8/8/2019 PRACTICE EXAM II **Maths 21a, O. Knill, Summer 2019**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The distance d between a point P and a line through two distinct points A, B is given by $d = |\vec{AP} \times \vec{AB}|/|\vec{AB}|$.

Solution:
You know the distance formula!

2) T F The set of all points in three dimensional space satisfying $x^2 + y^2 = 1$ define a circle.

Solution:
it is a cylinder.

3) T F The surface area of a surface parametrized by $\vec{r}(u, v)$ on a parameter region R is given by $\int\int_R |r_u \times r_v| \, dudv$.

Solution:
Yes, now it is the right formula! In the midterm there had been a square.

4) T F Given the curve $\vec{r}_1(t) = [\cos(3t), \sin(3t), \cos(5t)]^T$ and the curve $\vec{r}_2(t) = [\cos(6t), \sin(6t), \cos(10t)]^T$. Then the TNB-frame of of $\vec{r}_1$ at $t = 0$ is the same as the TNB frame of $\vec{r}_2$ at $t = 0$.

Solution:
The TNB frame does not depend on the parametrization, except if we would go the other way round.

5) T F The curve $\vec{r}(t)$ has at the point $P = \vec{r}(0)$ the curvature 2. Then, the curve $\vec{r}(2t)$ has at the same point P the curvature 4.

Solution:
curvature is independent of the parametrization.

6) T F For any vector field $\vec{F} = [P, Q, R]^T$ the identity $\text{grad}(\text{div}(\text{curl}(\vec{F}))) = \text{curl}(\text{grad}(\text{div}(\vec{F})))$ holds.

Solution:
Both are zero.

7) | T | F |

For any closed curve $r(\vec{t})$ with $t \in [0,1]$ of positive length for which $|\vec{r}(t)| = 1$ for all t there is a point, where the curvature is non-zero.

Solution:
If the curvature were zero, then we would have a line and the curve could not be closed

8) | T | F |

If $f(x,y)$ is non-zero, the integral $\iint_R f(x,y)\ dxdy$ is a volume and so positive.

Solution:
f can be negative

9) | T | F |

The double integral $\int_0^1 \int_y^1 2e^{-x^2}\ dxdy$ evaluates to $1 - 1/e$.

Solution:
Yes, change the order of integration.

10) | T | F |

If $\vec{v} = \vec{r}(0) = \vec{r}\,'(0)$ is non-zero and $\vec{r}\,''(t)$ is always parallel to $\vec{v}$, then $\vec{r}(t)$ moves on part of a line through the origin 0).

Solution:
Integrate to see that $\vec{r}'(t)$ is still parallel to v at all times.

11) | T | F |

Given a surface S with boundary C with compatible orientations, then Stokes theorem implies $\int_C |\vec{r}\,'(t)|\ dt = \int_S |\vec{r}_u \times \vec{r}_v|\ dudv$.

Solution:
There is no such simple relation between length of the boundary and surface area.

12) | T | F |

The equation $x^2 - (z-1)^2 + y^2 + 2y = -1$ represents a two-sheeted hyperboloid.

Solution:
It is a cone.

13) T F

Any incompressible and irrotational vector field $\vec{F} = [P, Q, R]^T$ is built by linear functions P, Q, R.

Solution:
There are examples like $[x^3 - 3xy^2, -6xy, 0]$ which are both incompressible and irrotational. In general, take $F = \nabla f$, where f satisfies $f_{xx} + f_{yy} + f_{zz} = 0$.

14) T F

The curvature of the curve $\vec{r}(t) = [\cos(t^2)/3, \sin(t^2)/3]^T$ at $t = 1$ is everywhere equal to 3.

Solution:

15) T F

The distance $d(S_1, S_2)$ between spheres S_1, S_2 is defined as the minimum of all $d(P_1, P_2)$, where $P_1 \in S_1$ and $P_2 \in S_2$. Given 3 spheres S_1, S_2, S_3, then $d(S_1, S_2) + d(S_2, S_3) \geq d(S_1, S_3)$ holds.

Solution:
Take three non-intersecting spheres $L = \{(x - 1.5)^2 + y^2 + z^2 = 1\}, M = \{x^2 + y^2 + z^2 = 1\}, K = \{(x + 1.5)^2 + y^2 + z^2 = 1\}$. Then $d(L, M) = d(M, K) = 0$ but $d(L, K) > 0$.

