7/25/2019 SECOND HOURLY **Maths 21a, O.Knill, Summer 2019**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- As usual, all functions are assumed to be differentiable unless stated otherwise.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

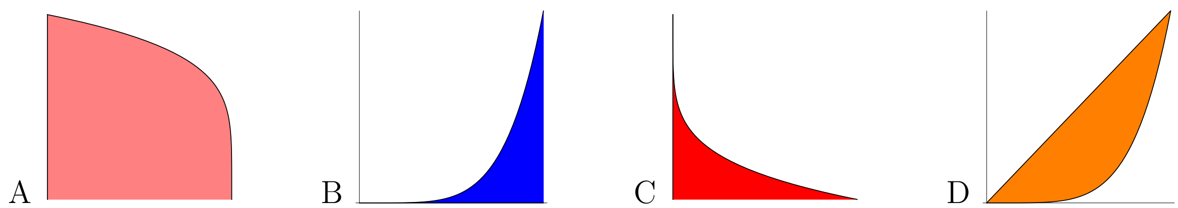
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points). No justifications needed.

1)	T	F	The relation $f(x,y) = \sin(x + \sin(y)) = 0$ defines a function $y = y(x)$ near $x = 0$. Its derivative is $y'(x) = -f_x(0,0)/f_y(0,0) = -1$.
2)	T	F	The point $(0,0)$ is the only critical point of $f(x,y) = x^9 - y^3$.
3)	T	F	The gradient of $f(x,y,z) = x^2+y^2-z$ at $(1,2,5)$ is perpendicular to $\vec{r}_u \times \vec{r}_v$, if $\vec{r}(u,v) = [v\cos(u), v\sin(u), v^2]^T$ is the parametrization of $f(x,y,z) = 0$.
4)	T	F	The surface area of a surface parametrized by $\vec{r}(u,v)$ over a domain R in the uv-plane is given by the integral $\|\int\int_R \vec{r}_u \times \vec{r}_v \; dudv\|$.
5)	T	F	If $f_{xy}(x,y) = 1$ everywhere, then f can only have saddle points.
6)	T	F	The surface area of a cone of height 2 and radius 1 is the same than the surface area of the unit sphere.
7)	T	F	The function $f(x,y) = x + x^2 - 7y^2$ has a global maximum under the constraint $x = 0$.
8)	T	F	Assume $(1,1)$ is a local maximum of g and $(1,1)$ is a local minimum of a function $f(x,y)$, then $(1,1)$ is a solution of the Lagrange equations.
9)	T	F	If $(0,0)$ is a saddle point for f with $D < 0$, then the directional derivative $D_{\vec{v}}f(0,0)$ takes positive and negative values if $\vec{v}$ ranges over all unit vectors.
10)	T	F	If $(0,0)$ is a saddle point for f, then $f_{xx}(0,0)$ and $f_{yy}(0,0)$ can have the same signs.
11)	T	F	For $\vec{u} = [1,0]^T$ and $\vec{v} = [0,1]^T$ then the discriminant D satisfies $D = (D_{\vec{u}}D_{\vec{u}}f)(D_{\vec{v}}D_{\vec{v}}f) - (D_{\vec{u}}D_{\vec{v}}f)^2$, where $D_{\vec{u}}, D_{\vec{v}}$ are directional derivatives.
12)	T	F	If $\vec{r}(t)$ is a curve on the surface $f(x,y,z) = 1$, then the unit tangent vector $\vec{T}(\vec{r}(t))$ is perpendicular to $\nabla f(\vec{r}(t))$.
13)	T	F	The function $f(x,y) = 4x^3y^2 + \sin(\sin(x^{10})) + \arctan(y^8)$ solves the partial differential equation $f_{xyxyxy} = 0$.
14)	T	F	If $f_x(x,y) = f_y(x,y)$ for all x,y, then $f_{xx}(x,y) = f_{yy}(y,x)$ for all (x,y).
15)	T	F	The double integral $\int_0^x \int_0^1 x^2 \; dxdy$ is a rational number.
16)	T	F	As we have learned from "gifted": $\int_{-\infty}^{\infty} e^{-x^2} \; dx = \sqrt{\pi}$.
17)	T	F	Using linearization we can estimate $\sqrt{101} \sim 10 + 1/20 = 10.05$.
18)	T	F	If $f(x,t)$ solves the heat equation, then $f(x,-t)$ solves the wave equation.
19)	T	F	There is a function f such that $D_{\vec{u}}f(0,0)$ is positive for all unit vectors $\vec{u}$.
20)	T	F	The differential equation $f_{xx} - f_x = f_t$ is called the wave equation.

