REVIEW BEYOND VECTOR FIELDS — Maths21a,O.Knill

INTEGRATION.

- **Line integral:** $\int_C F \cdot ds = \int_a^b F(r(t)) \cdot r'(t)\, dt$
- **Surface integral** $\int\int_S f dS = \int_a^b \int_c^d f(r(u,v)) |r_u(u,v) \times r_v(u,v)|\, dudv$
- **Flux integral:** $\int\int_S F \cdot dS = \int_a^b \int_c^d F(r(u,v)) \cdot r_u(u,v) \times r_v(u,v)\, dudv$
- **Double integral:** $\int\int_R f\, dA = \int_a^b \int_c^d f(x,y)\, dxdy$.
- **Triple integral:** $\int\int\int_R f\, dV = \int_a^b \int_c^d \int_o^p f(x,y,z)\, dxdydz$.

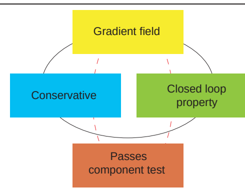
Area $\int\int_R 1 dA = \int\int_R 1\, dxdy$
Length $\int_a^b |r'(t)|\, dt$
Surface area $\int\int 1 dS = \int\int |r_u \times r_v|\, dudv$
Volume $\int\int\int_B 1\, dV = \int\int\int_B 1\, dxdydz$

DIFFERENTIATION.

- **Derivative:** $f'(t) = \dot{f}(t) = d/dt f(t)$.
- **Partial derivative:** $f_x(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)$.
- **Gradient:** $\text{grad}(f) = (f_x, f_y, f_z)$
- **Curl in 2D:** $\text{curl}(F) = \text{curl}((M,N)) = N_x - M_y$
- **Curl in 3D:** $\text{curl}(F) = \text{curl}(M,N,P) = (P_y - N_z, M_z - P_x, N_x - M_y)$
- **Div:** $\text{div}(F) = \text{div}(M,N,P) = M_x + N_y + P_z$.

IDENTITIES.
$\text{div}(\text{curl}(F)) = 0$
$\text{curl}(\text{grad}(f)) = (0,0,0)$
$\text{div}(\text{grad}(f)) = \Delta f$.

CONSERVATIVE FIELDS.
1) Gradient: $F = \text{grad}(f)$.
2) Closed curve property: $\int_C F \cdot dr = 0$ for any closed curve.
3) Conservative: C_i paths from A to B, then $\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$.
4) Mixed derivative property: $\text{curl}(F) = 0$ in simply connected regions.



TOPOLOGY. 1) **Interior** of region D: points which have a neighborhood contained in D.
2) **Boundary** of a curve: endpoints of curve, **boundary** of 2D region D: curves which bound the region, **boundary** of a solid D: surfaces which bound the solid.
3) **Simply connected** region D: a closed curve in D can be deformed within the interior of D to a point.
4) **Closed curve** Curve without boundary.
5) **Closed surface** surface without boundary.

LINE INTEGRAL THEOREM. If $C: r(t) = (x(t), y(t), z(t)), t \in [a,b]$ is a curve and f is a function either in 3D or the plane. Then

$$\int_C \nabla f \cdot ds = f(r(b)) - f(r(a))$$

CONSEQUENCES.
1) For closed curves the line integral $\int_C \nabla f \cdot ds$ is zero.
2) Gradient fields are **conservative**: if $F = \nabla f$, then the line integral between two points P and Q is path independent.
3) The theorem holds in any dimension. In one dimension, it reduces to the **fundamental theorem of calculus** $\int_a^b f'(x)\, dx = f(b) - f(a)$
4) The theorem justifies the name **conservative** for gradient vector fields.
5) The term "potential" was coined by George Green (1783-1841).

PROBLEM. Let $f(x,y,z) = x^2 + y^4 + z$. Find the line integral of the vector field $F(x,y,z) = \nabla f(x,y,z)$ along the path $r(t) = (\cos(5t), \sin(2t), t^2)$ from $t = 0$ to $t = 2\pi$.