16) T F

If $\vec{F}$ and $\vec{G}$ are vector fields for which the divergence is 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field.

Solution:
It would be the curl of a vector field.

17) T F

Let S is an ellipsoid, oriented outwards and $\vec{F}$ is a vector field in space which is the curl of an other vector field, then $\iint_S \vec{F} \cdot d\vec{S} = 0$.

Solution:
Yes, either by Stokes or by the divergence theorem

18) T F The parametrization $\vec{r}(u,v) = [u^3, 1, v^3]^T$ describes a plane.

Solution:
Yes, it is parallel to the xz-plane.

19) T F The flux of the curl of the field $\vec{F} = [0, x, 0]^T$ through the unit disc $\{(x,y,z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to π.

Solution:
The curl is constant $[0, 0, 1]$. As the length of that vector field is one, points in the direction of the normal and perpendicular to the surface, it is the surface area.

20) T F The flux of the vector field $\vec{F} = [x, 0, 0]^T$ through the unit disc $\{(x,y,z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to the area of the disc.

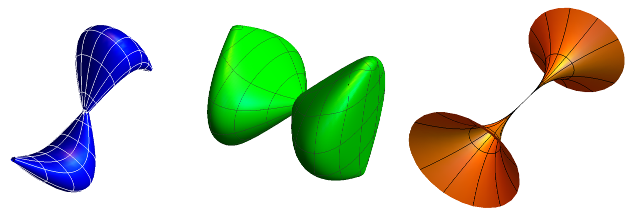
Solution:
The vector field is parallel to the disc, no fluid passes through. The flux is zero.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

Parametrized surface $\vec{r}(s,t)$	A-C
$[\cos(t)s^3, 2s, \sin(t)s^3]^T$	
$[\sin(t), \sin(2t)\sin(s), \sin(t)\cos(s)]^T$	
$[\cos(t)\sin(s), s, (2+\sin(t))\sin(s)]^T$	

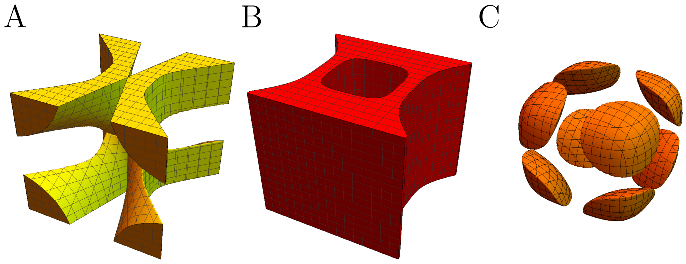
A B C



b) (2 points) Match the solids. There is an exact match.

Solid	A-C
$x^4 - y^4 < 4, x^4 + y^4 > 1$	
$x^4 z^4 > y^4, x^4 - y^4 < 1$	
$x^4 + y^4 + z^4 < 16, \lvert xyz\rvert > 1$	

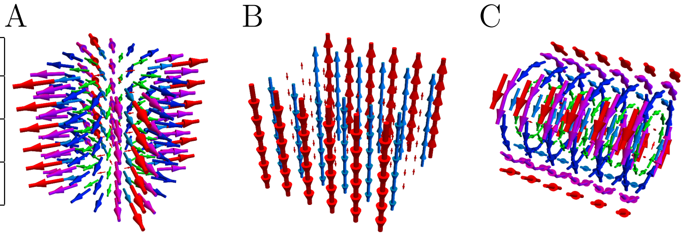
A B C



c) (2 points) The figures display vector fields. There is an exact match.

Field	A-C
$\vec{F}(x,y,z) = [\sin(x), \sin(y), 0]^T$	
$\vec{F}(x,y,z) = [0, 0, \sin(y)]^T$	
$\vec{F}(x,y,z) = [0, -\sin(z), \sin(y)]^T$	

A B C



d) (2 points) Recognize partial differential equations!