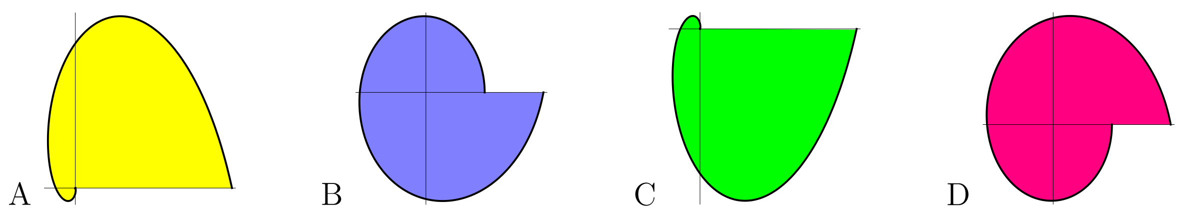
Problem 2) (10 points) No justifications are needed

a) (4 points) Match the following regions with their area computation. There is an exact match.



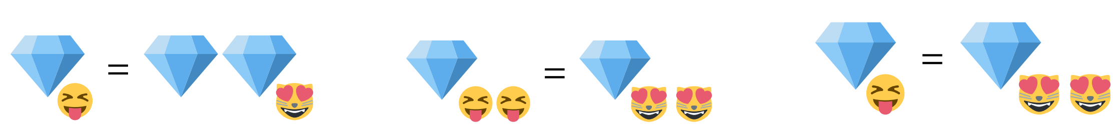
Enter A-D	Area Integral
	$\int_0^1 \int_{x^5}^{x} 1 \; dydx$
	$\int_0^1 \int_0^{x^5} 1 \; dydx$
	$\int_0^1 \int_0^{1-x^5} 1 \; dydx$
	$\int_0^1 \int_0^{1-y^5} 1 \; dxdy$

b) (4 points) Match the following regions with their area computation. There is an exact match.



Enter A-D	Area Integral
	$\int_0^{2\pi} \int_0^{\theta^3} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{(2\pi)^3-\theta^3} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{1+\theta^3} r \; drd\theta$
	$\int_0^{2\pi} \int_0^{1+(2\pi)^3-\theta^3} r \; drd\theta$

