7/25/2019 SECOND HOURLY PRACTICE I Maths 21a, O.Knill, Summer 2019

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- As usual, all functions are assumed to be differentiable unless stated otherwise.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

1) T F The gradient of $f(x,y) = x^2 + y^2$ at $(1,2)$ is perpendicular to the tangent line to $x^2 + y^2 = 5$ at $(1,2)$.

2) T F The surface area of a surface parametrized by $\vec{r}(u,v)$ over a domain R in the uv-plane is given by the integral $\int\int_R |\vec{r}_u \times \vec{r}_v|^2 \, dudv$.

3) T F If the partial derivative $f_x(x,y)$ is zero for all x, then f only depends on y.

4) T F The surface area of a cylinder of height 2 and radius 1 is the same than the surface area of a sphere of radius 1

5) T F The point $(1,1)$ is a critical point of $f(x,y) = x^5 - y^5$.

6) T F The function $f(x,y) = 4x^2 - 5y^2$ has a global minimum under the constraint $y = 0$.

7) T F Assume $(1,1)$ is a local maximum and $(0,0)$ is a local minimum of a function $f(x,y)$, then $\nabla f(0,0) = \nabla f(1,1)$.

8) T F If $(0,0)$ is a minimum for f with $D > 0$, then the second derivative $D_{\vec{v}} f(0,0)$ is negative for all unit vectors $\vec{v}$.

9) T F If $(0,0)$ is a local maximum for f, then $f_{xx}(0,0) \leq 0$.

10) T F For $u = [1,0]^T$ and $v = [0,1]^T$ we have $D_v D_u f = D_u D_v f$.

11) T F If $\vec{r}(t)$ is a curve on the surface $f(x,y,z) = 1$, then $\frac{d}{dt} f(\vec{r}(t)) = 0$ for all t.

12) T F The function $f(x,y) = x^2 - y^2$ solves the partial differential equation $f_x^2 + f_y^2 = 4f$.

13) T F If $f_x(x,y) = f_y(x,y)$ then $f(x,y) = f(y,x)$.

14) T F The identity $\int_0^2 \int_0^{x^2} 1 \, dydx = \int_0^4 \int_0^{y^2} 1 dxdy$ holds.

15) T F We have $\int_{-\infty}^{\infty} e^{x^2} \, dx = \pi$.

16) T F Using linearization we can estimate $(1.005)^2(1.0002)^3 \approx 1 + 2 \cdot 0.005 + 3 \cdot 0.0002$.

17) T F If $f(x,t)$ solves the wave equation, then $f(x,-t)$ solves the wave equation.

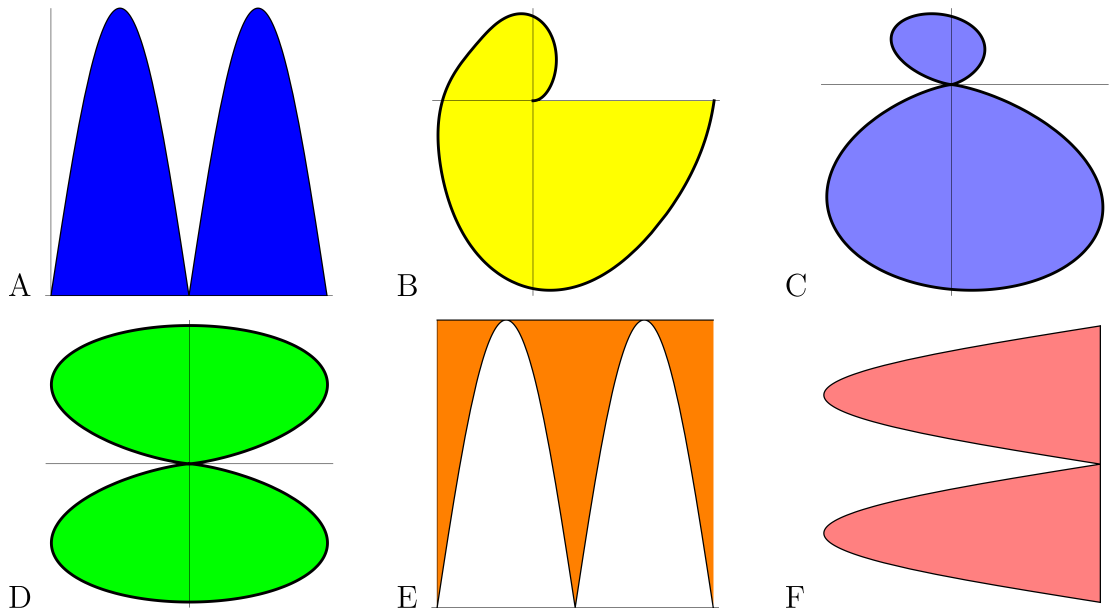
18) T F If the value $D_{\vec{u}} f(0,0)$ is independent of the choice of unit vector $\vec{u}$, then $(0,0)$ is a critical point of $f(x,y)$.

