Name:

- Start by writing your name in the above box.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Except for problems 1-4 and 9, give details.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have exactly 90 minutes to complete your work.

|		
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F The Cauchy-Schwartz inequality assures that $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$ for all $\vec{v}, \vec{w}$.

2) T F The two planes $3x + 4y + z = 4$ and $6x + 8y + 2z = 4$ intersect in a line.

3) T F If the distance between two points P and Q is zero, then $P = Q$.

4) T F The symmetric equations of $\vec{r}(t) = \langle 2t, 2t, t \rangle$ is $x = y = 2z$.

5) T F The arc length of a circle with constant curvature r is $2\pi r$.

6) T F The surface $x^2 - y^2 + 4y = z^2$ is a two sheeted hyperboloid.

7) T F There are two vectors $\vec{v}, \vec{w}$ in space for which the angle between the vectors is $3\pi/2$.

8) T F If the acceleration of a curve $\vec{r}(t)$ is zero at all times, then $\vec{r}(t) = \vec{r}(0)$.

9) T F The lines $\vec{r}(t) = \langle 3t, 4t, 5t \rangle$ and $\vec{r}(t) = \langle -4t, 3t, 0 \rangle$ intersect perpendicularly.

10) T F The point given in spherical coordinates as $\rho = 2, \phi = \pi, \theta = \pi$ is on the z-axes.

11) T F Given three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, then $|(\vec{u} \cdot \vec{v})||\vec{w}| = |\vec{u}||\vec{v} \cdot \vec{w}|$.

12) T F The surface given in spherical coordinates as $\cos(\phi) = \rho$ is a plane.

13) T F The arc length of the curve $\langle \sin(t), \cos(t), t \rangle$ from $t = 0$ to $t = 2\pi$ is larger than 2π.

14) T F The surface parametrized by $\vec{r}(u, v) = \langle v \sin(u), v \cos(u), 0 \rangle$ is a plane.

15) T F The length of the cross product of two non-parallel vectors $\vec{v}$ and $2\vec{w}$ is larger than the length of the cross product of the vectors $\vec{v}$ and $\vec{w}$.

16) T F It is possible that the intersection of an ellipsoid with a plane is a hyperbola.

17) T F The function $f(x, y) = x^4 y^4/(x^2 + y^2)$ is continuous at $(0, 0)$ in the sense we have defined continuity: there is a value one can assign to $f(0, 0)$ so that the function is continuous everywhere.

18) T F Given two curves $\vec{r}(t)$ and $\vec{s}(t)$ for which $\vec{r}\,''(t) = \vec{s}\,''(t)$ for all t and $\vec{r}(0) = \vec{s}(0)$, then $\vec{r}(t) = \vec{s}(t)$.

19) T F If $\vec{v} \times \vec{w} = \vec{w} \times \vec{v}$, then $\vec{v} = \vec{w}$ or $\vec{v} = -\vec{w}$.

20) T F The point $M = (A + B + C)/3$ is equidistant from A and B and C meaning: $d(A, M) = d(B, M) = d(C, M)$.
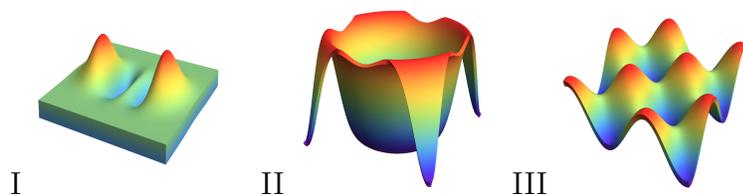
Total

Problem 2) (10 points) No justifications are needed in this problem.

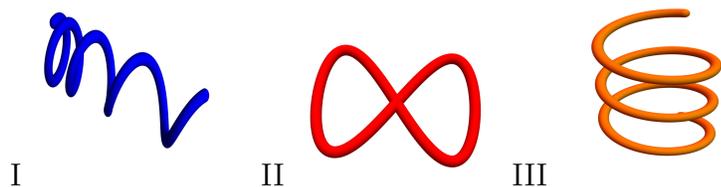a) (2 points) Match the contour surfaces $g(x, y, z) = 1$. Enter O, if there is no match.

I   II   III
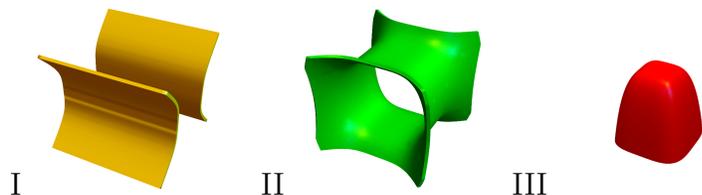
Function $g(x, y, z) =$	Enter O,I,II or III
$x^4 + z^4 + z$	
$x^4 - y^2 + z^2$	
$x - y$	
$x^4 - y^4$	

b) (2 points) Match the graphs of the functions $f(x, y)$. Enter O, if there is no match.

