8/8/2019 PRACTICE EXAM IV **Maths 21a, O. Knill, Summer 2019**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-2 do not require any justifications. Problem 3 only 1-2 words. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) No justifications are necessary

1) | T | F | The vectors $\vec{v} = [10,0,0]^T$ and $\vec{w} = [20,0,0]^T$ are perpendicular.

Solution:
Their dot product is not zero.

2) | T | F | If f has a maximum at $(0,0)$ under the constraint $g(x,y) = c$, and $\nabla f, \nabla g$ are both not zero at $(0,0)$ then the angle between $\nabla f(0,0)$ and $\nabla g(0,0)$ is zero or π.

Solution:
Indeed thats what the Lagrange equations tell

3) | T | F | The linearization of $f(x,y) = x^2 + y^3 - x$ at $(2,1)$ is $L(x,y) = 4 + 4(x-2) + 3(y-1)$.

Solution:
Almost, the correct one is $L(x,y) = 3 + 4(x-2) + 3(y-1)$.

4) | T | F | The flux of the gradient ∇f through a unit sphere $f(x,y,z) = x^2+y^2+z^2 = 1$ is 6 times the volume of the sphere.

Solution:
We compute $\text{div}(\nabla f) = 6$ which shows that the flux is 6 times the surface area.

5) | T | F | If $\vec{B}(t)$ is the bi-normal vector to a circle contained in the plane $x = 1$, then $\vec{B}(t)$ is always parallel to $[1,0,0]^T$.

Solution:
It is always the normal vector

6) | T | F | The parametrization $\vec{r}(u,v) = [u\cos(v), u, u\sin(v)]^T$ describes a cone.

Solution:
It is a cylinder.

7) T | **F**

Let E be unit cube with boundary surface S oriented outwards. If $\iint_S \vec{F} \cdot d\vec{S} = 0$, then $\text{div}(\vec{F})(x, y, z) = 0$ everywhere inside E.

Solution:
There could be a cancellation of divergence like for $\vec{F} = [x^2, 0, 0]^T$ with the unit sphere S.

8) **T** | F

If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) then $\iint_S \vec{F} \cdot d\vec{S} = 0$ for any closed surface S.

Solution:
This follows from the divergence theorem.

9) T | **F**

If $\vec{F}$ is a conservative vector field in space, then $\vec{F}$ is has zero divergence everywhere.

Solution:
The vector field $\vec{F}(x, y, z) = [x, 0, 0]^T$ is conservative but has divergence 1 everywhere.

10) **T** | F

The volume of the solid E is $\int \int_S [x, 2x + z, x - y]^T \cdot d\vec{S}$, where S is the surface of the solid E oriented outwards.

Solution:
Since the divergence of the vector field is 1, this follows from the divergence theorem.

11) T | **F**

The vector field $\vec{F}(x, y, z) = [4x + 4y, 4x - 4y, z]^T$ has zero curl and zero divergence everywhere.

Solution:
The divergence is not zero

12) **T** | F

If $\vec{F} = [x^2+y, x+y, x-y^2+z]^T$, then the flux of the vector field curl(curl(F)) through a sphere $x^2 + y^2 + z^2 = 1$ is zero.

Solution:
By the divergence theorem and using the fact that the vector field is $\mathrm{curl}(G)$ for some other vector field $G = \mathrm{curl}(F)$.

13) | T | F |

If the vector field $\vec{F}$ has zero curl everywhere then the flux of $\vec{F}$ through any closed surface S is zero.

Solution:
It would follows from the divergence theorem if $\vec{F}$ were incompressible. It would also follow that the line integral along any closed curve is zero.

14) | T | F |

The equation $\mathrm{div}(\mathrm{grad}(f)) = 0$ is an example of a partial differential equation for an unknown function $f(x, y, z)$.

Solution:
If we write it out, it is actually the Laplace equation.

15) | T | F |

The vector $(\vec{i} - 2\vec{j}) \times (\vec{i} + 2\vec{j})$ is the zero vector if $\vec{i} = [1, 0, 0]^T$ and $\vec{j} = [0, 1, 0]^T$.