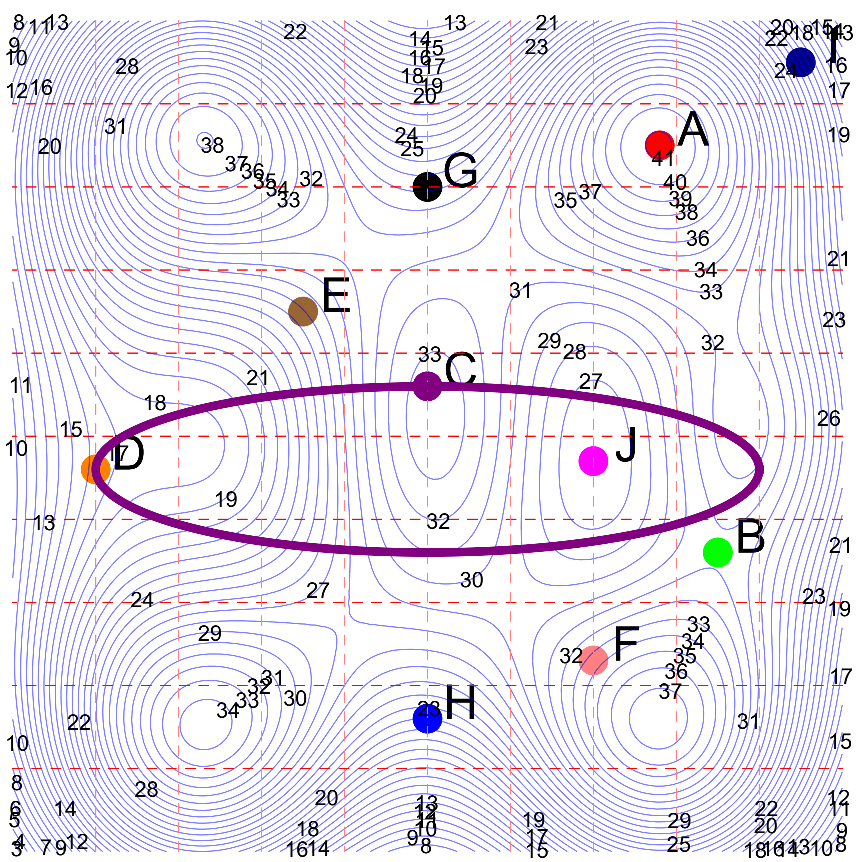
c) (2 points) Name all the three PDE's. The symbols are replaced by Emojis. Each Emoji is either a variable or a function. Remember that **who ever eats hamburgers during heat waves is transported to La la place!**



Problem 3) (10 points) (No justifications are needed.)

(10 points) For each question, there is exactly one of the 10 points A-J which fits the bill. Define $\vec{v} = [1,1]^T/\sqrt{2}, \vec{w} = [1,-1]^T/\sqrt{2}$. An ellipse $g(x,y) = c$ is a constraint for two questions.

	Enter your choice of A-J
A point with $f_x = 0, f_y < 0, f_{xx} > 0$	
A saddle point	
A maximum under the constraint $g = c$	
A minimum under the constraint $g = c$	
$\lvert f_{xx}\rvert$ is largest among A-J	
$\lvert\nabla f\rvert$ largest among A-J	
A local max of f	
A local min of f	
A point where $D_{\vec{v}} f > 0, D_{\vec{w}} f = 0$	
A point where $D_{\vec{w}} f > 0, D_{\vec{v}} f = 0$	



Redemption! Here is a chance to get one or the other points back. The answer must be perfect!
(1 point) Formulate the chain rule $\frac{d}{dt} f(\vec{r}(t)) = ...$
(1 point) What is the direction of maximal ascent for f at a non-critical point (x_0, y_0)?
(1 point) What does Fubini's theorem tell?

Problem 4) (10 points)

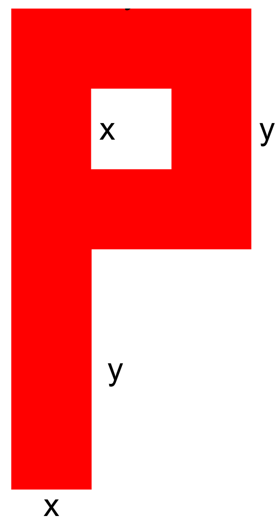
What is the optimal geometry for the **letter P**? In order to find out, we maximize a weighted area of the letter under the constraint that the boundary length is fixed. The area scales the bottom part 4 times.

$$f(x,y) = 4xy + y^2 - x^2 \ .$$

The total boundary length is

$$g(x,y) = 4x + 6y = 58 \ .$$

Use Lagrange to find the maximal f value under this constraint.



Problem 5) (10 points)

Meet

$$f(x,y) = x^3 - 2y^2 - 12xy$$

It is one of the millions overlooked functions in math. It is called "f" and it is sad, as nobody has looked at it yet and recognized its talent. This is the chance for 15 minutes of fame for "f".

a) (8 points) Find the maximum, minima or saddle points of "f" using the second derivative test.

b) (2 points) Does "f" have a global maximum or global minimum? Explain.