I   II   III

Function $f(x, y) =$	Enter O,I,II or III
$\sin(x)\sin(y)$	
$\sin(x^2 + y^2 - 3)$	
$x\cos(y^2)$	
$\exp(-x^2 - y^2)(x^4 - x^2)$	

c) (2 points) Match the space curves with the parametrizations. Enter O, if there is no match.
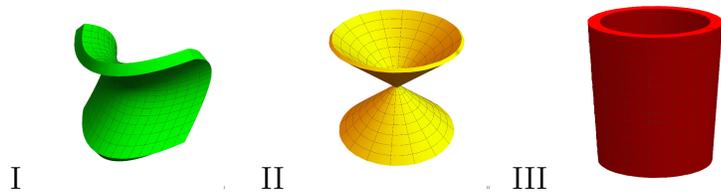
I   II   III

Parametrization $\vec{r}(t) =$	Enter O, I,II or III
$\langle t^2/5, \sin(4t), \cos(4t) \rangle$	
$\langle \cos(t), \sin(2t), \cos(t) \rangle$	
$\langle t, t, t \rangle$	
$\langle \sin(t), \cos(t), \sin(3t) \rangle$	

d) (2 points) Match the functions g with contour plots in the xy-plane. Enter O, if there is no match.

I   II   III

Function $g(x, y) =$	Enter O, I,II or III
$\sin(x^2 - y^2)$	
$\sin(x) - y$	
$|x| + y$	
$10x^2 + y^2$	

e) (2 points) Match the quadrics. Enter O if there is no match.

I   II   III

| Quadric	Enter O,I,II or III
$y = x^2 - z^2$	
$x^2 + y^2 = 1$	
$x^2 + y^2 + z^2 = 1$	
$y = 1$	
$x^2 + y^2 = z^2$	

Problem 3) (10 points)	(No justifications are needed in this problem)

a) (4 points) If you combine the left and upper statement you get a statement. Check the ones which are true for any two vectors $\vec{u}$ and $\vec{v}$.

Object	always 0	can be $\neq 0$	always $\vec{0} = \langle 0,0,0 \rangle$	can be nonzero vector
$(\vec{u} \times \vec{v}) \times (\vec{u} \times \vec{v})$				
$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v})$				
$\text{proj}_{\vec{u} \times \vec{v}} \vec{u}$				
$\text{proj}_{\vec{u} \times \vec{v}} \vec{v}$				

b) (3 points) The three curves parametrized here are each located on a surface. This helps you to match the curves.

| $\vec{r}(t) =$	Enter A-C
$\langle \cos(9t)\sin(t), \sin(9t)\sin(t), \cos(t) \rangle$	
$\langle (5 + (2 + \cos(3t))\sin(9t))\cos(t), (5 + (2 + \cos(3t))\sin(9t))\sin(t), (2 + \cos(3t))\cos(9t) \rangle$	
$\langle \cos(7t), \sin(7t), \sin(9t) \rangle$	

B   A   C
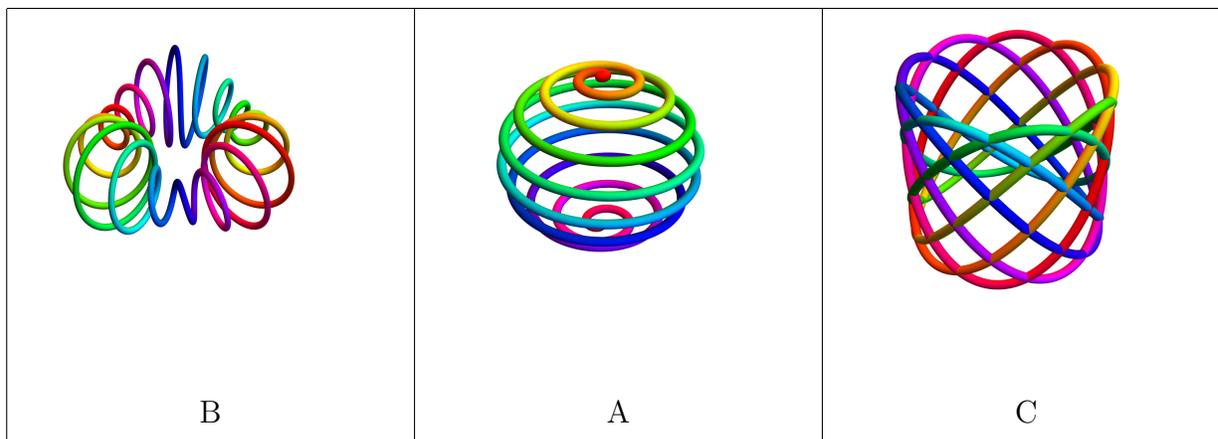
c) (3 points) Complete the "multiplication table". In each case, enter either $\vec{T}, \vec{N}, \vec{B}$ or $\vec{0}$ in each of the 9 boxes