Equation	A-F
Laplace	
Burger	
Wave	

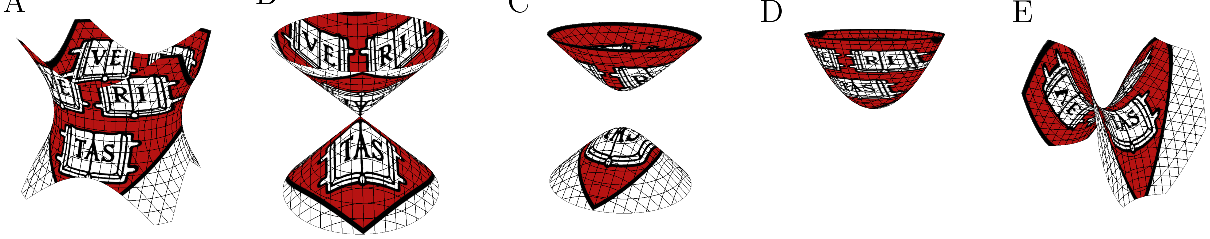
	PDE
A	$u_t^2 + u_x^2 = 1$
B	$u_t = u_x$
C	$u_{tt} = -u_{xx}$

	PDE
D	$u_t = u_x$
E	$u_{tt} = u_{xx}$
F	$u_t = -u u_x$

e) (2 points) Find the true quadrics! Veritas!

	Enter two letters from A-E each
Which ones are paraboloids?	
Which ones are hyperboloids?	

A B C D E



Solution:
a) CBA
b) BAC
c) ABC
d) CFE
e) (DE) are the parabolic and hyperbolic paraboloids and (AC) were the one and two sheeted hyperboloids.
Especially, this last part was wrong surprisingly often. Despite the Veritas texture.

Problem 3) (10 points) No justifications necessary

a) (5 points) Enter either equal, or inequality signs. In case, there should be no relation in general, enter the "not-equal sign" $\neq$. The vectors $\vec{v}, \vec{w}$ are vectors in three dimensional space and α is the angle between the two vectors. The letters $\vec{T}, \vec{N}, \vec{B}$ denote the TNB frame in a space curve.

Formula 1	Enter $=, \leq, \geq, \neq$,	Formula 2
$\|\vec{N}\|$		$\|\vec{B}\|$
$\|\vec{v} \cdot (\vec{v} \times \vec{w})\|$		$\|\vec{v} \times \vec{v}\|$
$\|\vec{v} - \vec{w}\|$		$\|\vec{v}\| + \|\vec{w}\|$
$\|\vec{v} \cdot \vec{w}\|$		$\|\vec{v}\| \ \|\vec{w}\|$
$\|\vec{v} \cdot \vec{w}\|$		$\|\vec{v}\|\|\vec{w}\| \cos(\alpha)$

This space is for relaxation purposes only. Take some rest.

b) (5 points) We list here **12 of the most important theorems in multivariable calculus** and identify what they are about, then vote which is our favorate. The statement "involves vector fields" means that the statement of the theorem involves a vector field. The statement "involves integrals" means that when writing down the theorem, there appears at least one integral.

Number	Theorem	Involves vector fields	Involves integrals
1)	Pythagoras theorem		
2)	Al Khashi theorem		
3)	Cauchy-Schwarz theorem		
4)	Clairaut theorem		
5)	Fubini theorem		
6)	Gradient theorem		
7)	Second derivative test		
8)	Lagrange theorem		
9)	Fundamental theorem of line integrals		
10)	Greens theorem		
11)	Stokes theorem		
12)	Divergence theorem		

Which is your favorite? (Your choice is not affecting your score of this problem).

My favorite theorem is theorem number $\boxed{}$.

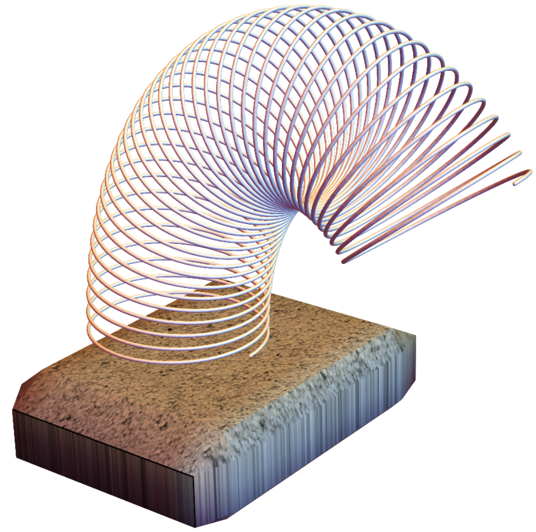
Solution:
a) Also this was solved surprisingly often wrong. One reason was probably that it was close to the problem in the practice exam. But it had subtle changes which changed some of the answers in an essential way.
$=$ because both are equal to 1.
$=$ because both are 0.
$\leq$ because of the triangle inequality.
$\leq$ because of Cauchy-Schwarz.
$\geq$ because without absolute sign on the left, it would be an identity.
b) The theorems which involve vector fields are the four theorems on vector fields The "Fundamental theorem of line integrals", Green, Stokes and Divergence theorems. The other theorms might involve vectors but not vector fields. The same applies for integrals. These are integral theorems. There was an other theorem, Fubini's theorem, which also involves integrals.

