8/8/2019 FINAL EXAM **Maths 21a, O. Knill, Summer 2019**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or scratch paper at the end for work. Put the final result on the page where the question is asked.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Even correct answers without derivation can not be given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) No justifications are necessary

1) T F The distance d between a point P and a plane through three points A, B, C is given by $d = |\vec{AP} \cdot \vec{BC}|/|\vec{BC}|$.

2) T F The set of all points in three dimensional space satisfying $x^2 + y^2 = z^2, x + y + z = 1$ defines a conic section.

3) T F The surface area of a surface parametrized by $\vec{r}(u,v) = [u, v, g(u,v)]$ is given by $\iint_R |g_u \times g_v| \; dudv$.

4) T F Given two curves $\vec{r}_1(t) = [t^3, t^2, t^4]^T$, and $\vec{r}_2(t) = [t^6, t^4, t^8]^T$ with curvature functions $k_1(t) = \kappa(r_1(t))$ and $k_2(t) = \kappa(r_2(t))$. Then $k_1(0.5) = k_2(0.5)$.

5) T F For any vector field $\vec{F} = [P, Q, R]^T$, the identity grad(div(curl($\vec{F}$)) = div(grad(div($\vec{F}$))) holds.

6) T F Given two different points A, B with distance 1 in space there exists a suitably parametrized curve $r(\vec{t})$ such that $\int_a^b |\vec{r}\,'(t)| \; dt = 2$.

7) T F If $R = \mathbf{R}^2$ is the entire plane, then the improper integral $\iint_R e^{-x^2-y^2} \; dxdy$ has the value π.

8) T F The equation div(grad(div(grad(f))) $= 0$ is always true for any smooth function f.

9) T F If $\int_a^b \int_c^d f(x,y) \; dxdy = \int_c^d \int_a^b f(x,y) dxdy$ for all a, b, c, d and f is continuous, then $f(x,y) = f(y,x)$.

10) T F If $\vec{v} = \vec{r}(0) = \vec{r}\,'(0)$ is zero, then $\vec{r}(t)$ is zero for all times $t > 0$.

11) T F A surface S has boundary C with compatible orientation then $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \; dt = \iint_S \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \; dudv$.

12) T F The equation $y^2 - z^2 + x^2 + 2y = -1$ represents a two-sheeted hyperboloid.

13) T F A gradient field $\vec{F} = \nabla f$ that is divergence free (=incompressible) has a potential f satisfying the PDE $f_{xx} + f_{yy} + f_{zz} = 0$.

14) T F The curvature of the curve $\vec{r}(t) = [25\cos(t^7), 25\sin(t^7)]^T$ at $t = 1$ is everywhere equal to 5.

15) T F The distance between lines satisfies the triangle inequality in the sense that $d(L, M) + d(M, K) \geq d(L, K)$ for any three lines K, L, M.

16) T F If $\vec{F}$ and $\vec{G}$ are smooth vector fields in $\mathbf{R}^2$ for which the curl is constant 1 everywhere. Then $\vec{F} - \vec{G}$ is a gradient field.

17) T F Let E is the solid given as the complement of a circle $\{x^2 + y^2 = 1, z = 0\}$ in $\mathbf{R}^3$, then this solid E is simply connected.

18) T F The parametrization $\vec{r}(u,v) = [u^2, u^4 + v^4, v^2]^T$ is (part of) a paraboloid.

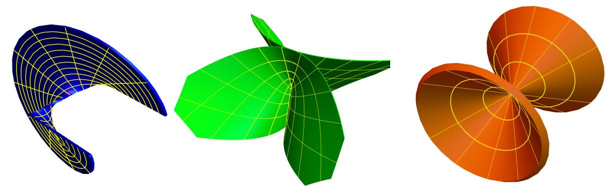
19) T F The flux of the curl of the field $\vec{F} = [x, y, z]^T$ through the unit sphere $\{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ oriented outwards is equal to 4π.

