7/24/2008 SECOND HOURLY **Maths 21a, O. Knill, Summer 2008**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Provide details to all computations except for problems 1-3.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The acceleration vector $\vec{r}\,''(t)$ and the jerk vector $\vec{r}\,'''(t)$ are always orthogonal to each other.

2) T F The function $f(x,y) = x^4 + y^4$ satisfies the partial differential equation $f_{xx} - f_{yy} = 0$.

3) T F If we know the speed of a curve at all times as well as the initial position $\vec{r}(0)$, then we can determine the position $\vec{r}(t)$.

4) T F There is a function $f(x,y)$ of two variables which has no critical points at all.

5) T F If $(2,3)$ is a local maximum for the function f with discriminant $D > 0$, then $f_{xx}(2,3) < 0$.

6) T F If f satisfies the partial differential equation $f_x + f_y = 0$ everywhere, then the discriminant D is zero at every critical point.

7) T F If $f(x,y)$ has a saddle point, then $-f(x,y)$ has a saddle point.

8) T F The value of the function $f(x,y) = xy$ at $(x,y) = (3.1, 5.2)$ can be estimated as $15 + 0.5 + 0.6$.

9) T F The curvature of the curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle\rangle = (10\sin(t), 10\cos(t), 20)$ is $1/20$.

10) T F The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

11) T F If the curvature of a curve is nonzero at some point, then the curve is not contained in a line.

12) T F The gradient of f at a point (x_0, y_0) is parallel to the level curve of f which contains (x_0, y_0).

13) T F If $(1,1)$ is a critical point of $f(x,y)$ and is not a critical point of $g(x,y)$ then $(1,1)$ can not be a critical point of f under the constraint g.

14) T F If an airship always moves in the direction opposite to the gradient of the pressure, then the pressure momentarily decreases.

15) T F At points, where the velocity of a curve is zero, the curvature is zero.

16) T F If D is the discriminant at a critical point and $D < 0$ then $f_{xy} \leq 0$.

17) T F The function $f(x,y) = x^4 y^4$ satisfies the partial differential equation $f_{yxyxyxyxy} = 0$.

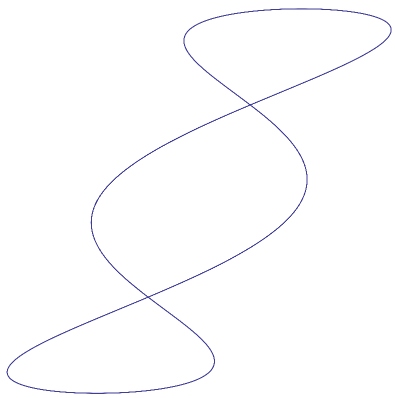
18) T F If $(0,0)$ is a critical point of $f(x,y)$ and $f_{xx}(0,0) > 0$ and $f_{yy}(0,0) > 0$, then $(0,0)$ can not be a local minimum.

19) T F The tangent plane to the graph $f(x,y) = z$ at a point (x_0, y_0) is parallel to $\langle 0, 0, 1 \rangle$ if (x_0, y_0) is a critical point.

20) T F If the curvature of a curve $\vec{r}(t)$ is constant 10 everywhere, then the curve is a circle.

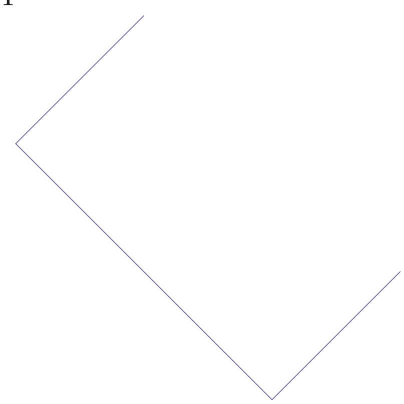
Problem 2) (10 points)

a) (4 points) Match the parameterizations with the curves. No justifications are needed.

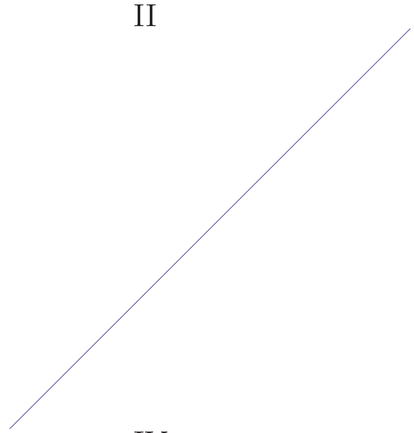


I

II



III



IV

Enter I,II,III,IV here	Parameterization
	$\vec{r}(t) = \langle 3t^5, t^5 + 4 \rangle$
	$\vec{r}(t) = \langle \lvert t-1 \rvert, \lvert t+1 \rvert \rangle$
	$\vec{r}(t) = \langle \sin(t)/t, t \rangle$
	$\vec{r}(t) = \langle \sin(t) + \cos(3t), \sin(t) \rangle$

b) (3 points) The linear estimates below obtained as in this example: for $f(x) = \sin(x)$ estimate $\sin(\pi + 1/10)$ by $\sin(\pi) + \cos(\pi)(1/10) = -1/10$. No justifications are needed.

expression	Enter A-C here
$1/2^{1/10}$	
$\sqrt{1 + 1/5}$	
$1 + 1/11$	

A	$1 + \frac{1}{10}$
B	$1 + 1/10 - \frac{1}{100}$
C	$1 - \frac{1}{10}$

c) (3 points) Match the arc length and curvature We know that the unit circle $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ parametrized from 0 to 2π has arc length 2π and curvature constant equal to 1. What happens if the radius changes?