Solution:
Either compute directly $[1, -2, 0]^T \times [1, 0, 0]^T = [0, 0, 2]^T$ or foil out in your head to get the result is $-2\vec{i} \times \vec{j}$ which is equal to $-2\vec{k}$.

16) | T | F |

Let L be the line $x = y, z = 0$ in the plane $\Sigma : z = 0$ and let P be a point. Then $d(P, L) \geq d(P, \Sigma)$.

Solution:
Yes, restricting to a line can make the distance not smaller.

17) | T | F |

The chain rule assures that $\frac{d}{dt} f(\vec{r}\,'(t)) = \nabla f(\vec{r}\,'(t)) \cdot \vec{r}\,''(t)$.

Solution:
Yes. It is just the usual chain rule applied to the parametrization $\vec{r}\,'(t)$.

18) T F

If K is a plane in space and P is a point not on K, there is a unique point Q on K for which the distance $d(P,Q)$ is minimized.

Solution:
This point Q is the projection of the point onto the plane. Every other point has larger distance as we can draw a triangle which has a right angle at Q.

19) T F

The parametrized surface $\vec{r}(u,v) = [u, v, u^2 + v^2]^T$ is everywhere perpendicular to the vector field $\vec{F}(x,y,z) = [x, y, x^2 + y^2]^T$.

Solution:
Since we do not see any theoretical reason why this should be true, lets experiment. At the point $[1,1,2]^T$, the vector field is $[1,1,2]^T$ and the normal vector to the surface is $[1,0,2]^T \times [0,1,2]^T = [-2,-2,1]^T$. The normal vector is not parallel to the field.

20) T F

Assume $\vec{r}(t)$ is a flow line of a vector field $\vec{F} = \nabla f$. Then $\vec{r}\,'(t) = \vec{0}$ if $\vec{r}(t)$ is located at a critical point of f.

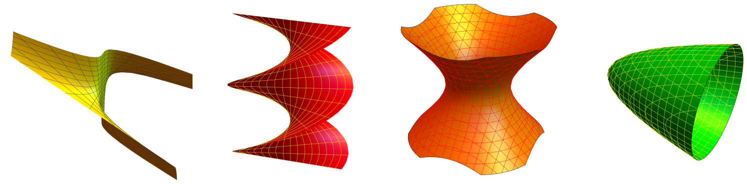
Solution:
By definition $\vec{r}'(t) = \vec{F}(\vec{r}(t)) = \nabla f(\vec{r}(t))$. At a critical point, this is zero.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Surface	1-4
$\vec{r}(u,v) = [u\cos(v), u\sin(v), v]^T$	
$\vec{r}(u,v) = [u^2 v, u, v]^T$	
$x^2 - y^2 = 1 - z^2$	
$2 + x - y^2 - z^2 = 0$	

1 2 3 4



b) (2 points) Match the expressions. There is an exact match.

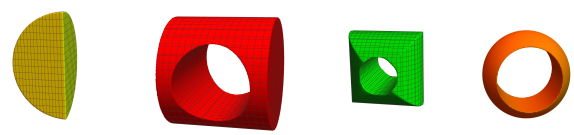
Integral	Enter A-D
$\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert \; dudv$	
$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dudv$	
$\int_C \lvert\vec{r}\,'(t)\rvert \; dt$	
$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \; dt$	

	Type of integral
A	line integral
B	flux integral
C	arc length
D	surface area

c) (2 points) Match the solids. There is an exact match.

Solid	a-d
$x^2 + z^2 \leq 3, x^2 + z^2 \geq 1, x^2 + y^2 + z^2 \leq 2$	
$x^2 < 8, y^2 + z^2 < 9, x^2 + z^2 > 4$	
$x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0$	
$x^2 + y^2 \leq 4, y^2 + z^2 \leq 4, x^2 + z^2 > 1$	

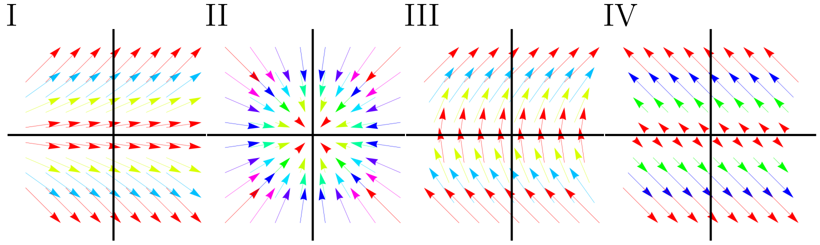
a b c d



d) (2 points) The figures display vector fields. There is an exact match.