20) T F The flux of the vector field $\vec{F} = [0, 0, 1]^T$ through the unit disc $\{(x,y,z) \mid x^2 + y^2 \leq 1, z = 0\}$ oriented upwards is equal to the area of the disc.

Problem 2) (10 points) No justifications are necessary.

a) (2 points) Match the following surfaces. There is an exact match.

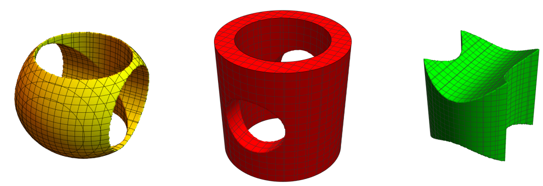
A B C



Parametrized surface $\vec{r}(s,t)$	A-C
$[v\cos(u), v\sin(u), u]^T$	
$[u\cos(v), u, u\sin(v)]^T$	
$[u^2v, uv^2, u^2-v^2]^T$	

b) (2 points) Match the solids. There is an exact match.

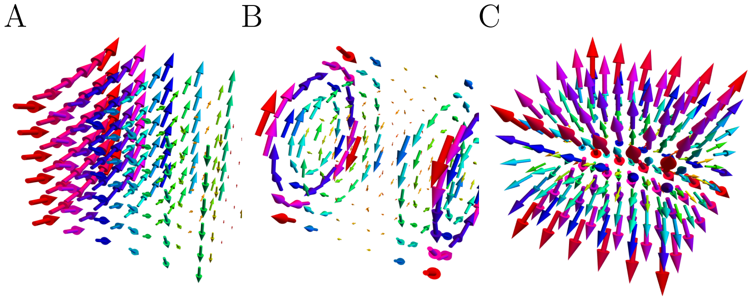
A B C



Solid	A-C
$x^2+y^2+z^2<9, x^2+y^2>4, y^2+z^2>4$	
$x^2-y^2<1, z^2-x^2<1, x^2+y^2<4$	
$x^2+y^2<4, x^2+y^2>2, x^2+z^2>1$	

c) (2 points) The figures display vector fields. There is an exact match.

A B C



Field	A-C
$\vec{F}=[0,-x\sin(z), x\sin(y)]^T$	
$\vec{F}=[0,1-x,x]^T$	
$\vec{F}=[x,2y,3z]^T$	

d) (2 points) Recognize partial differential equations!

Equation	A-F
Laplace	
Burger	
Wave	

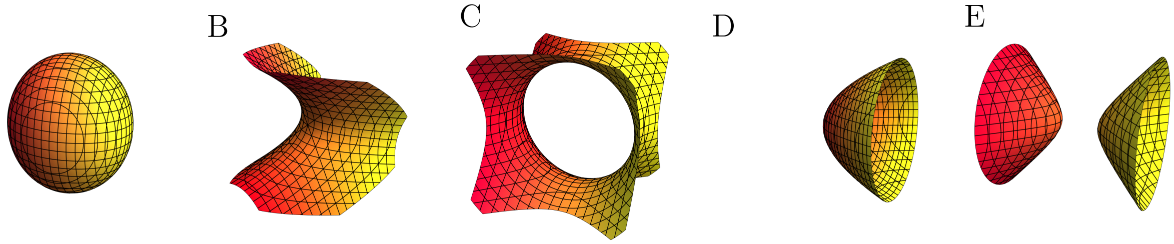
	PDE
A	$X_{tt}=-X_{xx}$
B	$X_t^2-1=X_x^2$
C	$X_t=X_x$

	PDE
D	$X_t=X_xX_{xx}$
E	$X_{tt}=X_{xx}$
F	$-XX_x=X_t$

e) (2 points) Some surfaces

	Enter a letter from A-E each
Pick the one sheeted hyperboloid	
Pick the one elliptic paraboloid	