Problem 4) (10 points)

It is fun to play with the **slinky** as it can climb down stairs while the arc length of the slinky stays the same. We compute here the motion $\vec{r}(t)$ of the top slinky part by giving the acceleration

$$\vec{r}\,''(t) = \begin{bmatrix} -100^2\cos(100t) \\ -100^2\sin(100t) \\ -1 \end{bmatrix}$$

and the initial position $\vec{r}(0) = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ and initial velocity $\vec{r}\,'(0) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$. Find the path $\vec{r}(t)$.



Solution:

Integrate twice and adjust the constants. $\vec{r}(t) = \begin{bmatrix} \cos(100t) + 2t \\ \sin(100t) - 100t - 2 \\ -t^2/2 + 3t + 5 \end{bmatrix}$. As usual, the main mistake was incorrect adjustements of the constants.

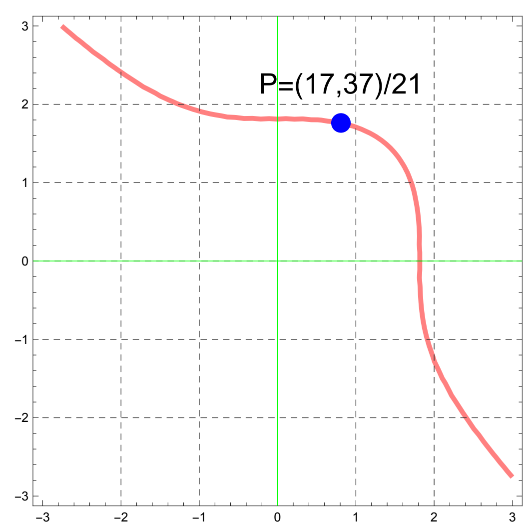
Problem 5) (10 points)

The book **"Magnificant Mistakes in Mathematics"** reports that Legendre conjectured in a book of 1794 that there are no rational solutions of the equation

$$x^3 + y^3 = 6 \ .$$

Henry Ernest Dudeny proved this wrong by stating $(17/21)^3 + (37/21)^3 = 6$.

a) (3 points) Find a nonzero vector $[a,b]^T$ perpendicular to the curve at $(17/21, 37/21)$.
b) (4 points) Find the tangent line to the curve at $(17/21, 37/21)$.
c) (3 points) Find the linearization $L(x,y)$ at that point.



Solution:

a) The gradient is $\nabla f(x,y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}$. Evaluated at the point gives $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3(17/21)^2 \\ 3(37/21)^2 \end{bmatrix}$.

b) The equation of the line is $ax + by = d$, where a, b were already computed in a) and where d is obtained by plugging in the point. The answer is

$$3(17/21)^2 x + 3(37/21)^2 y = 18$$

(The number on the right came from the fact that $x_0^3 + y_0^3 = 6$.) Any form which is correct was accepted of course. The simplest is $289x+1369y = 2646$. c) The linearizationi is

$$L(x,y) = 6+a(x-17/21)+b(y-37/21) = 6+3(17/21)^2(x-17/21)+3(37/21)^2(y-37/21) \; .$$

Problem 6) (10 points)

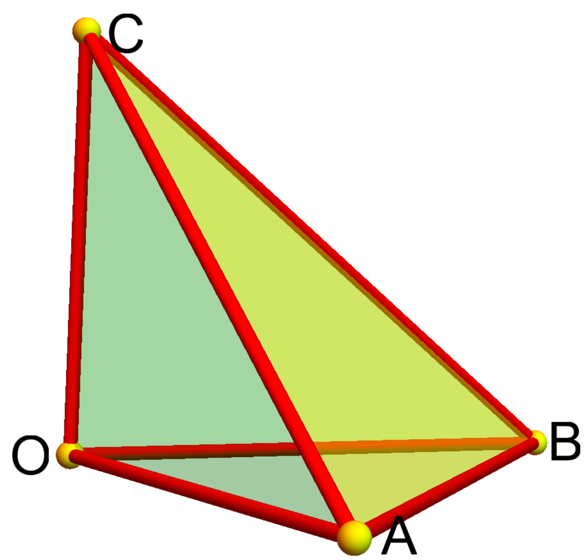
a) (2 points) Find the equation $ax+by+cz = d$ of the plane through $A = (2,0,0), B = (0,3,0), C = (0,0,5)$.