Field	I-IV
$\vec{F}(x,y) = [-x, -y]^T$	
$\vec{F}(x,y) = [1, y]^T$	
$\vec{F}(x,y) = [y, 1]^T$	
$\vec{F}(x,y) = [-y, y]^T$	

I II III IV



e) (2 points) Match the partial differential equations:

Equation	1-3
Laplace	
Black-Scholes	
Wave	

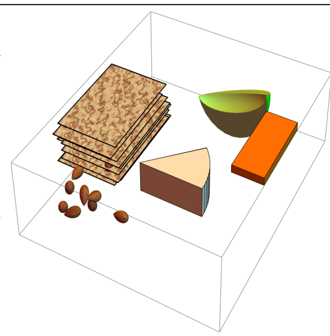
	Partial differential equations
1	$u_t = u - x u_x - x^2 u_{xx}$
2	$u_{tt} - u_{xx} = 0$
3	$u_{tt} + u_{xx} = 0$

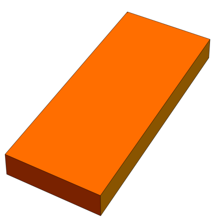
Super joker! If you write down a correct equation for the **Burgers equation**, you can regain 2 lost points in this problem. Of course, the maximal number of points to be gained in this problem is still 10.

Solution:
a) 2134
b) DBCA
c) dbac
d) II,I,III,IV
e) 312
$u_t = uu_x + u_{xx}$. Some have made use of the joker! Most who knew it, also aced the problems above although.

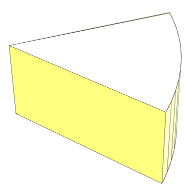
Problem 3) (10 points) No justifications necessary

a) (5 points) Oliver loves **cheese-fruit bistro boxes**. You rarely find any of them in coffee shops, because "Olli the bistro monster" is eating them all (for breakfast, lunch **and** dinner!). One of them contains apples, nuts, cheese and crackers. Lets match the objects and volume integrals:

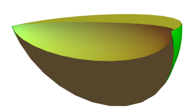




I

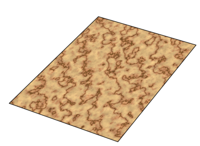


II



III

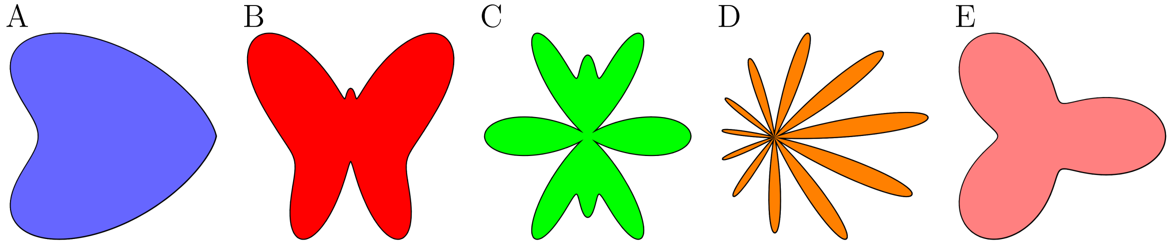
IV



V

Enter I-V	volume formula
	$\int_1^2 \int_0^{\pi} \int_0^{\pi/5} \rho^2 \sin(\phi) \; d\theta d\phi d\rho$
	$\int_{-6}^{6} \int_{-10}^{10} \int_{-0.1}^{0.1} 1 \; dz dx dy$
	$\int_0^{\pi/4} \int_0^{10} \int_{-5}^{5} r \; dz dr d\theta$
	$\int_{-1}^{1} \int_{-4}^{4} \int_{-8}^{8} 1 \; dx dy dz$
	$\int_0^{\pi} \int_0^{2\pi} \int_0^{\cos(2\phi)/4} \rho^2 \sin(\phi) \; d\rho d\theta d\phi$

b) (5 points) Biologist **Piet Gielis** once patented polar regions because they can be used to describe biological shapes like cells, leaves, starfish or butterflies. Don't worry about violating patent laws when matching the following polar regions:

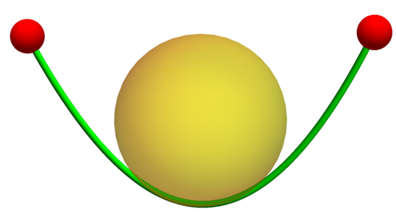


Enter A-E	polar region
	$r(t) \leq \|2 + \cos(3t)\|$
	$r(t) \leq \|\cos(5t) - 5\cos(t)\|$
	$r(t) \leq \|1 + \cos(t)\cos(7t)\|$
	$r(t) \leq \|8 - \sin(t) + 2\sin(3t) + 2\sin(5t) - \sin(7t) + 3\cos(2t) - 2\cos(4t)\|$
	$r(t) \leq \|\sin(11t) + \cos(t)/2\|$

Solution:
a) III,V,II,I,IV
b) E,A,C,B,D
This one was hard. Only a handful got that right. Symmetry can show the way. If the radial function is even, then there is a symmmetry along the x axes. This happens for A,C,E. Some can be figured out by counting. D has 11 petals. E has 3 petals. The symmetry has decided about B, but C and A were hard to distinguish. One way is to look at $t = \pi$ and $t = 0$.

Problem 4) (10 points)

A parabola is parametrized by $\vec{r}(t) = [t, t^2, 0]^T$, where $-1 \leq t \leq 1$. We are interested in some properties at the tip $\vec{r}(0) = [0, 0, 0]^T$.



a) (2 points) Find the speed $|\vec{r}\,'(t)|$ at $t = 0$.

b) (2 points) Find the acceleration vector $\vec{r}\,''(t)$ at $t = 0$.

c) (3 points) Find the curvature κ at $t = 0$.

d) (3 points) Write down the arc length integral of the curve. We have evaluated that in class. No need to evaluate it here.

Solution:
a) 1
b) $[0, 2, 0]^T$.
c) 2
d) $\int_{-1}^{1} \sqrt{1 + 4t^2}\, dt$.

Problem 5) (10 points)

Some chain rule related stuff:

a) (4 points) Estimate $1.002^6 \cdot 0.998^3$ using linear approximation.

b) (3 points) Find the tangent plane to $x^2 + y^4 + z^6 = 3$ at $(1, 1, 1)$.

c) (3 points) Find $D_{\vec{v}} f(1, 1)$ for $\vec{v} = \frac{[5,12]^T}{13}$ and $f(x, y) = x^{13} + y^{13}$.

Solution:
a) The function is $f(x, y) = x^6 y^3$. The gradient at $(1, 1)$ is $[6, 3]^T$. Since $f(1, 1) = 1$, we have

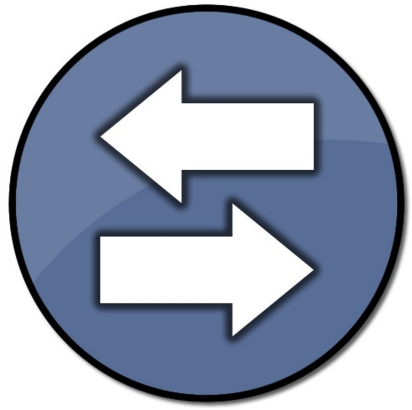
$$L(1, 1) = 1 + 60.0002 + 3(-0.0002) = 1.006 \ .$$

b) The gradient $[2x, 4y^3, 6z^5]^T$ at the point $(1, 1, 1)$ is $[1, 2, 3]^T$ so that the equation is $x + 2y + 3z = d$. We find out d by plugging in a point. The answer is $\boxed{x + 2y + 3z = 6}$.
c) The gradient is $[13x^{12}, 13y^{12}]^T$. At the point $(1, 1)$ it is $[13, 13]^T$. The directional derivative is $[13, 13]^T \cdot [5, 12]^T/13 = \boxed{17}$.