A B C D E



Problem 3) (10 points) No justifications necessary

Mathematicians live for ever as they are remembered for ever. But only if we do remember them. So, lets remember them to keep them alive.

a) There was a mathematician who's life inspired the Good Will hunting story. Who was it?

b) Stokes theorem was posed as a calculus exam problem. There was a student taking that exam, who would later become famous. Who was this student?

c) Who proved a theorem about changing the order of integration?

d) Who proved a theorem about changing the order of differentiation?

e) Which two mathematicians are associated to a basic inequality appearing in the dot product? As an alternative, name the inequality.

f) The Newtonian law of gravity as an inverse square force F can be derived from a partial differential equation $\text{div}(F) = 4\pi\sigma$. Who was the mathematician who first looked that way at this?

g) Who was first seeing the principle that at a maximum or minimum, the derivative ∇f must be zero. The person discovered it in one dimensions but the argument is the same in higher dimensions.

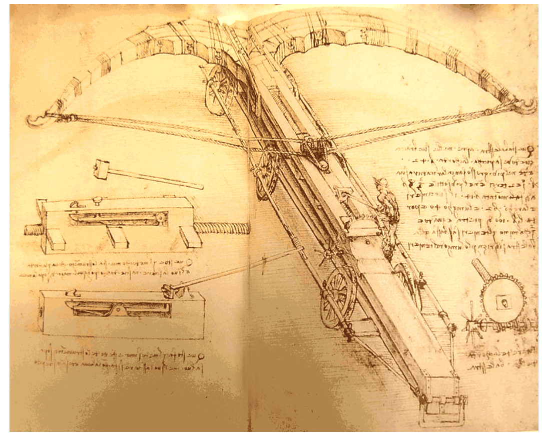
h) Which mathematician first looked at extremization problems with constraints?

i) There is a famous fractal in the plane. We saw how to dive deep into the realms up to 10^{-200} and also tried to get the area of that object. Who was the mathematician who found it?

j) There is a partial differential equation used in finance. You have found some solutions in a homework. Name at least one of the persons associated to it.

Bonus: you can regain here one of the lost point in problem 3) by telling, who first discovered the dot and cross product simultaneously by defining a product of 4-tuples.

Problem 4) (10 points)

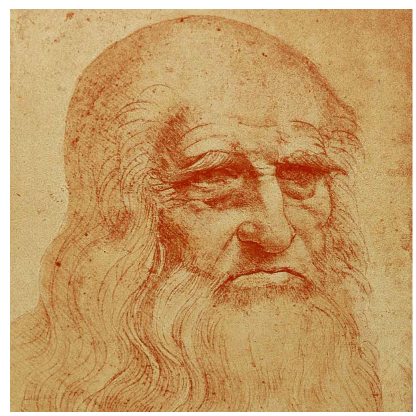


Leonardo Da Vinci designed a cross bow. It was heavy and needed to be cocked by mans of a worm gear. We don't believe it has ever been realized. (By the way, as for today, 2019, the best cross bow Scorpyd Aculeus can shoot bows with over 500 km/h, faster then any speed achieved so far by production car.) Assume now we have a cross bow at position $\vec{r}(0) = (2, 0, 50)$ and that the initial velocity is $\vec{r}\,'(0) = (100, 3, 0)$. Assume there is a gravitational and time dependent side wind force

$$\vec{r}\,''(t) = \begin{bmatrix} 1 \\ \sin(t) \\ -10 \end{bmatrix} .$$

Find the trajectory $\vec{r}(t)$ and at which point does it hit the ground $z = 0$?