19) T F $f(x,y) = e^{x^2+y} \cos(x-y)$ satisfies the partial differential equation $f_{xxyy} = f_{yyxx}$.

20) T F If $f_{xx} f_{yy} = f_{xy}^2$ for all x, y, then f is constant.

Problem 2) (10 points) No justifications are needed

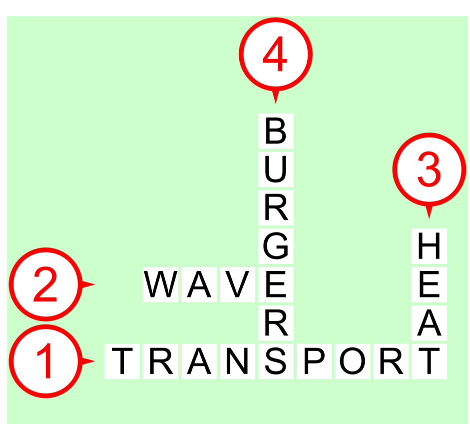
a) (6 points) Match the following regions with their area computation. There should be an exact match meaning that every entry A-F as well as the entry O (for no match) all appear exactly once.



Enter A-F or O	Area Integral
	$\int_0^{2\pi} \int_0^{\vert\sin(x)\vert} 1 \; dydx$
	$\int_0^{2\pi} \int_{1-\vert\sin(y)\vert}^{1} 1 \; dxdy$
	$\int_0^{2\pi} \int_0^{\vert\sin(y)\vert} 1 \; dxdy$
	$\int_0^{2\pi} \int_{\vert\sin(x)\vert}^{1} 1 \; dydx$
	$\int_0^{2\pi} \int_0^{\sin^4(\theta)} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{\theta\sin^4(\theta)} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{1-\sin^4(\theta)} r \; drd\theta$

b) (4 points) We design a crossword puzzle. Identify the PDEs. There will be two entries O and all the entries 1-4 appear exactly once.

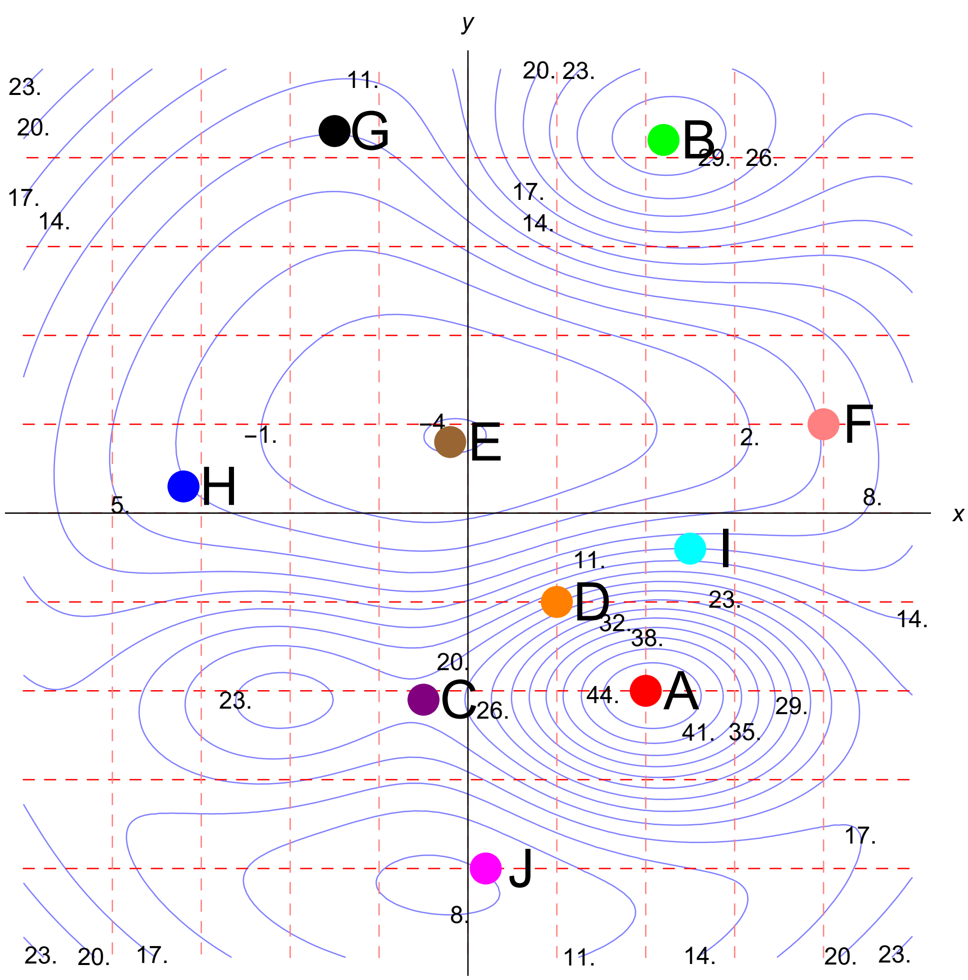
Enter 1-4 or O if no match	
	$f_t = f_{xx}$
	$f_t = f_x$
	$f_{tt} = -f_{xx}$
	$f_{tt} = f_{xx}$
	$f_t + ff_x = f_{xx}$
	$f_x^2 + f_y^2 = 1$



