8/8/2019 PRACTICE EXAM II **Maths 21a, O. Knill, Summer 2019**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) No justifications are necessary

1) T F For any vector field $\vec{F} = [P,Q,R]^T$, the identity $\text{div}(\text{curl}(\vec{F})) = |\text{grad}(\text{div}(\vec{F}))|$ holds.

2) T F The arc length of any curve C contained on the unit sphere $x^2+y^2+z^2=1$ is smaller or equal than 2π.

3) T F The integral $\iint_R 1 \; dudv$ is the surface area of a surface parametrized by $R \to S, (u,v) \to \vec{r}(u,v)$.

4) T F The equation $\text{div}(\text{grad}(f) = 0$ is an example of a partial differential equation.

5) T F A mass point moving on a curve $\vec{r}(t)$ for which the acceleration $\vec{r}\,''(t)$ is zero for all t remains either at a point or moves on a straight line.

6) T F Given a closed surface S and a constant vector field $\vec{F} = [2,3,4]^T$, then the flux of $\vec{F}$ through S is zero.

7) T F A vector field $\vec{F}$ which is incompressible is by Clairaut also irrotational.

8) T F The curvature of any point in the ellipse $x^2/4+y^2=1$ is everywhere smaller than 1.

9) T F Two curves are parametrized by $\vec{r}(t), \vec{s}(t)$. As they intersect at a point P, there exists a time t such that they collide $\vec{r}(t) = \vec{s}(t)$.

10) T F Given three curves L, M, K. If $d(L,M), d(M,K), d(L,K)$ are the distances between the curves, then the triangle inequality $d(L,M) + d(M,K) \geq d(L,K)$ holds.

11) T F If $\vec{F}$ and $\vec{G}$ are two vector fields for which the divergence is the same. Then $\vec{F} - \vec{G}$ is a constant vector field.

12) T F Let S is the unit sphere, oriented outwards and $\vec{F}$ is a vector field in space which is a gradient field then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

13) T F If $\vec{F}, \vec{G}$ are two vector fields which have the same curl, then $\vec{F} - \vec{G}$ is a gradient field.

14) T F The expression $\text{div}(\text{grad}(\text{div}(\text{curl}(\text{curl}(\text{grad}(f))))))$ is a well defined function of three variables if f is a function of three variables.

15) T F The parametrization $\vec{r}(u,v) = [u+v, u+v, u+v]^T$ describes a plane.

16) T F Any function $u(x,y)$ that obeys the differential equation $u_{xx} + u_y = 1$ has no local maxima.

17) T F If $\vec{F}(x,y,z)$ is a vector field so that $[P_x, Q_y, R_z]^T = [0,0,0]^T$, then it is incompressible.

18) T F If $f(x,y) = 0$ is a level set and $f(x, g(y)) = 0$, then $g_x = -f_x/f_y$ provided f_y is not zero.

19) T F By linear approximation we can estimate $\sqrt{10001} = 100 + 1/200$.

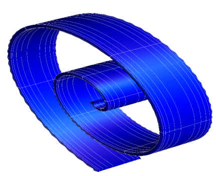
20) T F The flux of $\vec{F}(\vec{r}(u,v)) = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$ through an ellipsoid surface S parametrized by $\vec{r}(u,v)$ is the surface area of S.

Problem 2) (10 points) No justifications are necessary.

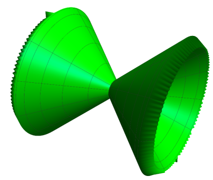
a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(u,v)$	A-C
$[\sin(v)+\sin(u), \cos(u), u]^T$	
$[v, v\sin(u), v\cos(u)]^T$	
$[v\sin(v), v\cos(v), \sin(u)]^T$	

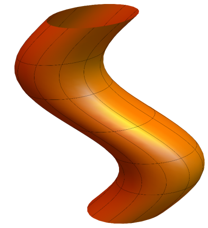
A



B



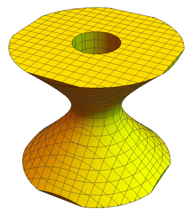
C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^2y^2+x^2z^2+y^2z^2<1$	
$x^2+y^2>1, x^2+y^2-z^2<1$	
$\|(x,y,z)\|<3, \|x\|+\|y\|+\|z\|>3$	