Problem 5) (10 points)



a) (5 points) Find an approximation for the number

$$2.001^2 * 7.003^2 * 9.999^3 .$$

by linearizing the function

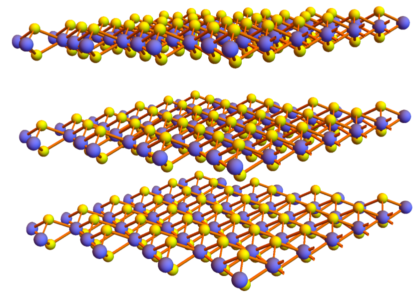
$$f(x, y, z) = x^2 y^2 z^3$$

at $(2, 7, 10)$.

b) (5 points) What is the equation $ax + by + cz = d$ of the tangent plane to the surface $f(x, y, z) = 196000 = 2^2 * 7^2 * 10^3$ at the point $(x_0, y_0, z_0) = (2, 7, 10)$?

Problem 6) (10 points)

Molybdenum disulfide MoS_2 consists of molybdenum Mo and sulfur S. Crystalized molybdenite can be found for example in the Swiss mountains, like in the Baltschieder region. The crystal layer structure is pretty cool. Similarly as graphite, the planes are held together by van der Waals forces. As it has low friction and is added as a solid lubricant. Assume one of the layers is the plane



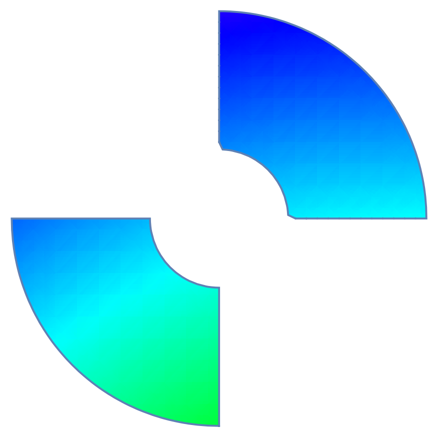
$$x + 2y + z = 4$$

and on the next layer is an atom with coordinate $P = (3, 4, 5)$.

a) (7 points) What is the distance of P to the plane?

b) (3 points) Please parametrize a line perpendicular to the plane passing through P.

Problem 7) (10 points)



a) (5 points) Evaluate the double integral

$$\int_0^4 \int_0^{x^2} \frac{y^4}{4 - \sqrt{y}} \, dydx$$

b) (5 points) Find the moment of inertia

$$\iint_G x^2 + y^2 \, dxdy$$

for the region $G = \{xy > 0, \;\; 4 < x^2 + y^2 < 9\}$.

Problem 8) (10 points)

In order to optimize a parachute of Da Vinci, we classify the critical points of the function

$$f(x,y) = 4x^2y + 2x^2 + y^2 \ .$$

The first part is related to the volume, the second part is related to residual volume dragged along during the fall. Don't worry about the derivation of the function $f(x,y)$. It is a Da Vinci thing. So:

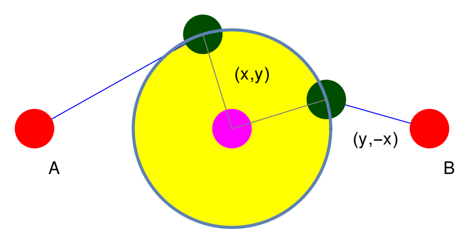
a) (8 points) Classify the critical points.

b) (2 points) Decide whether any of the points is a global maximum or global minimum.

Problem 9) (10 points)

Motivated by a drawing of Da Vinci, we look at the following problem: a wheel of radius 1 is attached by rubber bands to two points $A = (-2,0)$ and $B = (2,0)$. The point (x,y) connects to A and $(y,-x)$ to $(2,0)$. The point (x,y) is constrained to

$$g(x,y) = x^2 + y^2 = 1 \ .$$



The wheel will settle at the position, for which the potential energy $f(x,y) = [(x+2)^2 + y^2 + (y-2)^2 + x^2]/2$ which is

$$f(x,y) = x^2 + y^2 + 2x - 2y + 4$$

is minimal. Find that position using the method of Lagrange multipliers.

