7/24/2008 SECOND HOURLY **Maths 21a, O. Knill, Summer 2008**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** The acceleration vector $\vec{r}\,''(t)$ and the jerk vector $\vec{r}\,'''(t)$ are always orthogonal to each other.

Solution:
This is already false for an accelerated path on a line.

2) T **F** The function $f(x, y) = x^4 + y^4$ satisfies the partial differential equation $f_{xx} - f_{yy} = 0$.

Solution:
Check this by differentiating.

3) T **F** If we know the speed of a curve at all times as well as the initial position $\vec{r}(0)$, then we can determine the position $\vec{r}(t)$.

Solution:
If we would know the velocity and the initial position, it would be ok.

4) **T** F There is a function $f(x, y)$ of two variables which has no critical points at all.

Solution:
Take $f(x, y) = x + y$ for example.

5) **T** F If $(2, 3)$ is a local maximum for the function f with discriminant $D > 0$, then $f_{xx}(2, 3) < 0$.

Solution:

6) **T** F If f satisfies the partial differential equation $f_x + f_y = 0$ everywhere, then the discriminant D is zero at every critical point.

Solution:
Because $f_{xy} = f_{xx}$ and $f_{yx} = f_{yy}$, we have $D = f_{xx}f_{yy} - f_{xy}^2 = 0$.

7) T F If $f(x,y)$ has a saddle point, then $-f(x,y)$ has a saddle point.

Solution:
The same critical point is also a saddle point, if we turn it upside down.

8) T F The value of the function $f(x,y) = xy$ at $(x,y) = (3.1, 5.2)$ can be estimated as $15 + 0.5 + 0.6$.

Solution:
Use the formula for $L(x,y)$.

9) T F The curvature of the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \rangle = (10\sin(t), 10\cos(t), 20)$ is $1/20$.

Solution:
This is a circle of radius 10.

10) T F The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

Solution:
Unlike in the practice exam, this is now correct.

11) T F If the curvature of a curve is nonzero at some point, then the curve is not contained in a line.

Solution:
The curvature of a line is zero.

12) T F The gradient of f at a point (x_0, y_0) is parallel to the level curve of f which contains (x_0, y_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

13) T **F** If $(1,1)$ is a critical point of $f(x,y)$ and is not a critical point of $g(x,y)$ then $(1,1)$ can not be a critical point of f under the constraint g.

Solution:
It not only can, it always is. $\nabla f = \lambda \nabla g$ is solved with $\lambda = 0$.

14) **T** F If an airship always moves in the direction opposite to the gradient of the pressure, then the pressure momentarily decreases.

Solution:
Use the chain rule.

15) T **F** At points, where the velocity of a curve is zero, the curvature is zero.

Solution:
The curvature is not defined at such points.

16) T **F** If D is the discriminant at a critical point and $D < 0$ then $f_{xy} \leq 0$.

Solution:
Take $f(x,y) = -xy$.

17) **T** F The function $f(x,y) = x^4y^4$ satisfies the partial differential equation $f_{yxyxyxyxy} = 0$.

Solution:
By Clairot's theorem, we can change the order of integration. We know $f_y = 0$.

18) T **F** If $(0,0)$ is a critical point of $f(x,y)$ and $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) > 0$, then $(0,0)$ can not be a local minimum.

Solution:
Take $x^2 + y^2 - 100xy$ for example

19) | T | F | The tangent plane to the graph $f(x,y) = z$ at a point (x_0, y_0) is parallel to $\langle 0, 0, 1 \rangle$ if (x_0, y_0) is a critical point.

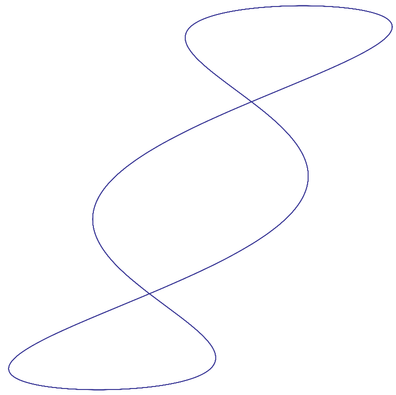
Solution:
The vector $(2, 3, 1)$ is perpendicular to the surface as is $-f_x, -f_y, 1$.