A



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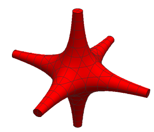

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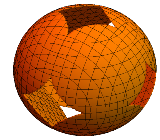
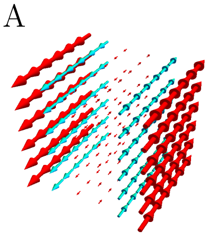


c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F}(x,y,z)=[x,y,0]^T$	
$\vec{F}(x,y,z)=[-y,x,0]^T$	
$\vec{F}(x,y,z)=[0,x,0]^T$	

A



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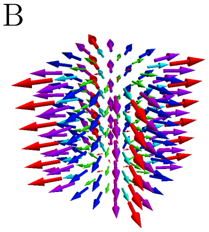

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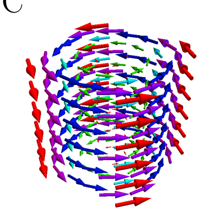


d) (2 points) Recognize partial differential equations!

Equation	A-F
Wave	
Transport	
Burger	

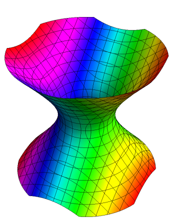
	PDE
A	$f_t+f_x=0$
B	$f_{tt}-f_{xx}=0$
C	$f_t+ff_x=0$

	PDE
D	$f_t-f_x=0$
E	$f_{tt}+f_{xx}=0$
F	$f_t^2+f_x^2=1$

e) (2 points) Find the quadrics!

	Enter A-E
Which one is the one sheeted hyperboloid?	
Which one is the hyperbolic parabolid?	

A



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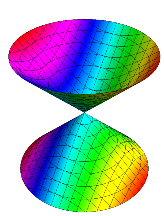

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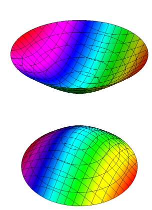
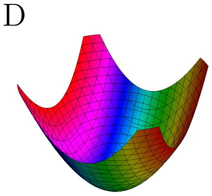
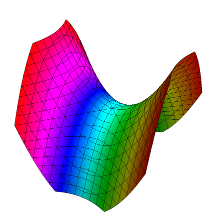


D



E



Problem 3) (10 points) No justifications necessary

a) (5 points) Complete the **signs**. Fill in either an **equal sign** ($=$) a **less or equal sign** ($\leq$) or **larger than equal sign** ($\geq$). Each correct answer is a point.

Formula		
$\vec{v} \cdot \vec{w}$		$\|\vec{v}\|\|\vec{w}\| \cos(\theta)$
$\vec{v} \cdot \vec{w}$		$\|\vec{v}\|\|\vec{w}\|$
$\|\vec{v}\| + \|\vec{w}\|$		$\|\vec{v} + \vec{w}\|$
$\int_0^1 \|\vec{r}\,'(t)\| \, dt$		$\| \int_0^1 \vec{r}\,'(t) \, dt\|$
$\|\vec{v} \times \vec{w}\|$		$\|\vec{v}\|\|\vec{w}\|$

We have seen several fundamental theorems. We want you to write down the results.

b) (1 point) What is the fundamental theorem of gradients?

c) (1 point) What is the fundamental theorem of line integrals?

d) (1 point) What is Clairaut's theorem?

e) (1 point) What is Fubini's result?

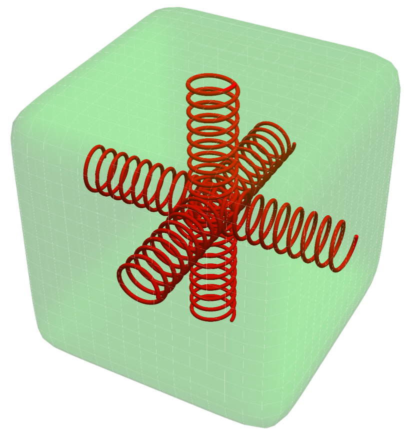
f) (1 point) Write down a relation between curl, grad and div which is always zero.