The curve	2 arc length	arc length/2	2 curvature	curvature/2
$\vec{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle$				
$\vec{r}(t) = \langle (1/2)\cos(t), (1/2)\sin(t) \rangle$				

Problem 3) (10 points)

Quantity	Check if it depends on parametrization of $\vec{r}$	Is a vector
Curvature of $\vec{r}(t)$		
Arc length of $\vec{r}(t)$ from 0 to 1		
Acceleration of $\vec{r}(t)$		
Jerk of $\vec{r}(t)$		
Speed of $\vec{r}(t)$		
Unit tangent of $\vec{r}(t)$		
Normal of $\vec{r}(t)$		
Binormal of $\vec{r}(t)$		
$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$		
$\vec{r}\,'(t) \times \vec{r}\,''(t)$		

Problem 4) (10 points)

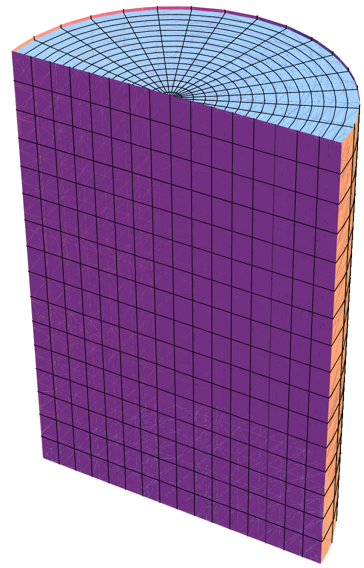
Oliver rides his bike along streets in the Massachusetts. Since the streets can be quite bumpy, he tries to avoid critical points which are maxima (bumps) or minima (potholes) but aims to drive over saddle points (mountain passes). Assume the street is the graph of the function

$$f(x,y) = \frac{x^4}{4} + \frac{y^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} .$$

List all critical points and classify them as local maxima, local minima and saddle points.

Problem 5) (10 points)

Its amazing how many new energy drinks pop up every year and each tries to be original. Beside the concept like "water from Norway" or "vitamin drinks" or "taurin", "coffeine bomb", it needs also a catchy name "pink bull", "muscle milk" etc, and a fancy shape. Summer school is always a good time to come up with business ideas. We launch in this midterm an energy drink called "sweet tooth" which contains a insanely strong "caffeine, taurin, ginseng, isotonic vitamin" combination. But what really makes it stand out from the crowd is its shape: its half a cylinder of radius x and height y. Its only for tough guys or girls although: one first has to bite off a corner of the can with the teeth, in order to drink it ...



For a fixed volume $y\pi x^2/2 = \pi/2$, we want to minimize the material cost $\pi xy + \pi x^2 + 2xy$. In other words, we want to minimize the function

$$f(x,y) = (2+\pi)xy + \pi x^2$$

under the constraint

$$g(x,y) = yx^2 = 1 \ .$$

Solve it with Lagrange method!

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle t^3+3, 2+\cos(t^3), 3+\sin(t^3)\rangle$$

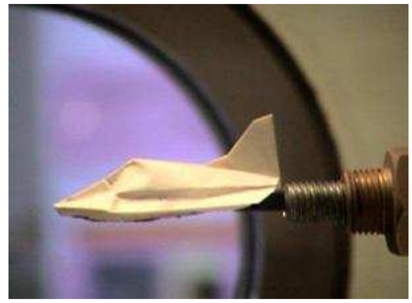
from $t=0$ to $t=\pi$.

Problem 7) (10 points)

Estimate $f(x,y) = \sqrt{x^3+y^3}$ for $x = 1.001, y = 2.01$ by linear approximation.

Problem 8) (10 points)

NASA will soon throw a 20 cm small origami paper airplane from the space station to the ground. The international space station is at height 350 km. Assume for a moment that there would be no atmosphere and that the airplane would just fall down to the earth. What would its velocity be at the ground if the astronouts decide to throw the paper plane away with at first. We set up the problem as follows: our initial position is $\vec{r}(0) = \langle 0, 0, 350 \rangle$ the initial velocity is $\vec{r}\,'(0) = \langle 0, 0, 1 \rangle$ and the acceleration is constant $\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$.



Problem 9) (10 points)

a) (5 points) Find the tangent plane to the surface $xy + x^2y + zx^2 = 20$ at the point $(2, 2, 2)$.
b) (5 points) Find the tangent line to the curve $x^2y^2 - xy + 3x = 3$ at the point $(1, 1)$.