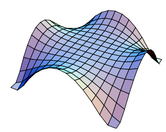
PARTIAL DERIVATIVES **Maths21a, O. Knill**

PARTIAL DERIVATIVE. If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x}f(x,y,z)$ is defined as the derivative of the function $g(x) = f(x,y,z)$, where y is fixed. The partial derivative with respect to y is defined similarly.

NOTATION. One also writes $f_x(x,y) = \frac{\partial}{\partial x}f(x,y)$ etc. For iterated derivatives the notation is similar: for example $f_{xy} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}f$.

REMARK. The notation for partial derivatives $\partial_x f, \partial_y f$ were introduced by Jacobi. Lagrange had used the term "partial differences". Partial derivatives measure the rate of change of the function in the x or y directions. For functions of more variables, the partial derivatives are defined in a similar way.

EXAMPLE. $f(x,y) = x^4 - 6x^2y^2 + y^4$. We have $f_x(x,y) = 4x^3 - 12xy^2, f_{xx} = 12x^2 - 12y^2, f_y(x,y) = -12x^2y + 4y^3, f_{yy} = -12x^2 + 12y^2$. We see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation**.



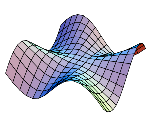
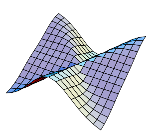
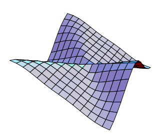
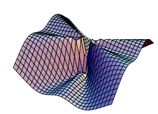
CLAIROT THEOREM. If f_{xy} and f_{yx} are both continuous, then $\boxed{f_{xy} = f_{yx}}$. Proof. Compare the two sides:

$dx f_x(x,y) \sim f(x+dx,y) - f(x,y)$ $dydx f_{xy}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y+dy) - (f(x+dx,y) - f(x,y))$	$dy f_y(x,y) \sim f(x,y+dy) - f(x,y)$. $dxdy f_{yx}(x,y) \sim f(x+dx,y+dy) - f(x+dx,y) - (f(x,y+dy) - f(x,y))$

CONTINUITY IS NECESSARY. Example: $f(x,y) = (x^3y - xy^3)/(x^2+y^2)$ contradicts Clairot:

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$,	$f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

$\boxed{f(x,y)}$ $\boxed{f_x(x,y)}$ $\boxed{f_y(x,y)}$ $\boxed{f_{xy}(x,y)}$

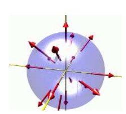





GRADIENT. If $f(x,y,z)$ is a function of three variables, then

$$\nabla f(x,y,z) = \left(\frac{\partial}{\partial x}f(x,y,z), \frac{\partial}{\partial y}f(x,y,z), \frac{\partial}{\partial z}f(x,y,z)\right)$$

is called the **gradient** of f. The symbol ∇ is called **Nabla**. It is named after an Egyptian harp, the Hebrew word "nevel"=harp seems to have the same aramaic origin). We will talk about the gradient in detail next week.

NORMAL. As we will see later, the gradient $\nabla f(x,y)$ is orthogonal to the level curve $f(x,y) = c$ and the gradient $\nabla f(x,y,z)$ is normal to the level surface $f(x,y,z)$. For example, the gradient of $f(x,y,z) = x^2 + y^2 - z^2$ at a point (x,y,z) is $(2x, 2y, -2z)$.



WHY ARE PARTIAL DERIVATIVES IMPORTANT?

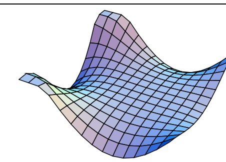
- Geometry. For example, the gradient $\nabla f(x,y,z)$ is a vector normal to a surface at the point (x,y,z). Tangent spaces.
- Approximations, linarizations.
- Partial differential equations: laws describing nature.
- Optimization problems, as we will see later.
- Solution to some integration problems using generalizations of fundamental theorem of calculus.
- In general helpful to understand and analyze functions of several variables.

PARTIAL DIFFERENTIAL EQUATIONS. An equation which involves partial derivatives of an unknown function is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

1) $f_{xx}(x,y) = f_{yy}(x,y)$ is an example of a partial differential equation 1) $f_x(x,y) = f_{xx}(x,y)$ is an example of an ordinary differential equation. The variable y can be considered as a parameter.

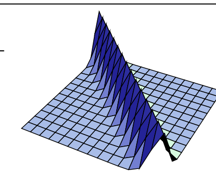
LAPLACE EQUATION

$\boxed{f_{xx} + f_{yy} = 0}$.
$f(x,t) = x^2 - y^2$.



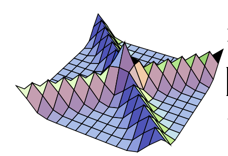
ADVECTION EQUATION

$\boxed{f_t = f_x}$.
$f(x,t) = g(x-t)$.



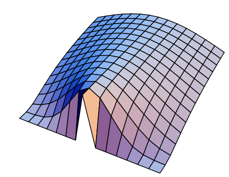
WAVE EQUATION

$\boxed{f_{tt} = f_{xx}}$. $f(t,x) = \sin(x-t) + \sin(x+t)$



HEAT EQUATION

$\boxed{f_t = f_{xx}}$
$f(t,x) = \frac{1}{\sqrt{t}}e^{-x^2/(4t)}$



"A great deal of my work is just **playing with equations** and seeing what they give. I don't suppose that applies so much to other physicists; I think it's a peculiarity of myself that I like to play about with equations, just **looking for beautiful mathematical relations** which maybe don't have any physical meaning at all. Sometimes they do." - Paul A. M. Dirac.

Dirac discovered a PDE describing the electron which is consistent both with quantum theory and special relativity. This won him the Nobel Prize in 1933. Dirac's equation could have two solutions, one for an electron with positive energy, and one for an electron with negative energy. Dirac interpretated the later as an **antiparticle**: the existence of antiparticles was later confirmed.