$\text{proj}_{\vec{T}}(\vec{T}) =$		$\text{proj}_{\vec{T}}(\vec{N}) =$		$\text{proj}_{\vec{T}}(\vec{B}) =$	
$\text{proj}_{\vec{N}}(\vec{T}) =$		$\text{proj}_{\vec{N}}(\vec{N}) =$		$\text{proj}_{\vec{N}}(\vec{B}) =$	
$\text{proj}_{\vec{B}}(\vec{T}) =$		$\text{proj}_{\vec{B}}(\vec{N}) =$		$\text{proj}_{\vec{B}}(\vec{B}) =$	

Problem 4) (10 points)

On October 16 of 1843, **Sir William Hamilton** discovered quaternions. This was also the birth date of the dot and cross product. The reason is that for quaternions without constant part, one has

$$(ai + bj + ck)(ui + vj + wk) = \langle a, b, c \rangle \cdot \langle u, v, w \rangle$$
$$+ \langle a, b, c \rangle \times \langle u, v, w \rangle$$
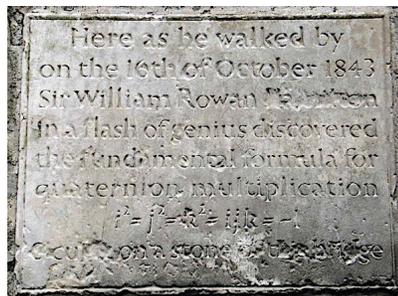
The quaternion plaque in Dublin shows that the postulate $i^2 = j^2 = k^2 = ijk = -1$ does the trick.

You can ignore the above story and compute some dot and cross products.
We work with

$$\vec{v} = \langle 3, 4, 5 \rangle = 3i + 4j + 5k$$

and

$$\vec{w} = \langle 2, 2, 1 \rangle = 2i + 2j + k \ .$$

a) (1 point) Compute $\vec{v} \cdot \vec{w}$	
b) (1 point) Compute $|\vec{v}|$	
c) (1 point) Compute $|\vec{w}|$	
d) (1 point) Compute $\vec{v} \times \vec{w}$	
e) (1 point) Compute $\vec{w} \times \vec{v}$	
f) (1 point) Compute $|\vec{v} \times \vec{w}|$	

The following problem is independent of a), b) and can be solved by ignoring quaternions:

There are two formulas involving an angle for the dot and the cross product which together assure that if both $\vec{v} \cdot \vec{w} = 0$ and $\vec{v} \times \vec{w} = \vec{0}$ then either $\vec{v} = \vec{0}$ or $\vec{w} = \vec{0}$. What are they?

g) (2 points) $\vec{v} \cdot \vec{w} =$	
h) (2 points) $|\vec{v} \times \vec{w}| =$	

Problem 5) (10 points)

Oliver learned from his grandmother how to knit. He even made once a pullover for himself. The knitting process got him interested in topology, knots and links.

Assume that the first needle is given by $\vec{v} = \langle 1, 1, 2 \rangle$ attached at $P = (-1, 0, 0)$ and the second needle is $\vec{w} = \langle -1, 2, 1 \rangle$ attached at $Q = (1, 0, 0)$. Find the distance between the lines defined by the needles.
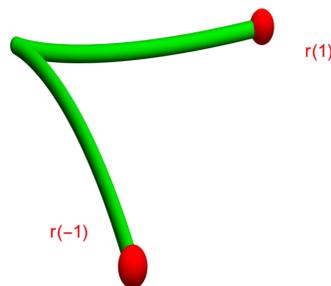
Problem 6) (10 points)

a) (4 points) Find the arc length of the curve $\vec{r}(t) = \langle 3t^2, 12t^3/3, 1 \rangle$ from $t = -1$ to $t = 1$. Hint: The integral is solvable. It is helpful to factor and split the integral.

b) (1 point) Find $\vec{r}\,'(1)$.

c) (1 point) Find $\vec{r}\,''(1)$.

d) (4 points) What is the curvature of the curve at time $t = 1$? Reminder: The formula for the curvature is

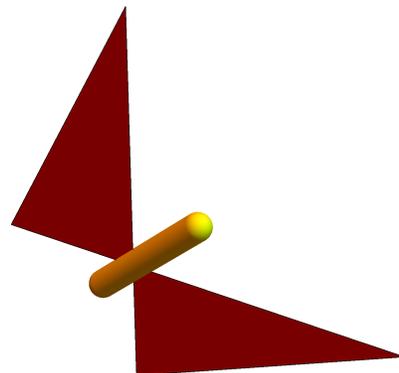$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}\ .$$