20) | T | F | If the curvature of a curve $\vec{r}(t)$ is constant 10 everywhere, then the curve is a circle.

Solution:
There are other curves. It can be a spiral too for example.

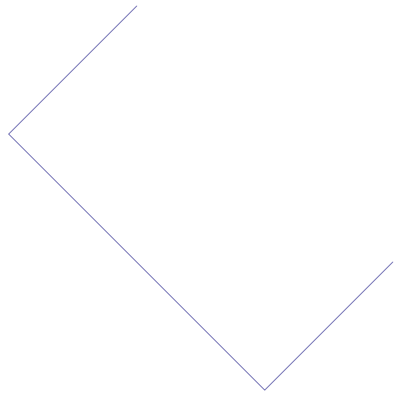
Problem 2) (10 points)

a) (4 points) Match the parameterizations with the curves. No justifications are needed.

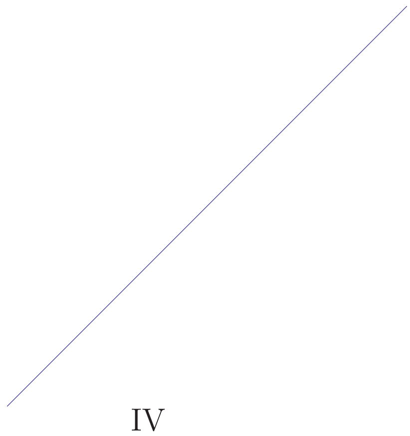


I

II



III



IV

Enter I,II,III,IV here	Parameterization
	$\vec{r}(t) = \langle 3t^5, t^5 + 4 \rangle$
	$\vec{r}(t) = \langle \lvert t-1 \rvert, \lvert t+1 \rvert \rangle$
	$\vec{r}(t) = \langle \sin(t)/t, t \rangle$
	$\vec{r}(t) = \langle \sin(t) + \cos(3t), \sin(t) \rangle$

Solution:

b) (3 points) The linear estimates below obtained as in this example: for $f(x) = \sin(x)$ estimate $\sin(\pi + 1/10)$ by $\sin(\pi) + \cos(\pi)(1/10) = -1/10$. No justifications are needed.

expression	Enter A-C here
$1/2^{1/10}$	
$\sqrt{1 + 1/5}$	
$1 + 1/11$	

A	$1 + \frac{1}{10}$
B	$1 + 1/10 - \frac{1}{100}$
C	$1 - \frac{1}{10}$

c) (3 points) Match the arc length and curvature We know that the unit circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ parametrized from 0 to 2π has arc length 2π and curvature constant equal to 1. What happens if the radius changes?

The curve	2 arc length	arc length/2	2 curvature	curvature/2
$\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$				
$\vec{r}(t) = \langle (1/2)\cos(t), (1/2)\sin(t) \rangle$				

Solution:

a) The curve $\langle |t-1|, |t+1| \rangle$ is piecewise linear, the curve $\langle \sin(t)/t, t \rangle$ is a graph, the curve $\langle \sin(t) + \cos(3t), \sin(t) \rangle$ is a closed curve, the curve $\langle 3t^5, t^5 + 4 \rangle$ is a line because $x/3 = y - 4$. The solution code is IV, III, II, I.

b) For $2^{-1/10}$, we look at the function $f(x) = x^{-1/10}$ which has derivative $x^{-11/10}/10$. We estimate $2^{-1/10} = 1 + (-1/10) = 1 - 1/10$.

For $\sqrt{1 + 1/5}$ we use the function $f(x) = \sqrt{x}$ which has the derivative $1/(2\sqrt{x})$. Linear estimation gives $1 + (1/2)1/5 = 1 + 1/10$.