Problem 6) (10 points)

Evaluate the following triple integral computing a volume in cylindrical coordinates:

$$\int_0^{\pi^2} \int_{\sqrt{r}}^{\pi} \int_0^{\sin(\theta)/(r\theta^2)} r \; dz d\theta dr \; .$$



Solution:
After evaluating the most inner integral

$$\int_0^{\pi^2} \int_{\sqrt{r}}^{\pi} \sin(\theta)/\theta^2 \; d\theta dr \; .$$

Switch the order of integration (The picture was supposed to be a hint),

$$\int_0^{\pi} \int_0^{\theta^2} \sin(\theta)/\theta^2 dr d\theta \; .$$

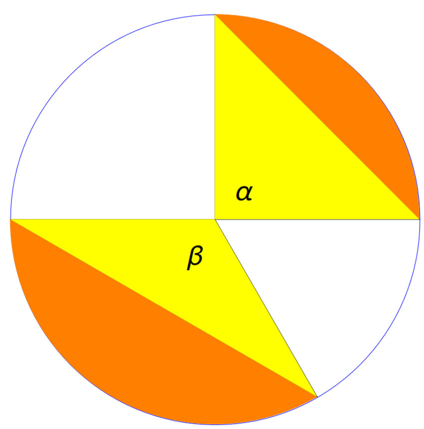
We get

$$\int_0^{\pi} \sin(\theta) \; d\theta = 2 \; .$$

Problem 7) (10 points)

The difference between the total area of two triangles and the area of the two sector regions attached to them in a disc of radius 2 is

$$f(\alpha, \beta) = 2\sin(\alpha) - \alpha + 2\sin(\beta) - \beta \; .$$



a) (5 points) Find the maximum of f using the second derivative test.

b) (5 points) Use Lagrange to find the maximum of f under the constraint $g(\alpha, \beta) = \alpha + \beta = \pi/2$.

Solution:
a) The gradient is $[f(\alpha, \beta) = [2\cos(\alpha) - 1, 2\cos(\beta) - 1]^T$. The roots are $\alpha = \pi/3, 5\pi/3$ or $\beta = \pi/3, 5\pi/3$. This gives 4 critical points. As the geometric concept can suggest to disregard the large angles, it was ok to just look at $(\pi/3, \pi/3)$. Since the discriminant is $4\sin(\alpha)\cos(\alpha) > 0$ and $f_{\alpha\alpha} = -2\sin(\alpha) < 0$, we have a maximum. The maximal value is $2\sqrt{3} - 2\pi/3$.
b) The Lagrange equations are $2\cos(\alpha) - 1 = \lambda = 2\cos(\beta) - 1 = \lambda$. We have $\cos(\alpha) = \cos(\beta)$ and so $(\pi/4, \pi/4)$ as the maximum. The maximal value is $2\sqrt{2} - \pi/2$.

Problem 8) (10 points)

This problem was written while watching the space opera **"Jupiter ascending"**, with a plot so predictable that it is a perfect movie to do some math on the side:

a) (5 points) The genetically engineered soldier Channing Tatum surfs with velocity

$$\vec{v}(t) = [\sin(t), \sin(2t), \cos(t)]^T .$$

He starts at the origin. Where has he surfed to at time $t = 2\pi$?

b) (5 points) Find a parametrization of the tangent line to the curve at $t = 2\pi$.