Problem 3) (10 points) (No justifications are needed.)

There is an exact match here meaning that each of the points $A - J$ appears exactly once. Of course, every correct answer gives one point. We use below the unit vectors $v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $w = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.

	Choose one A-J
At which point is $f_x = 0$ and $f_y > 0$?	
Which point is a local but not global maximum on the region?	
Which point is a saddle point?	
At which point is $f_x < 0$ and $f_y < 0$?	
At which point is $f_x > 0$ and $f_y = 0$?	
Which point is a global maximum on the region?	
At which point is the length of the gradient maximal among $A - J$?	
Which point is a global minimum in the region?	
At which point is $f_x = 0$ and $f_y < 0$?	
At which point is $D_v f > 0, D_w f = 0$?	



Problem 4) (10 points)

On the top of a MIT building there is a radar dome in the form of a spherical cap. Insiders call it the **"Death star" radar dome**. We know that with the height h and base radius a, we have volume and surface area given by $V = \pi r h^2 - \pi h^3/3$, $A = 2\pi r h = \pi$. This leads to the problem to extremize

$$f(x,y) = xy^2 - \frac{y^3}{3}$$

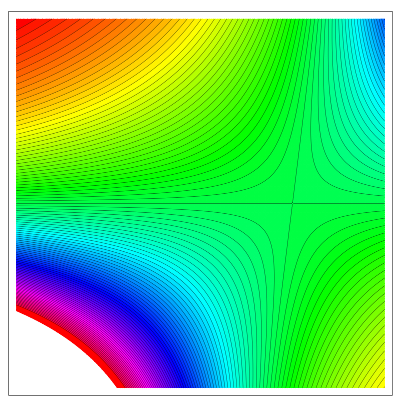
under the constraint

$$g(x,y) = 2xy = 1 \ .$$

Find the minimum of f on this constraint using the Lagrange method!

Problem 5) (10 points)

a) (5 points) Find and classify the critical points of $f(x,y) = xy$. We **know you know** the answer but we also **know** that you might get **knocked out** by the quotient rule in b) or by **"you-know-who"**. So here is the place to **show you know your know-how**. Even so the answer might be obvious to you, you have to give all details.



b) (5 points) Find and classify the critical point of the function

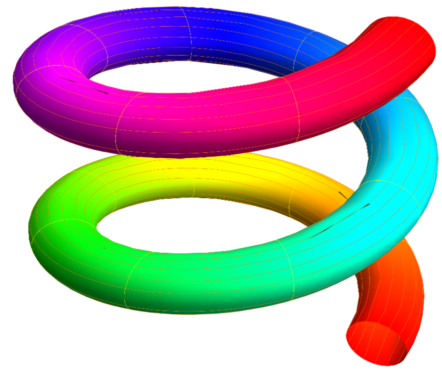
$$f(x,y) = \frac{2e^{xy}}{e^y + 1} \ .$$

There is one critical point. You are assured that the discriminant D is not zero. You do not need to compute D to find the nature of the critical point. If you want, you can still compute D and proceed as usual; we just let you know that you should be able to see after computing f_{xx}, what the nature of the critical point is, if you know $D \neq 0$. Some background information: this function is important as it can be written as $f(x,y) = \sum_{n=0}^{\infty} E_n(x) \frac{y^n}{n!}$, where $E_n(x)$ are the **Euler polynomials**.