Problem 10) (10 points)

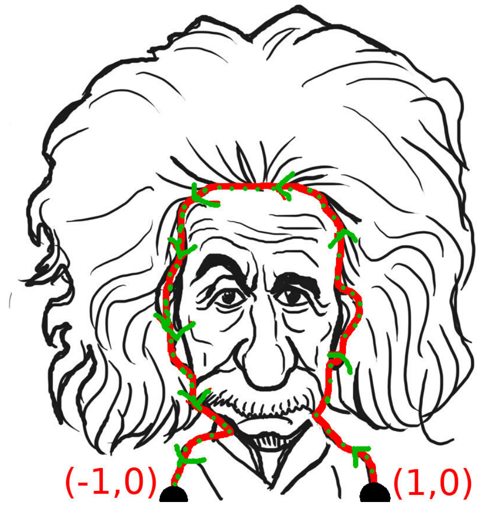
What is the line integral

$$\int_0^1 \vec{F}(r(t)) \cdot \vec{r}\,'(t)\; dt$$

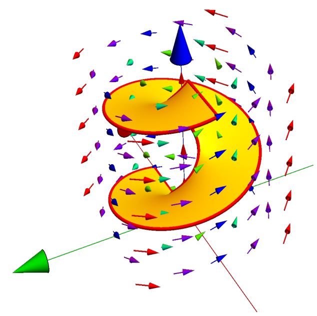
along the **Einstein curve** shown in the picture? The curve goes from $\vec{r}(0) = (1,0)$ to $\vec{r}(1) = (-1,0)$. The vector field is

$$\vec{F}(x,y) = \begin{bmatrix} 4x^3 + y + y^2 \\ 1 + x + 2xy \end{bmatrix} .$$

Don't ask for the formula of the Einstein curve. Only Einstein knows.



Problem 11) (10 points)



We build a model of the **Da Vinci helicopter**. The helicopter blade is a helix S parametrized as

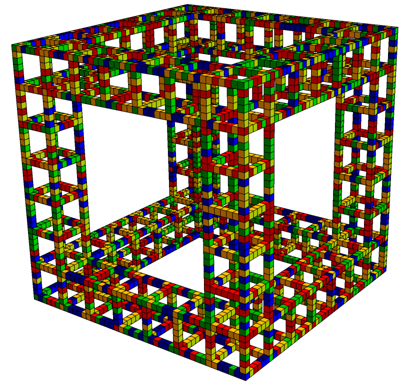
$$\vec{r}(u,v) = \begin{bmatrix} u\cos(v) \\ u\sin(v) \\ v \end{bmatrix}$$

with $0 \leq v \leq 3\pi$ and $1 \leq u \leq 5$. Its boundary C consists of 4 parts. A path going radially out, then the helix up, going radially in and then along the axes down. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix} .$$

Find the line integral of $\vec{F}$ along the curve C. The curve is oriented so that it is compatible with the surface orientation, which is the orientation given by the parametriation.

Problem 12) (10 points)



We love the **Menger sponge**. For an exhibit hall, we build a variant, where a cube is divided into 7 parts and the middle 5×5 cylinders are cut out, we end up with a fractal of dimension $\log(68)/\log(7) = 2.1684....$ The second iteration E shown in the picture consists of $68^2 = 4624$ cubes of side length 1. What is the flux of the vector field

$$\vec{F}(x,y,z) = \begin{bmatrix} 4x + e^{\cos(y)} \\ z^5 - y \\ y^5 - z \end{bmatrix}$$

through the boundary S of the solid E assuming as usual that the surface S is oriented outwards?

Problem 13) (10 points)

The **"Easy-going region"** D enclosed by the curve

$$\vec{r}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) - \cos(t) \end{bmatrix}$$

with $0 \le t \le 2\pi$ is called the "laidback disk". You can just call it the "Dude" or "His Dudeness" if you are not into that whole brevity thing. What is the area of the dude D?