b) (2 points) Determine its distance h of the plane to the origin $O = (0,0,0)$.

c) (2 points) Compute the area T of the triangle ABC.

d) (2 points) What is the volume $V = T \cdot h/3$ of the tetrahedron?

e) (2 points) The **3D Pythagoras theorem** states that the square of the area of ABC is the sum of the squares of the areas of the triangles OAB, OBC and OCA (which are each half of a rectangle). Check this in the current situation.



Solution:
a)

$$\vec{AB} \times \vec{AC} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 6 \end{bmatrix} .$$

The equation of the plane is $15x + 10y + 6z = 30$.
b) The distance formula is $\vec{AO} \cdot (\vec{AB} \times \vec{AC})/|\vec{AB} \times \vec{AC}|$. This is $30/19$.
c) The area is half the value of $|\vec{AB} \times \vec{AC}|$ which is $19/2$.
d) The volume is $(30/19)(19/2)/3 = 5$.
e) The areas of the side triangles are $3, 5$ and $15/2$. We see $3^2 + 5^2 + (15/2)^2 = (19/2)^2$.

Problem 7) (10 points)

We design a perfume line "phoenix" with the slogan **"Its bold scent embodies the maveric spirit. Be reborn everyday!"**

The bottle E is a solid given by the region

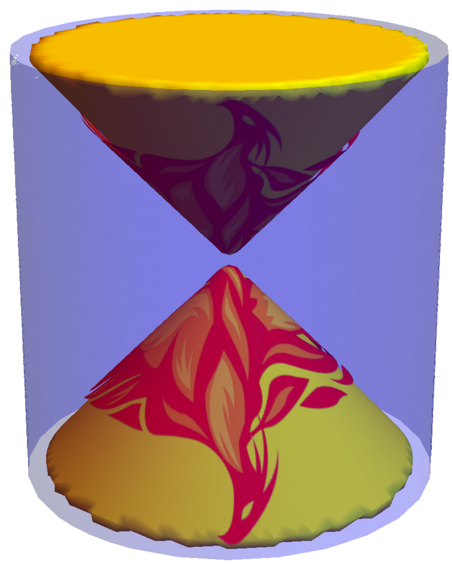
$$x^2 + y^2 \geq z^2,\ x^2 + y^2 \leq 1,\ -1 \leq z \leq 1$$

which is the complement of a cone in a cylinder. The perfume density is $x^2 + y^2$. Find the total mass

$$\int\int\int_E x^2 + y^2 \; dzdxdy .$$

of the perfume.

Remark: This slogan is plagiarized from an actual three letter cosmetic line.



Solution:
If R is the unit disk, then the integral is

$$\int\int_R \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} x^2 + y^2 \; dzdxdy = \int\int_R 2(x^2 + y^2)^{3/2} \; dxdy .$$

This double integral is of course best solved in polar coordinates.

$$\int_0^{2\pi} \int_0^1 2r^3 \cdot r \; drd\theta = 4\pi/5 .$$

Problem 8) (10 points)

We want to build a station on our **moon base** and find a place where the terrain is optimal. Given a height function $g(x,y) = xy + x^2 + y^2$, we can look at the place where the magnitude of the gradient $f(x,y) = |\nabla g|^2 = g_x^2 + g_y^2$ is extremal.