Solution:
a) Integrate $\vec{v}(t)$ and adjust the constant to get $\vec{r}(t) = 1 - \cos(t), 1/2(1 - \cos(2t), \sin(t)]^T$. We have $\vec{r}(2\pi) = [0, 0, 0]^T$. (Indeed, there is a lot of looping around in that movie).
b) We have the velocity $\vec{v}(2\pi) = [0, 0, 1]^T$ and position $\vec{r}(2\pi) = [0, 0, 0]^T$. So that the line is $\vec{R}(t) = [0, 0, t]^T$. There is a scene, where Channing holds Jupiter in its arms, moving up in a sort of “light elevator”. Guess, thats why it is called “Jupiter ascending” ...

Problem 9) (10 points)

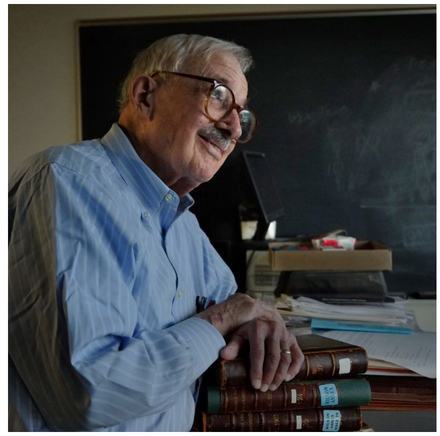
About 40 years ago, economists like **John Chipman** (who taught at Harvard) defined consumer surplus as a line integral $\int_C \vec{F}(r(t)) \cdot r'(t) dt$, where $C : \vec{r}(t)$ is a path in a prize xy-plane. The vector field $\vec{F} = (P, Q)$ consists of demand functions P, Q. Find the consumer surplus for

$$\vec{r}(t) = [t, t^2]^T$$

and prize field

$$\vec{F}(x, y) = [\sin(x) - y \sin(xy), \cos(y) - x \sin(xy)$$

from $t = 0$ to π.



Remark: Books written by economists are jargon and math heavy. You read in one of these books that "the constancy of the marginal utility of income utilizes Hicks-Slutsky partial differential equations." Having taken Math21a, and looking at this PDE, you understand what it implies: "$\vec{F}$ is conservative."

Solution:
The vector field is conservative (as one can check with $Q_x - P_y = 0$). There is therefore a potential. It is $f(x, y) = -\cos(x) + \cos(xy) + \sin(y) + C$. By the fundamental theorem of line integrals, the result is $f(\pi, \pi^2) - f(0, 0) = 1 + \cos(\pi^3) + \sin(\pi^2)$. It was also possible to do the line integral directly, but with much more pain.

Problem 10) (10 points)

A **band aide strip** has the shape of the region enclosed by the curve

$$\vec{r}(t) = [\cos^3(t), \sin(t) + \cos(t)]^T \ ,$$

where $0 \leq t \leq 2\pi$. Find its area!

Solution:
Take $[-y, 0]^T$ and use Greens theorem. We compute the line integral $\int_0^{2\pi} [(-\sin(t) - \cos(t)), 0]^T \cdot (-3\cos^2(t)\sin(t), \cos(t) - \sin(t)]^T \ dt$ using double angle formulas and get $3\pi/4$.

Problem 11) (10 points)

Der **Titan Aerospace Solara 50** is a drone with 150 feet wing span being able to carry a payload of 250 pounds possibly for 5 years. Google bought the company and aims to use it to bring internet to new regions. Assume Solara flies along a closed loop C:

$$\vec{r}(t) = [10\cos(t), 10\sin(t), 100]^T$$

with $0 \le t \le 2\pi$ under the influence of the wind force

$$\vec{F}(x,y,z) = [\sin(x^3) - 3y, 2x + \cos(y^5), \sin(z^2) + z^{11}]^T \ .$$

How much work $\int_C \vec{F} \cdot \vec{dr}$ does it have to do?

Solution:
The curl of $\vec{F}$ is $[0,0,5]^T$. We use Stokes theorem by using the surface $\vec{r}(u,v) = [u,v,100]^T$ which has the normal vector $[0,0,1]^T$. The line integral is the flux of the curl through that surface parametrized by the disk R given by $u^2 + v^2 < 100$. We get $\int\int_R [0,0,5]^T \cdot [0,0,1]^T dudv$ which is 5 times the area of R or 500π.

