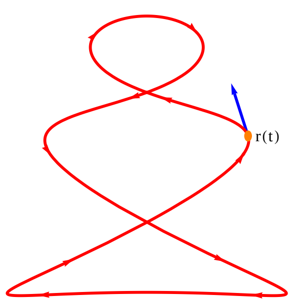
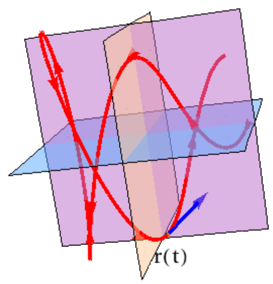
7: Parametrized curves

A **parametrization** of a curve is a map $\vec{r}(t) = \langle x(t), y(t) \rangle$ from a **parameter interval** $R = [a, b]$ to the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of the parametrization is called a **parametrized curve** in the plane. In three dimensions, the parametrization is $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ and the image of r is a **parametrized curve** in space.





We always think of the **parameter** t as **time**. For a fixed time t, we have a vector $\langle x(t), y(t), z(t) \rangle$ in space. As t varies, the end point of this vector moves along the curve. The parametrization contains more information about the curve then the curve alone. It tells for example, how fast we go along the curve.

1 1) The parametrization $\vec{r}(t) = \langle \cos(3t), \sin(5t) \rangle$ describes a curve in the plane. It is an example of a **Lissajous curve**.

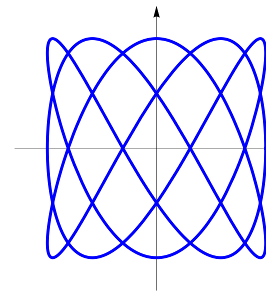
2 2) If $x(t) = t$, $y(t) = f(t)$, the curve $\vec{r}(t) = \langle t, f(t) \rangle$ traces the **graph** of the function $f(x)$. For example, for $f(x) = x^2 + 1$, the graph is a parabola.

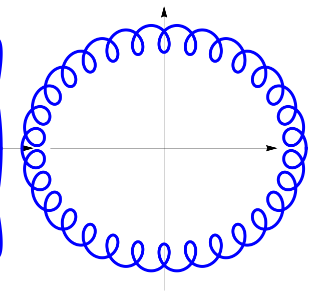
3 3) With $x(t) = 2\cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows an **ellipse**. We can see this from $x(t)^2/4 + y(t)^2 = 1$. We can can overlay an other circular motion to get an epicycle $\vec{r}(t) = \langle \cos(t) + \cos(31t)/4, 4\sin(t) + \sin(31t)/4 \rangle$.

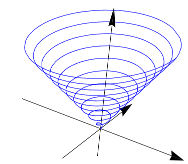
4 4) With $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$ we get the parametrization of a **space curve** $\vec{r}(t) = \langle t\cos(t), t\sin(t), t \rangle$. It traces a **helix** which has a radius changing linearly.

5 If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in the previous example but the curve is traversed **faster**. The **parameterization** of the curve has changed.

6 If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve again but we traverse it in the **opposite direction**.







7 If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u - a), b + t(v - b), c + t(w - c) \rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q. For example, $\vec{r}(t) = \langle 1 + t, 1 - t, 2 + 3t \rangle$ connects the points $P = (1, 1, 2)$ with $Q = (2, 0, 1)$.

Sometimes it is possible to eliminate the time parameter t and write the curve using equations. We need one equation to do so in two dimensions and two equations in three dimensions.

Some curves can be written as the intersection of two surfaces. The pair of equations $f(x, y, z) = 0, g(x, y, z) = 0$ is called an implicit description of a curve.

8 The symmetric equations describing a line $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$ can be seen as the intersection of two surfaces.

9 If f and g are polynomials, the set of points satisfying $f(x, y, z) = 0, g(x, y, z) = 0$ is an example of an **algebraic variety**. An example is the set of points in space satisfying $x^2 - y^2 + z^3 = 0, x^5 - y + z^5 + xy = 3$. An other example is the set of points satisfying $x^2 + 4y^2 = 5, x + y + z = 1$ which is an ellipse in space.

10 For $x(t) = t\cos(t), y(t) = t\sin(t), z(t) = t$, then $x = t\cos(z), y = t\sin(z)$ and we can see that $x^2 + y^2 = z^2$. The curve is located on a cone. We also have $x/y = \tan(z)$ so that we could see the curve as an intersection of two surfaces. Detecting relations between x, y, z can help to understand the curve.

11 Curves describe the paths of particles, celestial bodies, or quantities which change in time. Examples are the motion of a star moving in a galaxy, or economical data changing in time.. Here are some more places, where curves appear:

Strings or knots	are closed curves in space.
Large Molecules	like RNA or proteins can be modeled as curves.
Computer graphics:	surfaces are represented by mesh of curves.
Typography:	fonts represented by Bezier curves.
Space time:	curve in space-time describes the motion of an object
Topology:	space filling curves, boundaries of surfaces or knots.