For $1 + 1/11$ we look at the function $f(x) = 1 + 1/x$ which has derivative $-1/x^2$. We can estimate $1 + 1/11 = 1 + 1/10 - 1/100$.

c)

The curve	2 arc length	arc length/2	2 curvature	curvature/2
$\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$	x			x
$\vec{r}(t) = \langle (1/2)\cos(t), (1/2)\sin(t) \rangle$		x	x	

Problem 3) (10 points)

Quantity	Check if it depends on parametrization of $\vec{r}$	Is a vector
Curvature of $\vec{r}(t)$		
Arc length of $\vec{r}(t)$ from 0 to 1		
Acceleration of $\vec{r}(t)$		
Jerk of $\vec{r}(t)$		
Speed of $\vec{r}(t)$		
Unit tangent of $\vec{r}(t)$		
Normal of $\vec{r}(t)$		
Binormal of $\vec{r}(t)$		
$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$		
$\vec{r}\,'(t) \times \vec{r}\,''(t)$		

Solution:

Curvature, arclength, T, N, B all do not depend on the parametrization, the others do and replacing $r(t)$ by $r(t^2)$ for example changes the quantities.

Acceleration, Jerk and the T, N, B vectors as well as $\vec{r}\,'(t) \times \vec{r}\,''(t)$ are vectors, the others are scalars.

Problem 4) (10 points)

Oliver rides his bike along streets in the Massachusetts. Since the streets can be quite bumpy, he tries to avoid critical points which are maxima (bumps) or minima (potholes) but aims to drive over saddle points (mountain passes). Assume the street is the graph of the function

$$f(x,y) = \frac{x^4}{4} + \frac{y^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} \ .$$

List all critical points and classify them as local maxima, local minima and saddle points.

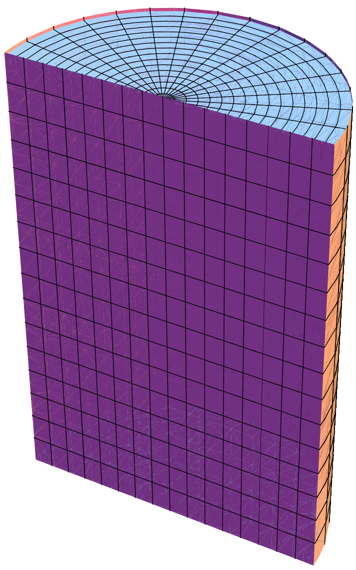
Solution:

$\nabla f(x,y) = x^3 - x, y^3 + y = (0,0)$ so that the critical points are $(-1,0), (0,0), (1,0)$. We have $D = (3x^2-1)(3y^2+1)$ and $f_{xx} = -1+3x^2$.

Point	D	f_{xx}	type
$(-1,0)$	$D=2$	2	min
$(0,0)$	$D=-1$	-1	saddle
$(1,0)$	$D=2$	2	min

Problem 5) (10 points)

Its amazing how many new energy drinks pop up every year and each tries to be original. Beside the concept like "water from Norway" or "vitamin drinks" or "taurin", "coffeine bomb", it needs also a catchy name "pink bull", "muscle milk" etc, and a fancy shape. Summer school is always a good time to come up with business ideas. We launch in this midterm an energy drink called "sweet tooth" which contains a insanely strong "caffeine, taurin, ginseng, isotonic vitamin" combination. But what really makes it stand out from the crowd is its shape: its half a cylinder of radius x and height y. Its only for tough guys or girls although: one first has to bite off a corner of the can with the teeth, in order to drink it ...



For a fixed volume $y\pi x^2/2 = \pi/2$, we want to minimize the material cost $\pi xy + \pi x^2 + 2xy$. In other words, we want to minimize the function

$$f(x,y) = (2+\pi)xy + \pi x^2$$

under the constraint

$$g(x,y) = yx^2 = 1 \ .$$

Solve it with Lagrange method!