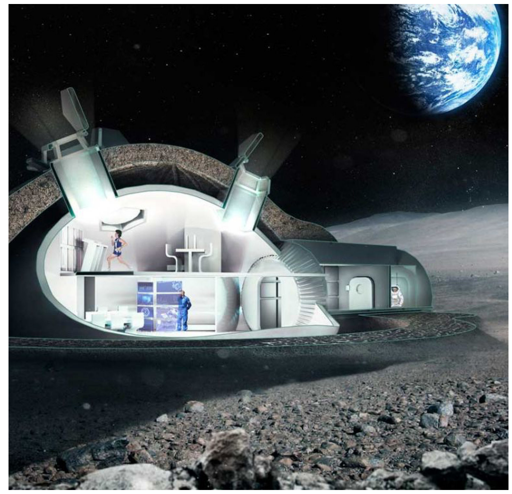


Image credit: ESA

a) (8 points) Classify the critical points of this function

$$f(x,y) = 5x^2 + 8xy + 5y^2$$

using the second derivative test.

b) (2 points) You should have found either a maximum or minimum or a saddle point. Is there a global maximum or global minimum among them?

Solution:

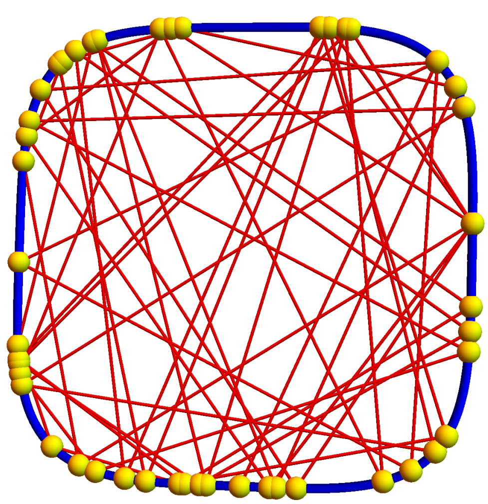
a) The gradient is $\nabla f(x,y) = \begin{bmatrix} 10x + 8y \\ 8x + 10y \end{bmatrix}$. It is zero only if $(x,y) = (0,0)$. We have only one critical point. We have $D = f_{xx}f_{yy} - f_{xy}^2$. which is equal to $10 * 10 - 8^2 = 36 > 0$ and $f_{xx} = 10 > 0$. The critical point is a local minimum.

b) We can write $f(x,y) = 5(x+y)^2 - 2xy \geq 5(x+y)^2 - 2xy - (x-y)^2 = 5(x+y)^2 + x^2 + y^2 \geq 0$. Therefore, the function is always non-negative and $(0,0)$ is a global minimum. (We did not require such a rigorous explanation to prove $f \geq 0$ here.)

Problem 9) (10 points)

The l^p **billiard table** for $p = 4$ is given by $g(x,y) = x^4 + y^4 = 1$. Finding trajectories amounts of extremizing the length of the trajectory. The picture to the right shows a few bounces of this billiard. It is an open question whether it shows chaos on a set of initial conditions which have positive area.

We solve the simpler problem to find the points on that curve, where the distance to the origin is minimal. To do so, we extremize the function $f(x,y) = x^2 + y^2$ under the constraint $g(x,y) = 1$. Find the maxima and minima using the Lagrange multiplier method.



Solution:
The Lagrange equations are

$$\begin{aligned} 2x &= \lambda 4x^3 \\ 2y &= \lambda 4y^3 \\ x^4 + y^4 &= 1 \, . \end{aligned}$$

Combining the first two equations gives $8xy^3 = 8yx^3$. This means either $x = 0$ or $y = 0$ or $x = y$. In the first case we get from the third equation the solutions $(0,1),(0,-1)$. In the second case $(1,0),(-1,0)$. In the third case we get the four solutions $(1,1)/2^{1/4}$,$(-1,1)/2^{1/4}$, $(1,-1)/2^{1/4}$, $(-1,-1)/2^{1/4}$. These are 8 solutions. The minimum is on the first four, the maximum on the later four.

Problem 10) (10 points)

According to **traditional Chinese medicine**, the human body contains 14 meridians. Each meridian is a pathway for life-energy known as "Qi". We don't know how and why they work but lets model them with vector fields and assign the **Qi energy** $\int_C \vec{F} \cdot \vec{dr}$, where C the meridian between acupuncture point $A = (1,2,1) = \vec{r}(0)$ and $B = (2,2,10) = \vec{r}(1)$. The spleen vector field is $\vec{F}$, the meridian C is parametrized by $\vec{r}(t)$ with $0 \leq t \leq 1$. The formulas are

$$\vec{F}(x,y,z) = \begin{bmatrix} x^3+y \\ y^3+x \\ z^5 \end{bmatrix}, \; \vec{r}(t) = \begin{bmatrix} 1+t \\ 2+\sin(15\pi t) \\ 9t+\cos(2\pi t^2) \end{bmatrix}$$

Find the Qi energy!