Problem 12) (10 points)

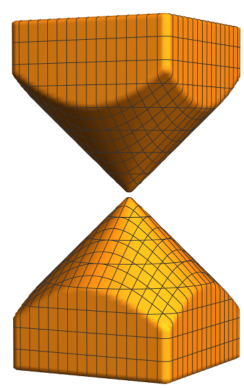
We build a new **nozzle** for a 3D printer. It has the shape of the solid region obtained by intersecting the solid cone

$$x^2 + y^2 \le z^2$$

with the cuboid

$$-1 \le x \le 1, -1 \le y \le 1, -2 \le z \le 2 \ .$$

Find its volume.



Solution:
The integral is $2\int_{-1}^{1}\int_{-1}^{1}\int_{\sqrt{x^2+y^2}}^{2} = 2\int_{-1}^{1}\int_{-1}^{1} 2-\sqrt{x^2+y^2}\; dxdy = 16-2\int_{-1}^{1}\sqrt{x^2+y^2}\; dxdy$. Getting this correctly got already 7 points. Many would directly try to set it up in cylindrical coordinates and then integrate the radius from 0 to 1. This would correspond to a rotationally symmetric situation, we have a square base. Polar coordinates is still your friend: the integral is then is $16-8\int_{-\pi/4}^{\pi/4}\int_0^{1/\cos(\theta)} r^2\; drd\theta = 16-(8/8)\int_{-\pi/4}^{\pi/4}\sec^3(\theta)d\theta$. Getting here already gave full marks. Only one student was able to integrate this correctly until the end. It works by parts $u=\sec(\theta), dv=\sec^2(\theta)d\theta$. The result is $16-(8/8)[\sqrt{2}+\log((\sqrt{2}+1)/(\sqrt{2}-1))/2] = 9.878....$

Problem 13) (10 points)

On July 31, 2015, a **blimp** flew over the science center. It was a windy day and the air ship E with ellipsoid surface hull S parametrized by

$$\vec{r}(u,v) = [10\cos(v)\sin(u), 3\sin(v)\sin(u), 2\cos(u)]^T$$

and volume $V(E) = 80\pi$ had to fight a rather turbulent wind velocity field

$$\vec{F}(x,y,z) = [x+\cos(y^2z), y+\sin(xz^2), z+\sin(x^2y)\ .$$

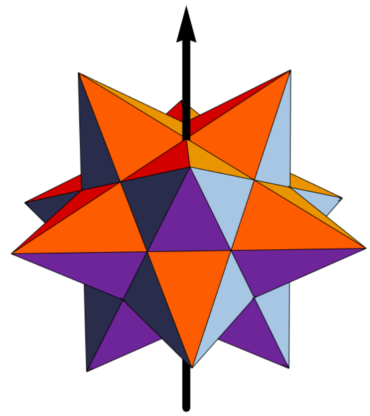
The flux of $\vec{F}$ through S is the total force acting onto the ship. Find that flux, assuming that the surface S is oriented outwards.

Photo: Oliver, July 31, 2015.

Solution:
The divergence of the field is constant 3. The result is by the divergence theorem 3 times the volume of the blimp. Which is 240π.

Problem 14) (10 points)

The polyhedron E in the figure is called **small stellated Dodecahedron**. The solid E has volume 10. Its moment of inertia $\iiint_E x^2 + y^2 \; dxdydz$ around the z-axis is known to be 1. Let S be the boundary surface of the polyhedron solid E oriented outwards.



a) (5 points) What is the flux of the vector field

$$\vec{F}(x, y, z) = [y^5 + x, z^5 + y, x^5 + z]^T$$

through S?

b) (5 points) What is the flux of the vector field

$$\vec{G}(x, y, z) = [x^3/3, y^3/3, 0]^T$$

through S?

Solution:

a) We use the divergence theorem. The divergence of the vector field is 3. The flux therefore is 3 times the volume of the solid which is 30.

b) Again, we use the divergence theorem. The divergence is x^2+y^2. The integral $\int\int\int_E x^2+y^2 \; dxdydz$ is 1.