Problem 4) (10 points)

A modern design shows a lamp in which three carbon springs hold together a thin silk hull. We advertise it as **"new materials and natural fabric in a symbiotic equilibrium".** Assume the velocity of the curve is given by

$$\vec{r}\,'(t) = [-10\sin(10t), 10\cos(10t), 1]^T$$

and that $\vec{r}(0) = [0, 10, 0]^T$.



a) (2 points) What is the unit tangent vector $\vec{T}(0)$ at $t = 0$?

b) (4 points) Find the parametrization $\vec{r}(t)$ of the curve.

c) (4 points) Find the total arc length of this curve if $-2\pi \leq t \leq 2\pi$.

Problem 5) (10 points)

a) (3 points) A new **t-shirt** shows the curve

$$x^2 + (y - x^{2/3})^2 = 1\ .$$

What is the tangent line at the point $(1, 1)$?

b) (3 points) Estimate $1.001^2 + (1.01 - 1.001^{2/3})^2$ using linear approximation.

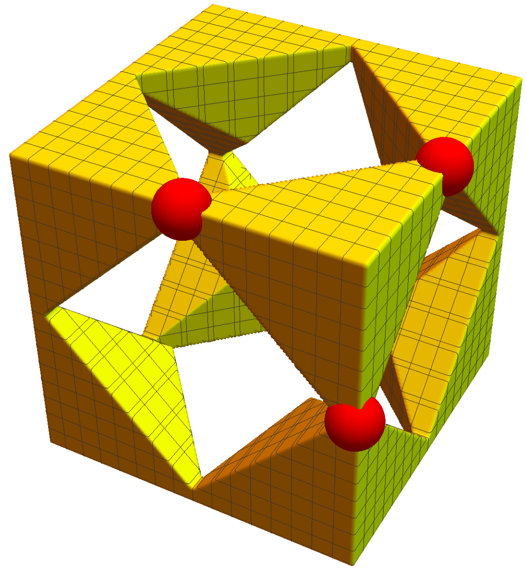
c) (4 points) A 3D implementation of the heart is the surface

$$x^2 + (y - x^{2/3})^2 + 16z^2 = 1\ .$$

The point $(1, 1, 0)$ is on this surface. What is the tangent plane to the surface at this point?

Problem 6) (10 points)

The **connection cube** is the complement of the truncated cube within the cube. Having realized it as a pasta, we cut it along the plane containing the points $A = (1,1,0), B = (1,0,1)$ and $C = (0,1,1)$.



a) (3 points) Find the equation of that plane.

b) (3 points) What is the area of the triangle ABC?

c) (4 points) Find the distance of that plane to the center $(0,0,0)$.

Problem 7) (10 points)

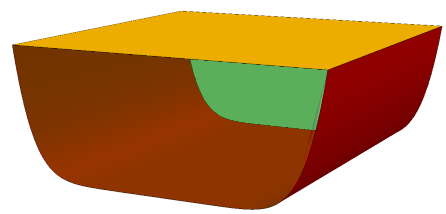
A Boston **duck tour boat** is bound from below by the function

$$f(x,y) = x^8$$

and from the top by the function

$$g(x,y) = 1 \ .$$

It is also bound by the planes $y = -1$ and $y = 1$.



a) (6 points) Find the volume of the boat.

b) (4 points) Write down the double integral which gives the surface area of the bottom part, the graph of the function f above the square. You don't have to evaluate the integral.

Problem 8) (10 points)

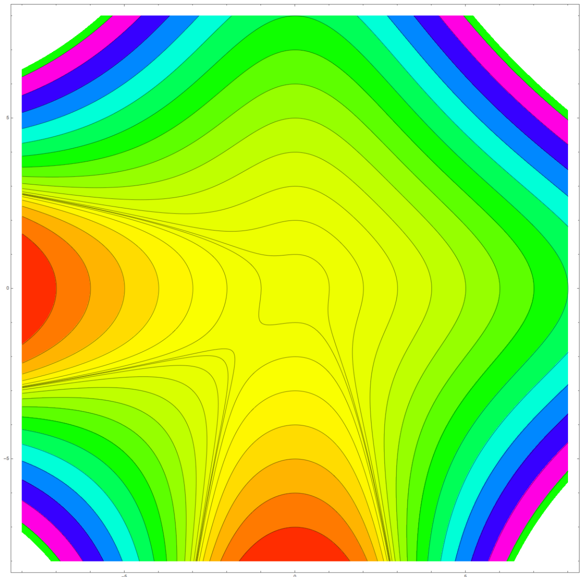
The **Laplacian** of a function $g(x,y)$ is defined as

$$\Delta g = g_{xx} + g_{yy} \; .$$

Let $g(x,y) = x^2y^2 + x^3 + y^3$.

a) (3 points) Write down the function $f(x,y)$ which is the Laplacian of g.

b) (7 points) Classify all critical points of f using the second derivative test.