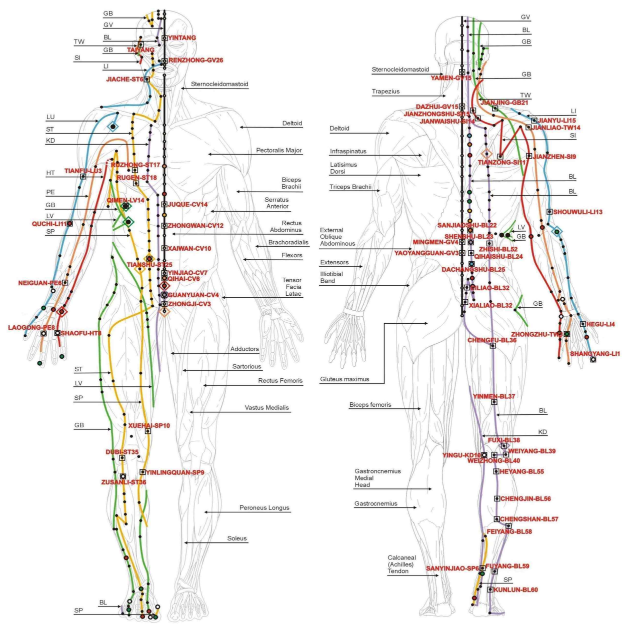


Image source: Wikipedia

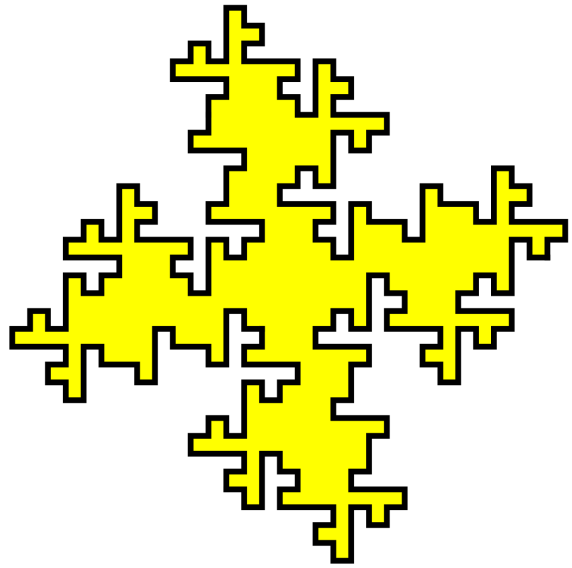
Solution:
The vector field is a gradient field. One can either check this with the Clairaut test or just boldly try to get a function f. This is therefore a problem for the fundamental theorem of line integrals. The potential function is $f(x,y,z) = x^4/4 + xy + y^4/4 + z^6/6$. The answer is $f(2,2,10) - f(2,2,1) = 500036/3 - 73/6 = 333333/2$. Leaving the fractions was of course ok.

Problem 11) (10 points)

Find the line integral of the vector field

$$\vec{F}(x,y) = \begin{bmatrix} x+y \\ 3x+3y^2 \end{bmatrix}$$

along the boundary C of the **Koch island** shown to the right. The curve is oriented counter clockwise and encloses a region G which has 289 unit squares. The length of the curve C is 324.



Remark. The curve was computed using an **L-system** construction, given by rules ”$F+F+F+F$”, ”F” $\to$ ”$F-F+F+FFF-F-F+F$” which is not explained further here. You have all the information needed to solve the problem.

Solution:
This is a problem for Green's theorem. The curl of the vector field is constant 2. The result is $\int\int_G 2\,dxdy$ which is $2Area(G) = 2*289 = 578$. The length of the curve had just been given as a distraction.