Problem 6) (10 points)

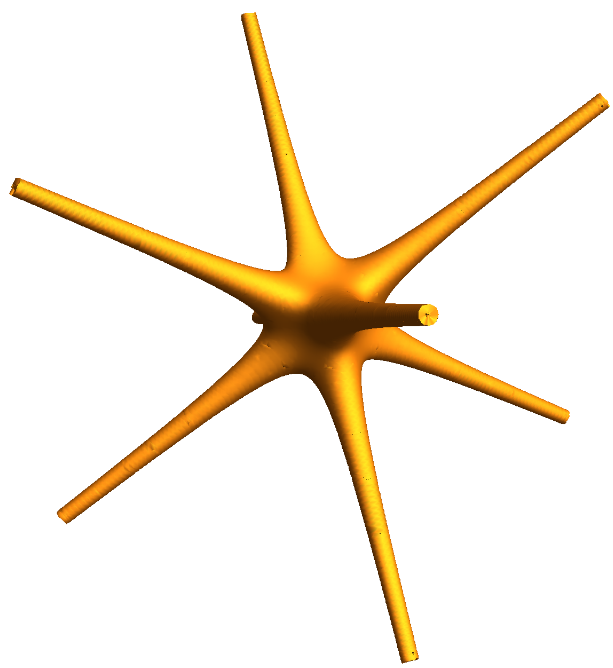
A **spring** is parametrized as

$$\begin{aligned} x &= 5\cos\theta + \cos\alpha\cos\theta \\ y &= 5\sin\theta + \cos\alpha\sin\theta \\ z &= \theta + \sin\alpha \ , \end{aligned}$$



where $0 \leq \theta \leq 4\pi$ and $0 \leq \alpha \leq 2\pi$ are angles. Find the surface area of the spring surface (without the two discs closing it off).
You are given and can use without justification that $|\vec{r}_\alpha \times \vec{r}_\theta|^2 = (5+\cos\alpha)^2$.

Problem 7) (10 points)



a) (6 points) Find the tangent plane to the surface

$$f(x,y,z) = x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 = 1 \ .$$

at the point $(0,1,1)$.
We call it the **octo star** evenso we have not spotted it in the drawings of **Ernst Haeckel** about "art forms from the abyss". A page from the marvellous book is displayed at the end of this exam. Examining these wonderful creatures can either calm your nerves (or freak you out).

b) (4 points) Find the linearization $L(x,y,z)$ of f at $(0,1,1)$.

Problem 8) (10 points)

The probability density of the location of the sunken ship **Dmitrii Donskoi** is given by an unknown function $f(x, y)$. The ship sank off the **Ulleungdo island** in 1905 between South Korea and Japan. Eager to get our hands on the **200 tons of gold**, we want to know more about the probability density function.

a) (4 points) Given two directions $v = \begin{bmatrix} 5 \\ 12 \end{bmatrix}/13$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we know $f(2, 3) = 8$ and the directional derivatives $D_v f(2, 3) = 1$, $D_w f(2, 3) = -1$. Find the gradient $\begin{bmatrix} a \\ b \end{bmatrix}$ of f at $(2, 3)$.

b) (3 points) Estimate $f(2.01, 3.0001)$ using linear approximation at $(2, 3)$.

c) (3 points) Find the tangent line at the point $(2, 3)$ to the level curve $f(x, y) = 8$.

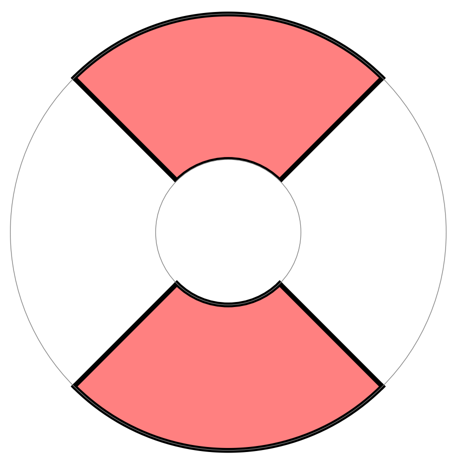
Problem 9) (10 points)