Problem 7) (10 points)

A **butterfly wing** is modeled by a simple stroke $ABCD$, where

$$A = (2, 1, 0), \quad B = (-1, 1, 0),$$

$$C = (0, 1, 2) \text{ and } D = (0, 1, -1).$$

As you see in the picture, this produces two triangles touching at $O = (0, 0, 0)$. Find the wing area which is the sum of the areas of the two congruent triangles.
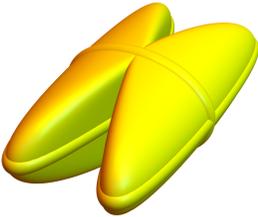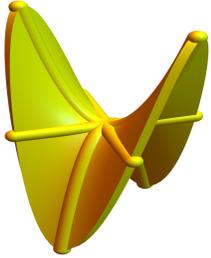

Problem 8) (10 points)

A **pinball machine** is tilted in such a way that a ball in the xy-plane experiences a constant force $\vec{F} = \langle 0, -2 \rangle$. A ball of mass 1 is hit the left flipper at the point $\vec{r}(0) = \langle -1, 0 \rangle$ with velocity $\vec{r}\,'(0) = \langle 1/2, 5 \rangle$.

a) (4 points) Find the velocity $\vec{r}\,'(t)$.

b) (4 points) What trajectory $\vec{r}(t) = \langle x(t), y(t) \rangle$ does the ball follow?

c) (2 points) As the ball hits the line $y = 0$, is it reachable by the player? In other words, does it hit $y = 0$ within the interval $x \in [1, 2]$?
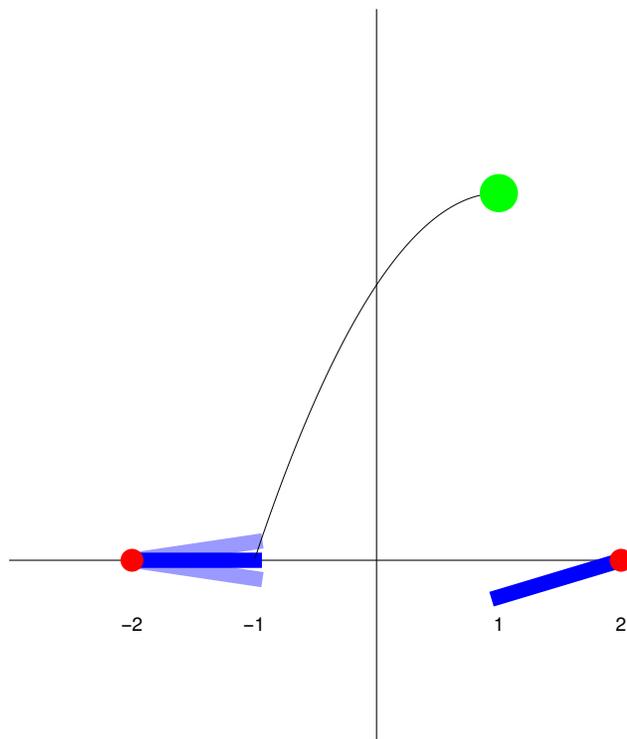
An MIT start-up has designed some **pasta**, which takes its shape when putting in water. This is 4D printing technology. We diversify our math-candy company and work on some pasta, parametrized by surfaces. We ask you to complete the parametrization (2 points each).

a)		The **connelloni** $$x^2 + y^2 = z^2$$ is parametrized by $$\vec{r}(\theta, z) = \langle \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots\ldots\ldots, z \rangle$$
b)		The **eightellini** $$x^2(1 - x^2) = y^2$$ is parametrized by $$\vec{r}(\theta, z) = \langle \cos(\theta), \sin(2\theta), \ldots\ldots\ldots \rangle$$
c)		The **lemnisagna** $$4x^2(1 - z^2 - x^2) = y^2(1 - z^2)$$ is parametrized by $$\vec{r}(\theta, \phi) = \langle \cos(\theta)\sin(\phi), \sin(2\theta)\sin(\phi), \ldots\ldots\ldots \rangle$$
d)		The **saddelloni** $$x^2 - y^2 = z$$ is parametrized by $$\vec{r}(x, y) = \langle \ldots\ldots\ldots, \ldots\ldots\ldots, x^2 - y^2 \rangle$$
e)		The **hörnli** (a Swiss pasta type) $$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$$ is parametrized by $$\vec{r}(\theta, \phi) = \langle (2 + \cos(\phi))\cos(\theta), \ldots\ldots\ldots\ldots\ldots, \ldots\ldots\ldots \rangle$$