Problem 12) (10 points)

A famous photograph shows **Nikola Tesla** reading within a firework of sparks produced by **Tesla coils**. The electric field is given

$$\vec{F}(x,y,z) = \begin{bmatrix} x^3 \\ y^3 \\ z^3 \end{bmatrix} .$$

Find the flux $\int\int_S \vec{F}\cdot d\vec{S}$ of the vector field through outwards oriented sphere

$$S : x^2+y^2+z^2 = 9 .$$

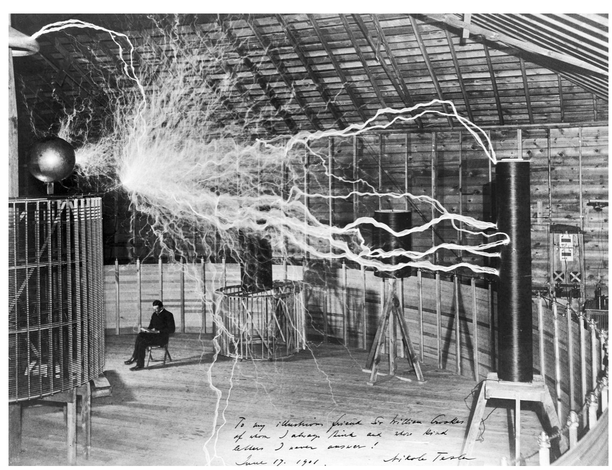


Image source: TeslasAutobiography

Solution:
This is a problem for the divergence theorem. The divergence is $3x^2 + 3y^2 + 3z^2 = 3\rho^2$. We have to integrate this over the sphere of radius 3 which we do in spherical coordinates

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{3} 3\rho^2 \rho^2 \sin(\phi)\; d\rho d\phi d\theta$$

This is $4\pi 3^6/5$.

Problem 13) (10 points)

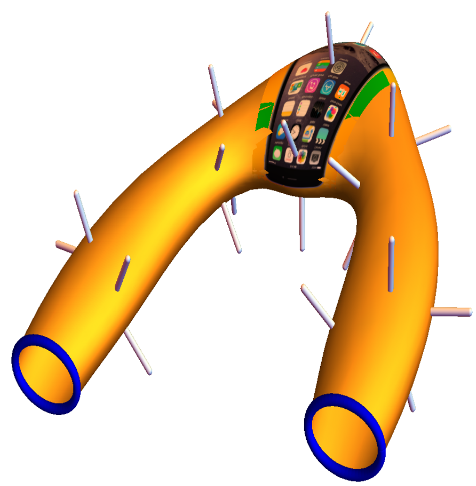
Inspired by electronically connected cars like Tesla's Model 3, we build a **high tech triathlon bike** handle S with integrated electronics. This surface S is given by

$$\vec{r}(t,s) = \begin{bmatrix} (9 + (2 + \sin(5s)/2)\cos(t))\sin(s) \\ (9 + (2 + \sin(5s)/2)\cos(t))\cos(s) \\ (9 + \sin(5s)/2)\sin(t)/3 \end{bmatrix}$$

where $0 \leq t \leq 2\pi$ and $0 \leq s \leq \pi$. The built-in phone measures the flux $\int\int_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$ of the curl of the earths magnetic field

$$\vec{F} = \begin{bmatrix} z \\ 0 \\ y \end{bmatrix}$$

through the surface S which is oriented outwards and has two boundary parts: one,when $s = 0$ and one when $s = \pi$. Compute this flux.



Solution:
This is a problem for Stokes theorem. Instead of computing the flux we compute the line integrals along the boundaries. There are two boundary parts. But we have to make sure that the orientation is right. The curve to the left is $\vec{r}_1(t) = \begin{bmatrix} 0 - 9 - 2\cos(t) + 3\sin(t) \end{bmatrix}$ which is oriented right. The second one is $\vec{r}_2(t) = \begin{bmatrix} 0 - 9 + 2\cos(t) + 3\sin(t) \end{bmatrix}$ which is going counter clockwise and has not the right orientation (remember that the surface has to be to the left) so that this sign needs to be changed. The flux is the sum of two line integrals:

$$\int_0^{2\pi} \begin{bmatrix} 3\sin(t) \\ 0 \\ 9 + 2\cos(t) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -9 - 2\sin(t) \\ 3\cos(t) \end{bmatrix} \, dt - \int_0^{2\pi} \begin{bmatrix} 3\sin(t) \\ 0 \\ -9 - 2\cos(t) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2\sin(t) \\ 3\cos(t) \end{bmatrix} \, dt \; .$$

Using the double angle formula for evaluating $\int_0^{2\pi} \cos^2(t) \; dt = \pi$, this evaluates to $\boxed{6\pi + 6\pi = 12\pi}$.