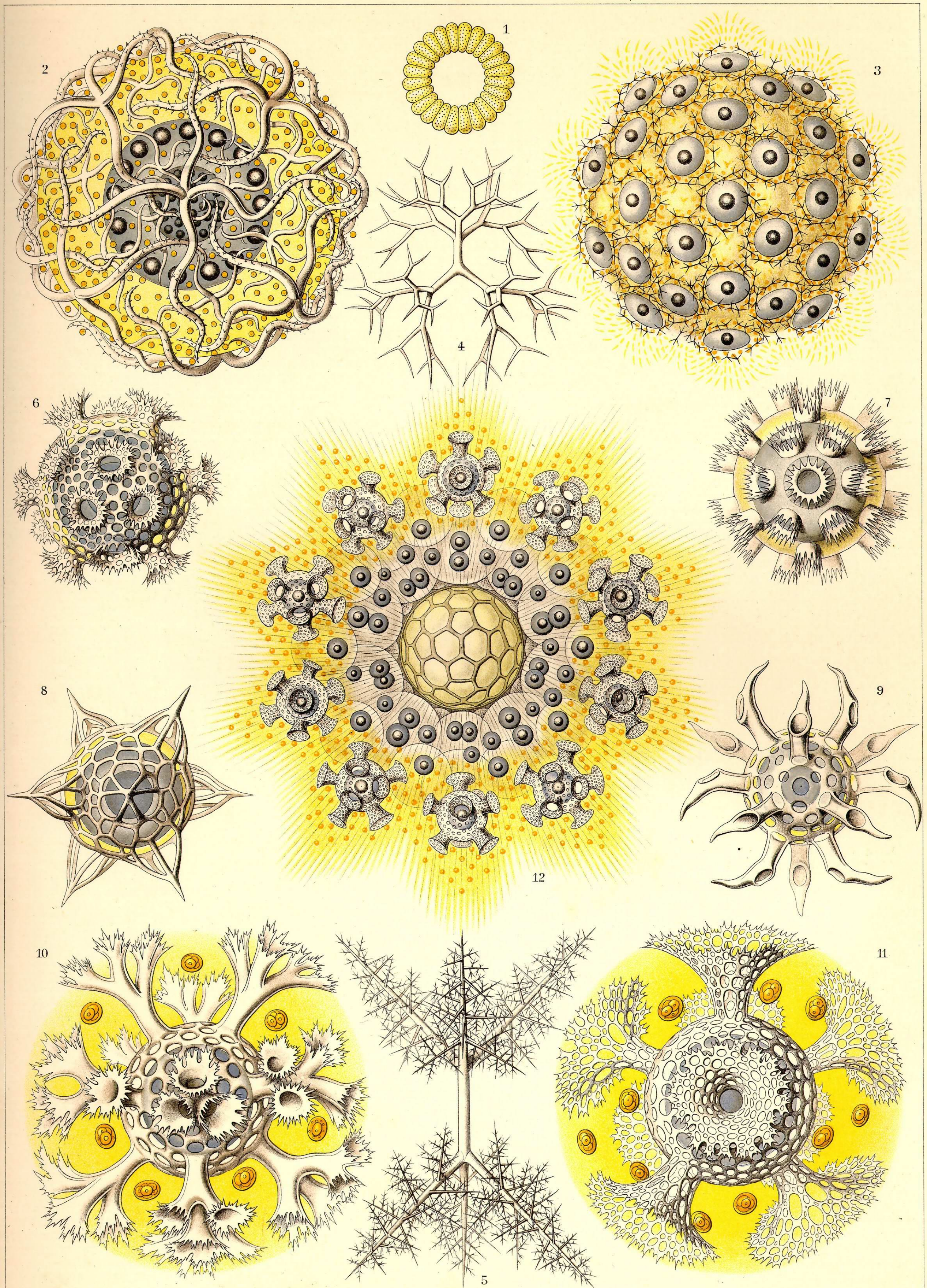
a) (5 points) Evaluate the double integral

$$\int_{-1}^{1} \int_{x^2}^{1} \frac{e^y}{\sqrt{y}} \, dydx \ .$$

b) (5 points) Let $R = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 9, y^2 > x^2\}$. Integrate

$$\int \int_R 2e^{-x^2-y^2} \, dxdy \ .$$





Polycyttaria. — Vereins-Strahlinge.