Problem 9) (10 points)

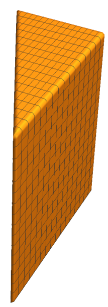
When you eat **raclette**, a swiss speciality, you place the cheese near the open fire. The part exposed to the fire melts. You scape that off and eat it with potatos, pearl onions (”petit poireau antillais”) and pickels (”cornishons”) and white wine. Our **raclette cheese** has volume

$$f(x,y) = x^2y/2$$

and fixed surface area

$$g(x,y) = x^2 + xy = 3$$

exposed to the fire. Using Lagrange, find the cheese parameters x, y which has maximal volume.



Problem 10) (10 points)

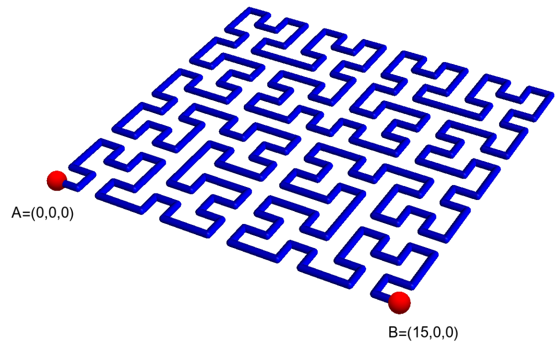
Peano curves and **Hilbert curves** are famous in topology. The allow for example to prove that there is a continuous map from the interval to the square which reaches every point of the square!

Lets look at the Hilbert curve $C = H(4)$ displayed in the picture. The curve is parametrized by $\vec{r}(t)$. It starts at $A = (0,0,0) = \vec{r}(0)$ and ends at $B = (15,0,0) = \vec{r}(1)$.

What is the line integral $\int_C \vec{F} \cdot \vec{dr}$ of the vector field

$$\vec{F}(x,y,z) = [x^4 + 2xy^2z^2, y^4 + 2x^2yz^2, z^4 + 2x^2y^2z]^T$$

along the curve C?



P.S. A comedian once defined "Mathematics as the science where you prove things which are obvious or things which are obviously false." An example of the later is the continuous surjection from the interval and the square.

Problem 11) (10 points)

At Porter square in Cambridge, at the intersection of Roseland Street and Mass Ave, a cool **bike locking arc** has recently been installed. Mathematically it represents a "link".

We place one of the arcs into the plane and model it with the curve

$$\vec{r}(t) = \begin{cases} \left[\ 2\cos(t), \sin(t) - \sin^2(t)\ \right], & 0 \le t \le \pi \\ \left[\ 2\cos(t), \sin(t)\ \right], & \pi \le t \le 2\pi \end{cases}$$

Find the area of the two-dimensional region G enclosed by this curve.

Problem 12) (10 points)

A **model train** can be built from a copper coil in which a battery and magnet moves. A problem related to that is to find the flux of the curl of the magnetic vector field

$$\vec{F}(x,y,z) = [-y, x, 0]^T$$

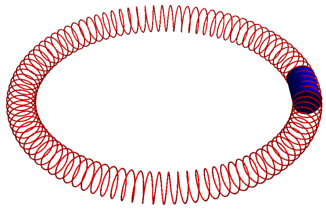
through the surface parametrized by

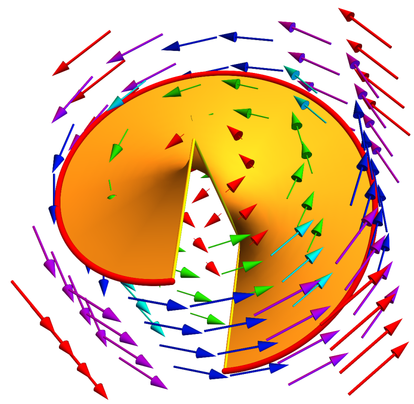
$$S : \vec{r}(s,t) = [s\cos(t), s\sin(t), t]^T ,$$

where $0 \leq t \leq 2\pi, 0 \leq s \leq 1$. The surface is identified on the top and bottom so the **boundary curve of the surface** is

$$C : \vec{r}(1,t) = [\cos(t), \sin(t), t]^T , 0 \leq t \leq 2\pi .$$

You are required to use an integral theorem. No credit for direct flux computations will be given.





