

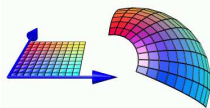
PARAMETRIC SURFACES

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PARAMETRIC SURFACES. The image of a map

$$\vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

defines a **surface**. \vec{r} is called the **parametrisation** of the surface. The surface is defined by three functions $x(u, v), y(u, v), z(u, v)$ each is a function of two variables. If we fix one of the variables, say $v = v_0$, then $u \mapsto \vec{r}(u, v_0)$ is a curve on the surface. Similarly, if we fix $u = u_0$, then $v \mapsto \vec{r}(u_0, v)$ is a curve on the surface.

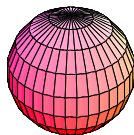
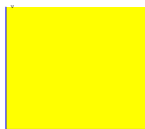


TWO WAYS TO REPRESENT A SURFACE.

- I) Solutions of an equation $g(x, y, z) = 0$.
- II) Parameterizations as image of $\vec{r}(u, v)$.

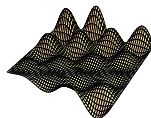
EXAMPLE GRAPH. The graph of a function $f(x, y)$ can be parametrized as $(u, v) \mapsto \vec{r}(u, v) = (u, v, f(u, v))$ and can also be written as a level surface: $\{g(x, y, z) = z - f(x, y) = 0\}$.

EXAMPLE SPHERE. The sphere is the set of (x, y, z) for which $g(x, y, z) = x^2 + y^2 + z^2 = 1$. The sphere is also the image of the parametrisation $\vec{r}(\theta, \phi) = (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi))$. (The upper half sphere is the graph of the function $f(x, y) = \sqrt{1 - x^2 - y^2}$. We could also write the sphere as a graph of $\rho(\theta, \phi) = r$ in spherical coordinates).



NOTE. There are surfaces, which can not be represented as an image of a single parameterization \vec{r} and where different patches are needed. To do so, Mathematicians introduced the notion of a "manifold".

EXAMPLES. of parameterized surface $(u, v) \mapsto \vec{r}(u, v)$



Graphs

$$\begin{bmatrix} u \\ v \\ f(u, v) \end{bmatrix}$$

Sphere

$$\begin{bmatrix} \cos(u) \sin(v) \\ \sin(u) \sin(v) \\ \cos(v) \end{bmatrix}$$

Plane through points P, Q, R

$$\begin{matrix} P+ \\ u(Q - P) + \\ v(R - P) \end{matrix}$$

Dini surface

$$\begin{bmatrix} \cos(u) \sin(v) \\ \sin(u) \sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix}$$

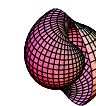
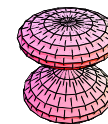
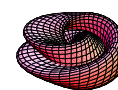
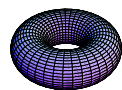
SURFACE OF REVOLUTION.

When spinning a graph $y = f(x)$ around the x-axis, we obtain a **surface of revolution**. Keeping $u = x$ as one of the parameters and v as the angle of rotation and $g(u)$ as the radius, we get $x(u, v) = u, y(u, v) = g(u) \cos(v), z(u, v) = g(u) \sin(v)$. Therefore, $\vec{r}(u, v) = (u, g(u) \cos(v), g(u) \sin(v))$. For example, for $g(x) = x$, we obtain a cone, for $g(x) = x^2$, we obtain a paraboloid, the surface with $g(x) = 1 + \sin(x)$ is plotted.



GRID CURVES. If we keep u constant, then $v \mapsto \vec{r}(u, v)$ is a curve on the surface. Similarly, if v is constant, then $u \mapsto \vec{r}(u, v)$ is a curve on the surface. These curves are called **grid curves**. If you plot a surface with a computer, the pictures usually show these grid curves.

MORE EXAMPLES. Parameterized surfaces $(u, v) \mapsto \vec{r}(u, v)$



Torus

Klein bottle

Eight surface

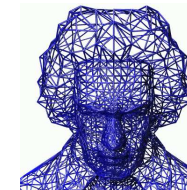
Snail surface

This is a homework

$$\begin{bmatrix} (\frac{3}{2} + \cos(u) \sin(2v) - \sin(u) \sin(4v)) \cos(2u) \\ (\frac{3}{2} + \cos(u) \sin(2v) - \sin(u) \sin(4v)) \sin(2u) \\ \frac{3}{2} \sin(u) \sin(2v) + \cos(u) \sin(4v) \end{bmatrix} \begin{bmatrix} \cos(u) \sin(v) \\ \sin(u) \sin(v) \\ \cos(v) + \log(\tan(\frac{v}{2})) + \frac{u}{5} \end{bmatrix} \begin{bmatrix} \cos(u) \sin(2v) \\ \sin(u) \sin(2v) \\ \sin(v) \end{bmatrix}$$

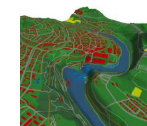
WHERE DO SURFACES OCCUR?

Computer graphics. i.e. modeling of faces, terrains, cars, spaceships etc. The map \vec{r} is called a **u-v map**. Such maps are used to apply textures to three-dimensional models. There are software utilities, which allows an artist to draw directly onto the R domain. The software then applies the map automatically onto the surface. You can paint like this on surfaces in space.



Level surfaces. A level surface of a function of three variables is the set $g(x, y, z) = c$, where c is a constant. Examples are **isotherms**, surfaces of constant temperature in the ocean, **isobars**, surfaces of constant pressure in the atmosphere.

Graphs. For example the height function $h(x, y)$ which gives the height $h(x, y)$ at a point (x, y) .



Geometric Intuition. Intuition for higher dimensional surfaces or "manifolds" is often obtained from 2-dimensional surfaces in three dimensional space. Higher dimensional surfaces appear everywhere. Our universe for example is modeled as a space time manifold in general relativity. The planetary motion in our the solar system is modeled by a motion on a $9 \cdot 6 - 10 = 44$ dimensional surface. Even so, we can not draw these objects, calculus allows to work with such objects as in two or three dimensions.

D-branes. In super-string theory physicists consider surfaces called **Dp-branes**. The letter D stands for "Dirichlet". $D1$ -branes are called **D-strings**. $D2$ -branes are two dimensional surfaces.

Art. Last but not least, there is the artistic or esthetic aspect. Examples are paintings by Fomenko, a Russian topologist. An other example is the surface "topological" at the entrance of the science center visible on the photo. You pass this surface every day.

