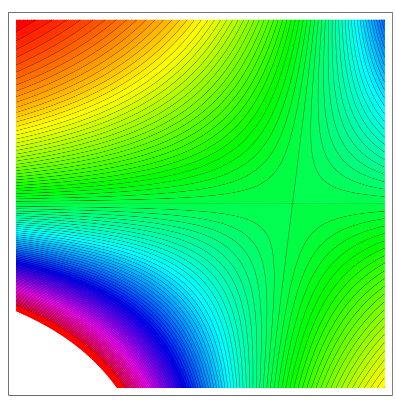
Find the minimum of f on this constraint using the Lagrange method!

Solution:

The Lagrange equations lead to $x = y$ and so $x = y = 1/\sqrt{2}$.

Problem 5) (10 points)

a) (5 points) Find and classify the critical points of $f(x, y) = xy$. We **know you know** the answer but we also **know** that you might get **knocked out** by the quotient rule in b) or by **"you-know-who"**. So here is the place to **show you know your know-how**. Even so the answer might be obvious to you, you have to give all details.



b) (5 points) Find and classify the critical point of the function

$$f(x, y) = \frac{2e^{xy}}{e^y + 1} \ .$$

There is one critical point. You are assured that the discriminant D is not zero. You do not need to compute D to find the nature of the critical point. If you want, you can still compute D and proceed as usual; we just let you know that you should be able to see after computing f_{xx}, what the nature of the critical point is, if you know $D \neq 0$. Some background information: this function is important as it can be written as $f(x, y) = \sum_{n=0}^{\infty} E_n(x) \frac{y^n}{n!}$, where $E_n(x)$ are the **Euler polynomials**.

Solution:

a) The gradient is $[y, x]^T$ so that the only critical point is $(0, 0)$. The discrminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 1 = -1$. It is a saddle point.

b) The gradient is $[2ye^{xy}/(e^y + 1), (2xe^{xy}/(e^y + 1) - 2e^{xy}e^y)/(e^y + 1)^2]^T$. We have used the quotient rule. As predicted, many did not no more master that one fully.

> Personal tip: if you forget, use the product rule for f and (1/g). This gives $f'/g - fg'/g^2$ This is equivelent to the quotient fule $(f'g - fg')/g^2$.

The first component f_x is zero only if $y = 0$. Plugging this into the second gives $x = 1/2$. Because $f_{xx} = 2y^2e^{xy}/(e^y + 1) = 0$, the knowledge of $D \neq 0$ forces $D < 0$. We have a saddle point.

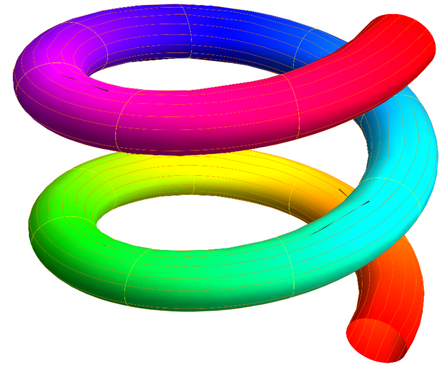
Problem 6) (10 points)

A **spring** is parametrized as

$$\begin{aligned} x &= 5\cos\theta + \cos\alpha\cos\theta \\ y &= 5\sin\theta + \cos\alpha\sin\theta \\ z &= \theta + \sin\alpha \ , \end{aligned}$$

where $0 \leq \theta \leq 4\pi$ and $0 \leq \alpha \leq 2\pi$ are angles. Find the surface area of the spring surface (without the two discs closing it off).

You are given and can use without justification that $|\vec{r}_\alpha \times \vec{r}_\theta|^2 = (5 + \cos\alpha)^2$.

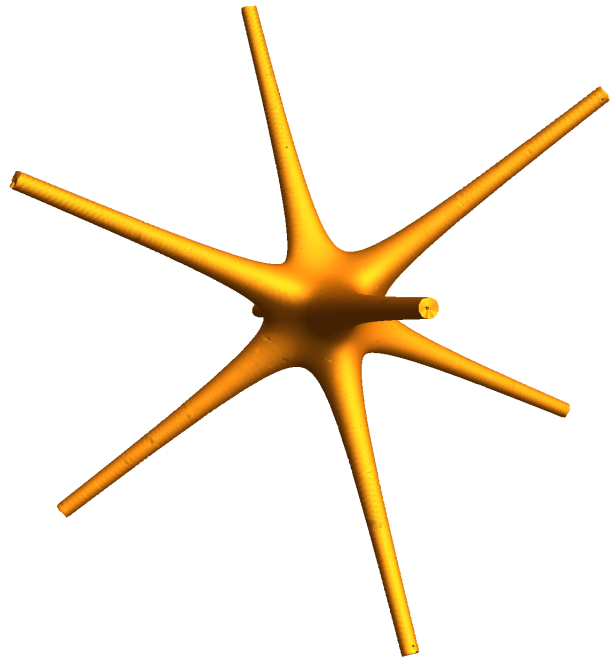


Solution:

$\int_0^{2\pi} \int_0^{4\pi} (5 + \cos(\alpha)\ d\theta d\alpha = (4\pi)(2\pi)5 = 40\pi^2$.

The main mistake was here that many integrated $(5 + \cos(\alpha))^2$ and not $(5 + \cos(\alpha)$.

Problem 7) (10 points)



a) (6 points) Find the tangent plane to the surface

$$f(x,y,z) = x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 = 1 .$$

at the point $(0,1,1)$.

We call it the **octo star** evenso we have not spotted it in the drawings of **Ernst Haeckel** about "art forms from the abyss". A page from the marvellous book is displayed at the end of this exam. Examining these wonderful creatures can either calm your nerves (or freak you out).

b) (4 points) Find the linearization $L(x,y,z)$ of f at $(0,1,1)$.

Solution:

a) The gradient is $[a,b,c]^T = [0,2,2]^T$. The tangent plane is $2y + 2z = d$. The constant d is obtained by plugging in $x = 0, y = 1, z = 1$ which gives $2y + 2z = 4$.
b) The linearization is $L(x,y,z) = f(0,1,1) + a(x-0) + b(y-1) + c(z-1) = 1 + 2(y-1) + 2(z-1)$.

Problem 8) (10 points)

The probability density of the location of the sunken ship **Dmitrii Donskoi** is given by an unknown function $f(x,y)$. The ship sank off the **Ulleungdo island** in 1905 between South Korea and Japan. Eager to get our hands on the **200 tons of gold**, we want to know more about the probability density function.

" DIMITRI DONSKOI "

a) (4 points) Given two directions $v = \begin{bmatrix} 5 \\ 12 \end{bmatrix}/13$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we know $f(2,3) = 8$ and the directional derivatives $D_v f(2,3) = 1$, $D_w f(2,3) = -1$. Find the gradient $\begin{bmatrix} a \\ b \end{bmatrix}$ of f at $(2,3)$.

b) (3 points) Estimate $f(2.01, 3.0001)$ using linear approximation at $(2,3)$.

c) (3 points) Find the tangent line at the point $(2,3)$ to the level curve $f(x,y) = 8$.

Solution:

a) Writing down the equation gives We know $a \cdot 5 + b \cdot 12 = 13$ and $b \cdot 1 = -1$. The second equation gives $b = -1$. The first equation gives $a = 5$. The gradient is $[5,-1]^T$.
b) The estimation is $8 + 5 \cdot 0.01 + (-1) \cdot 0.001$.
c) The tangent line is $L(x,y) = 5x + (-1)y = d$, where d is the constant obtained by plugging in the point $(2,3)$. It is $d = 7$. The line is $5x - y = 7$.

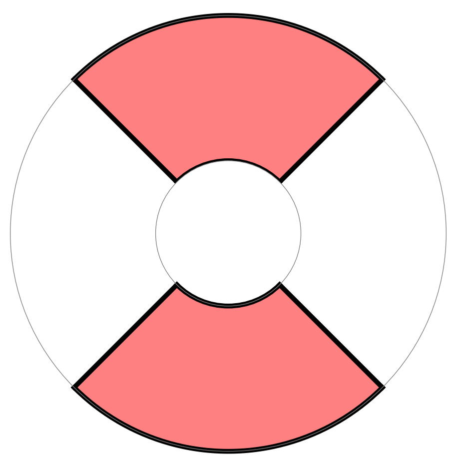
Problem 9) (10 points)

a) (5 points) Evaluate the double integral

$$\int_{-1}^{1} \int_{x^2}^{1} \frac{e^y}{\sqrt{y}} \, dy dx \ .$$

b) (5 points) Let $R = \{(x,y) \mid 1 \leq x^2 + y^2 \leq 9, y^2 > x^2\}$. Integrate

$$\int\int_R 2e^{-x^2-y^2} \, dxdy \ .$$



Solution:

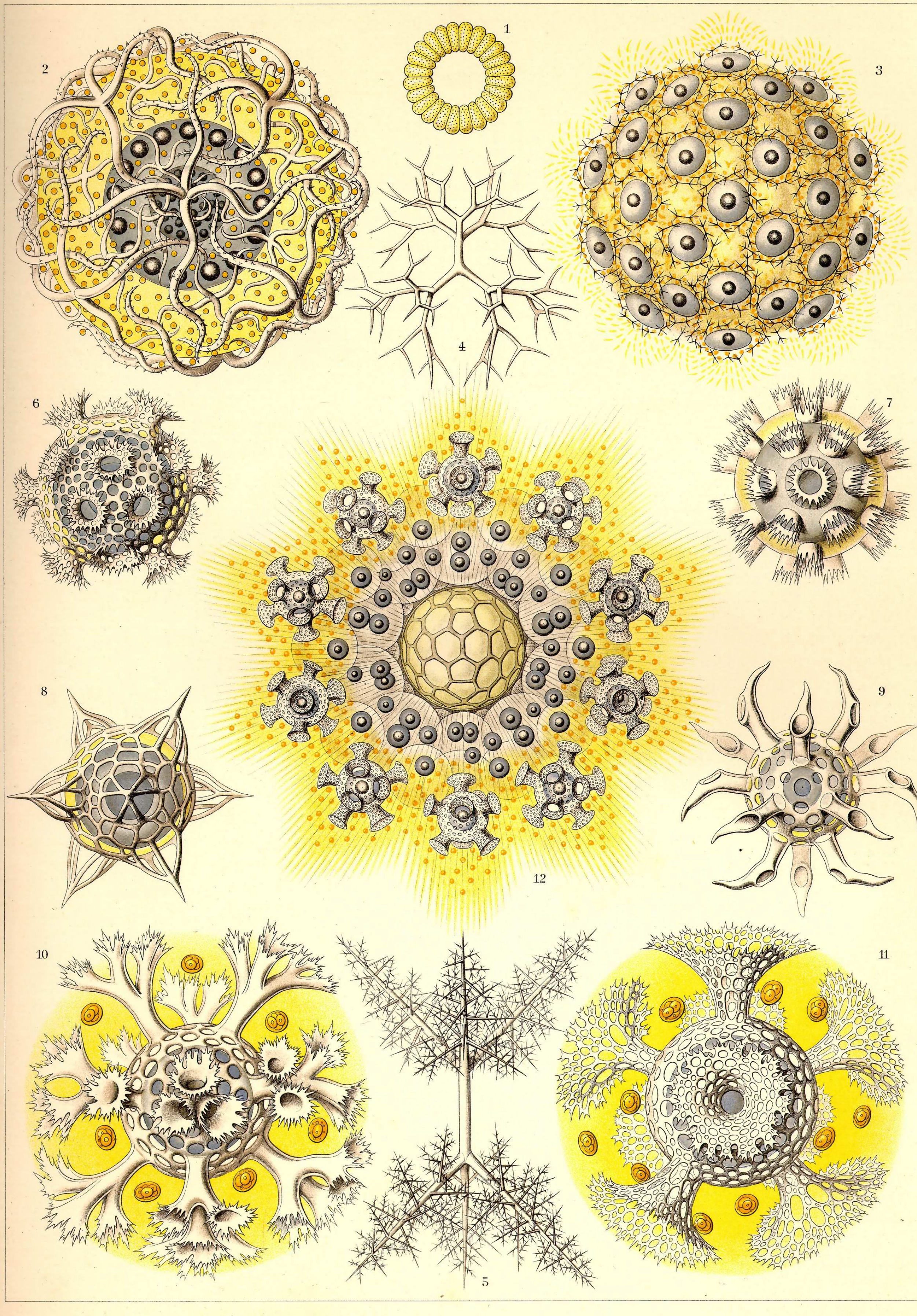
a) Change the order of integration. Almost universally, if no picture was drawn, the problem was not solved correctly.

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \frac{e^y}{\sqrt{y}} \, dxdy = 2(e^1 - e^0) = 2e - 2 \; .$$

b) Polar of course. We can integrate θ from $\pi/4$ to $3\pi/4$ and count everything twice. The integral is

$$2 \int_{\pi/4}^{3\pi/4} \int_1^3 2e^{-r^2} r \; drd\theta = 2\pi/2(e^{-1} - e^{-9}) = \pi(1/e - 1/e^9) \; .$$

We used substitution $r^2 = u$, $2rdr = du$.



Polycyttaria. — Vereins-Strahlinge.