The top and bottom are identified so that the boundary consists of the thickened curve only. The three straight lines which additionally appear to bound the surface are ignored. We don't make a mistake by ignoring them as their line integrals contribute nothing.

Problem 13) (10 points)

Archimedes computed the volume of the intersection of three cylinders. The **Archimedes Revenge** is the problem to determine the volume V of the solid R defined by

$$x^2+y^2-z^2 \leq 1, y^2+z^2-x^2 \leq 1, z^2+x^2-y^2 \leq 1 .$$

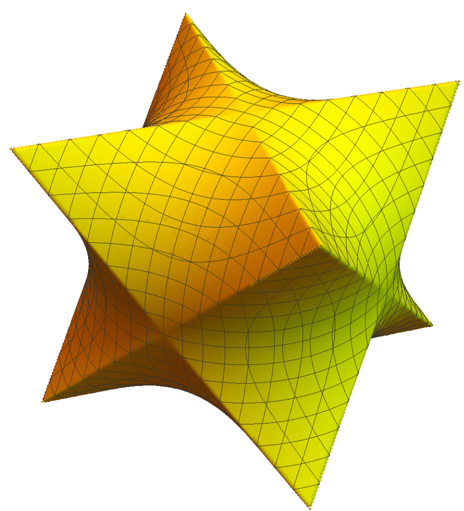
Archimedes Revenge is brutal! It is definitely to hard for this exam. We give you therefore the volume $V = \log(256)$. Now to the **actual exam problem**: find the flux

$$\iint_S \vec{F} \cdot d\vec{S}$$

of

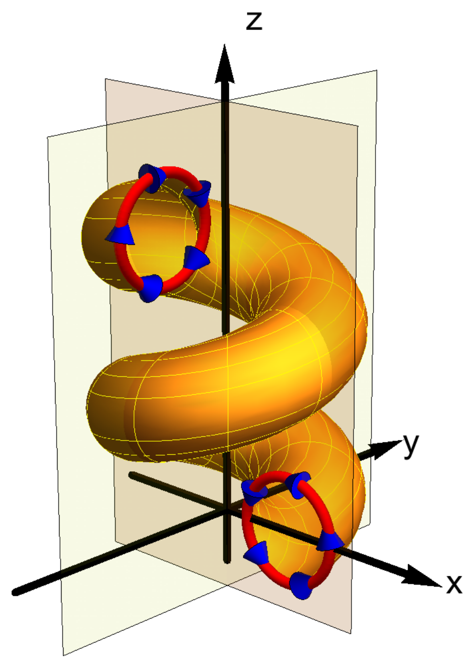
$$\vec{F}(x,y,z) = [2x+y^2+z^2, x^2+2y+z^2, x^2+y^2+2z]^T$$

through the boundary surface S of R, assuming that S is oriented outwards.



If you have been part of this Summer 2017 course and send me a handwritten correct solution of the **Archimedes Revenge** problem until December 31, 2017, you will earn a small prize (gift card) and admiration. The rules are that you have to solve the problem on your own and that no part of the solution should rely on computer algebra systems.

Problem 14) (10 points)



Find the **flux** of the curl of the vector field

$$\vec{F}(x,y,z) = [-z, z+\sin(xyz), x-3]^T + [x^5, y^7, z^4]^T$$

through the **twisted surface** oriented inwards and parametrized by

$$\vec{r}(t,s) = [(3+2\cos(t))\cos(s), (3+2\cos(t))\sin(s), s+2\sin(t)]^T$$

where $0 \leq s \leq 7\pi/2$ and $0 \leq t \leq 2\pi$.

Hint: This parametrization leads correctly already to a vector $\vec{r}_t \times \vec{r}_s$ pointing inwards. The boundary of the surface is made of two circles $\vec{r}(t,0)$ and $\vec{r}(t,7\pi/2)$. The picture gives the direction of the velocity vectors of these curves (which in each case might or might not be compatible with the orientation of the surface).