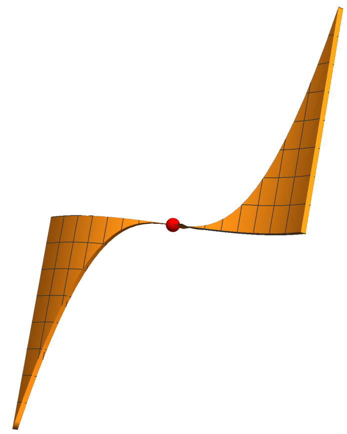
Give it up for "f"! We already reserved a star at the Hollywood hall of fame.

Problem 6) (10 points)

a) (5 points) To compute the surface area of a **propeller**, compute the surface area of

$$\vec{r}(u,v) = \begin{bmatrix} v \\ \frac{u^3}{3} + v \\ 2u \end{bmatrix}$$

with $0 \leq v \leq 1$ and $v^{1/3} \leq u \leq 1$. (The propeller will have twice the area).



b) (5 points) Now solve the integral and don't forget to use what you have learned about double integrals.

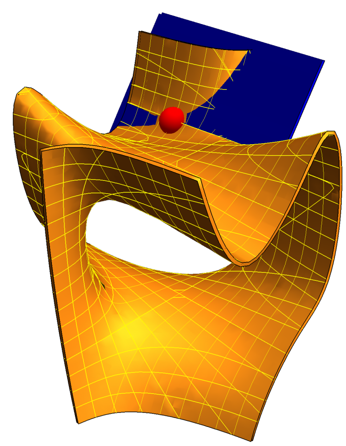
Problem 7) (10 points)

Knowing tangent planes is important in **ray tracing**, where the computer bounces light around in the scene.

a) (5 points) Find the tangent plane to the surface

$$xyz - x^5y + z^2 = 1$$

at the point $(1,1,1)$.

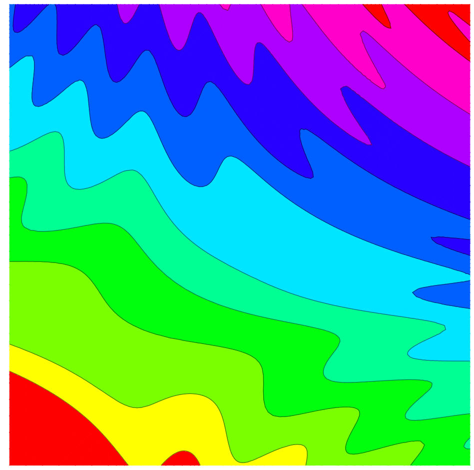


b) (5 points) Near $(x,y) = (1,1)$, we can write $z = g(x,y)$. Find the gradient

$$\nabla g(1,1) = \begin{bmatrix} g_x(1,1) \\ g_y(1,1) \end{bmatrix} .$$

Problem 8) (10 points)

As an **AI experiment**, we ask our computer to draw some art. Here is what it came up with: it painted the contour map of a function f and called it **"Shaken, but not stirred, 2019"**. What strange ideas computers come up with! It must have watched some of the movies which were on the harddrive.



a) (5 points) Find the linearization $L(x,y)$ of

$$f(x,y) = 1 + x + y^2 + \sin(xy)$$

at $(0,0)$.

b) (5 points) Estimate $f(0.01, 0.03)$ using linear approximation. The function is the same as the function in a).

Problem 9) (10 points)

In order to solve the following problems, remember all the **advise** we have given about solving double integrals.

a) (5 points) Find

$$\iint_G \frac{5y}{x^2+y^2} \, dxdy \; ,$$

where G is the region $1 \le x^2 + y^2 \le 25$, $y > 0$.

b) (5 points) Solve the following double integral

$$\int_1^2 \int_{e^y}^{e^2} \frac{x-1}{\log(x)-1} \, dxdy \; .$$