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle$ is called the **velocity** at time t. Its length $|\vec{r}\,'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

Any vector parallel to the velocity vector $\vec{r}\,'(t)$ is called **tangent** to the curve at $\vec{r}(t)$.

Here are where velocities, acceleration and jerk are computed:

Position	$\vec{r}\,(t)$	$= \langle \cos(3t), \sin(2t), 2\sin(t) \rangle$
Velocity	$\vec{r}\,'(t)$	$= \langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$
Acceleration	$\vec{r}\,''(t)$	$= \langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$
Jerk	$\vec{r}\,'''(t)$	$= \langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$

Lets look at some examples of velocities and accelerations:

Signals in nerves:	40 m/s	Train:	0.1-0.3 m/s^2
Plane:	70-900 m/s	Car:	3-8 m/s^2
Sound in air:	Mach 1=340 m/s	Free fall:	1G = 9.81 m/s^2
Speed of bullet:	1200-1500 m/s	Space shuttle:	$3G = 30m/s^2$
Earth around the sun:	30'000 m/s	Combat plane F16:	9G m/s^2
Sun around galaxy center:	200'000 m/s	Ejection from F16:	14G m/s^2.
Light in vacuum:	300'000'000 m/s	Electron in vacuum tube:	10^{15} m/s^2

The **addition rule** in one dimension $(f+g)' = f'+g'$, the **scalar multiplication rule** $(cf)' = cf'$ and the **Leibniz rule** $(fg)' = f'g+fg'$ and the **chain rule** $(f(g))' = f'(g)g'$ generalize to vector-valued functions because in each component, we have the single variable rule.

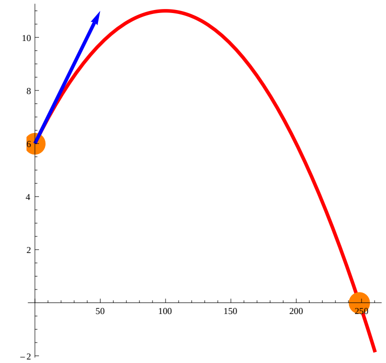
$(\vec{v}+\vec{w})' = \vec{v}\,' + \vec{w}\,'$, $(c\vec{v})' = c\vec{v}\,'$, $(\vec{v}\cdot\vec{w})' = \vec{v}\,'\cdot\vec{w} + \vec{v}\cdot\vec{w}\,'$ $(\vec{v}\times\vec{w})' = \vec{v}\,'\times\vec{w} + \vec{v}\times\vec{w}\,'$ $(\vec{v}(f(t)))' = \vec{v}\,'(f(t))f'(t)$.

The process of differentiation of a curve can be reversed using the **fundamental theorem of calculus**. If $\vec{r}\,'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ at all times as well as initial velocity and position $\vec{r}\,'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

The **free fall** is the case when acceleration is constant. The direction of the constant force defines what is "down". If $\vec{r}\,''(t) = \langle 0,0,-10\rangle$, $\vec{r}\,'(0) = \langle 0,1000,2\rangle$, $\vec{r}(0) = \langle 0,0,h\rangle$, then $\vec{r}(t) = \langle 0, 1000t, h+2t-10t^2/2\rangle$.

If $r''(t) = \vec{F}$ is constant, then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) - \vec{F}t^2/2$.



Homework

1 Sketch the plane curve $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t^3, t^2\rangle$ for $t \in [-1,1]$ by plotting the points for different values of t. Calculate its velocity $\vec{r}\,'(t)$ as well as its acceleration $\vec{r}\,''(t)$ at the point $t = 2$.

2 A device in a car measures the acceleration $\vec{r}\,''(t) = \langle \cos(t), -\cos(3t)\rangle$ at time t. Assume that the car is at the origin $(0,0)$ at time $t = 0$ and has zero speed at $t = 0$, what is its position $\vec{r}(t)$ at time t?

3 Verify that the curve $\vec{r}(t) = \langle t\cos(t), 2t\sin(t), t^2\rangle$ is located on the **elliptical paraboloid**

$$z = x^2 + \frac{y^2}{4} \; .$$

Use this fact to sketch the curve.

4 Find the parameterization $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ of the curve obtained by intersecting the elliptical cylinder $x^2/9 + y^2/4 = 1$ with the surface $z = xy$. Find the velocity vector $\vec{r}\,'(t)$ at the time $t = \pi/2$.

5 Consider the curve

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = \langle t^2, 1+t, 1+t^3\rangle \; .$$

Check that it passes through the point $(1,0,0)$ and find the velocity vector $\vec{r}\,'(t)$, the acceleration vector $\vec{r}\,''(t)$ as well as the jerk vector $\vec{r}\,'''(t)$ at this point.