Solution:
The Lagrange equations are

$$\begin{aligned}(2+\pi)y + 2\pi x &= 2xy\lambda &(1)\\ (2+\pi)x &= \lambda x^2 &(2)\\ yx^2 &= 1 &(3)\end{aligned}$$

From the first two equations (divide the first by the second using that $x = 0$ is not compatible with the third equation, gives $y = 2\pi x/(2+\pi)$. Plugging this into the third equation gives the relation $x = [(2+\pi)/(2\pi)]^{1/3}$ and $y = [2\pi/(2+\pi)]^{2/3}$.

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle t^3 + 3, 2 + \cos(t^3), 3 + \sin(t^3)\rangle$$

from $t = 0$ to $t = \pi$.

Solution:
The velocity is $\vec{r}\,'(t) = 3t^2\langle 1, -\sin(t), \cos(t)\rangle$. The speed is $3t^2\sqrt{2}$. The integral

$$L = \int_0^\pi 3t^2\sqrt{2} = \pi^3\sqrt{3}.$$

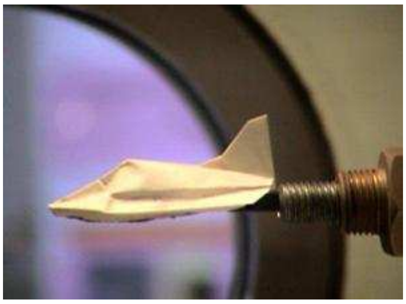
Problem 7) (10 points)

Estimate $f(x,y) = \sqrt{x^3 + y^3}$ for $x = 1.001, y = 2.01$ by linear approximation.

Solution:
$\nabla f(x,y) = \langle \frac{3x^2}{2\sqrt{x^3+y^3}}, \frac{3x^y}{2\sqrt{x^3+y^3}}\rangle$. At the point $(1,2)$ where $f(1,2) = 3$, we have and $\nabla f(1,2) = \langle 1/2, 2\rangle$. We have $f(1.001, 2.01) \sim 3 + (1/2)\cdot 0.001 + 2\cdot 0.01 = 3.010005$.

Problem 8) (10 points)

NASA will soon throw a 20 cm small origami paper airplane from the space station to the ground. The international space station is at height 350 km. Assume for a moment that there would be no atmosphere and that the airplane would just fall down to the earth. What would its velocity be at the ground if the astronouts decide to throw the paper plane away with at first. We set up the problem as follows: our initial position is $\vec{r}(0) = \langle 0, 0, 350 \rangle$ the initial velocity is $\vec{r}\,'(0) = \langle 0, 0, 1 \rangle$ and the acceleration is constant $\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$.



Solution:
Integrating $\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$ gives $\vec{r}\,'(t)\langle 0, 0, -10t \rangle + \langle 0, 0, 1 \rangle$. Integrating again gives $\vec{r}(t) = \langle 0, 0, -5t^2 \rangle + \langle 0, 0, t \rangle + \langle 0, 0, 350 \rangle = \langle 0, 0, -5t^2 + t + 350$. We have $\vec{r}(t) = 0$ for $t = (1 + \sqrt{7001})/10$. At that time, the velocity is $\langle 0, 0, \sqrt{7001} \rangle$.

Problem 9) (10 points)

a) (5 points) Find the tangent plane to the surface $xy + x^2y + zx^2 = 20$ at the point $(2, 2, 2)$.
b) (5 points) Find the tangent line to the curve $x^2y^2 - xy + 3x = 3$ at the point $(1, 1)$.

Solution:
a) The gradient is $\langle y+2xy+2xz, x+x^2, x^2 \rangle$. At the point $(2, 2, 2)$ the gradient is $\langle 18, 6, 4$. The plane has the form $18x + 6y + 4z = d$. By pluggin in the point $(2, 2, 2)$ we get $18x + 6y + 4z = 56$.
b) The gradient is $\langle 2xy^2 - y + 3, 2x^2y \rangle$. At the point $(1, 1)$ it is $4, 1$. The line is $4x + y = d$. By plugging in the point $(1, 1)$ we get $4x + y = 5$.