SOLUTION. $r(0) = (1,0,0)$ and $r(2\pi) = (1,0,4\pi^2)$ and $f(r(0)) = 1$ and $f(r(2\pi)) = 1 + 4\pi^2$. FTLI gives $\int_C \nabla f\, ds = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

GREEN'S THEOREM. If R is a region with boundary C and $F = (M,N)$ is a vector field, then

$$\int\int_R \text{curl}(F)\, dA = \int_C F \cdot ds$$

REMARKS.
1) Useful to swap 2D integrals to 1D integrals or the other way round.
2) The curve is oriented in such a way that the region is to your left.
3) The region has to has piecewise smooth boundaries (i.e. it should not look like the Mandelbrot set).
4) If $C: t \mapsto r(t) = (x(t), y(t))$, the line integral is $\int_a^b (M(x(t), y(t)), N(x(t), y(t)) \cdot (x'(t), y'(t))\, dt$.
5) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862).
6) If $\text{curl}(F) = 0$ in a simply connected region, then the line integral along a closed curve is zero. If two curves connect two points then the line integral along those curves agrees.
7) Taking $F(x,y) = (-y, 0)$ or $F(x,y) = (0,x)$ gives **area formulas**.

PROBLEM. Find the line integral of the vector field $F(x,y) = (x^4 + \sin(x) + y, x + y^3)$ along the path $r(t) = (\cos(t), 5\sin(t) + \log(1 + \sin(t)))$, where t runs from $t = 0$ to $t = \pi$.

SOLUTION. $\text{curl}(F) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $r(t) = (-t, 0), t = [-1,1]$, which has velocity $r'(t) = (-1, 0)$. The line integral is $\int_{-1}^1 (t^4 - \sin(t), -t) \cdot (-1, 0)\, dt = -t^5/5|_{-1}^1 = -2/5$.

REMARK. We could also find a potential $f(x,y) = x^5/5 - \cos(x) + xy + y^5/4$. It has the property that $\text{grad}(f) = F$. Again, we get $f(0,-1) - f(0,1) = -1/5 - 1/5 = -2/5$.

STOKES THEOREM. If S is a surface in space with boundary C and F is a vector field, then

$$\int\int_S \text{curl}(F) \cdot dS = \int_C F \cdot ds$$

REMARKS.
1) Stokes theorem implies Greens theorem if F is z independent and S is contained in the z-plane.
2) The orientation of C is such that if you walk along C and have your head in the direction, where the normal vector $r_u \times r_v$ of S, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903). 4) The flux of the curl of a vector field does not depend on the surface S, only on the boundary of S. This is analogue to the fact that the line integral of a gradient field only depends on the end points of the curve.
5) The flux of the curl through a closed surface like the sphere is zero: the boundary of such a surface is empty.

PROBLEM. Compute the line integral of $F(x,y,z) = (x^3 + xy, y, z)$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.

SOLUTION. The path C bounds a surface $S: r(u,v) = (u,v,0)$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\text{curl}(F)(x,y,z) = (0,0,-x)$ through S. The normal vector of S is $r_u \times r_v = (1,0,0) \times (0,1,0) = (0,0,1)$ so that $\int\int_S \text{curl}(F)\, dS = \int_0^2 \int_0^1 (0,0,-u) \cdot (0,0,1)\, dudv = \int_0^2 \int_0^1 -u\, dudv = -2$.

GAUSS THEOREM. If S is the boundary of a region B in space with boundary S and F is a vector field, then

$$\int\int\int_B \text{div}(F)\, dV = \int\int_S F \cdot dS$$

REMARKS.
1) Gauss theorem is also called **divergence theorem**.
2) Gauss theorem can be helpful to determine the flux of vector fields through surfaces.
3) Gauss theorem was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields F, the flux through a closed surface is zero. Such fields F are also called **incompressible** or **source free**.

PROBLEM. Compute the flux of the vector field $F(x,y,z) = (-x, y, z^2)$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
SOLUTION. By Gauss theorem, the flux is equal to the triple integral of $\text{div}(F) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z\, dxdydz = (3-0)(2-(-1))(4-1) = 27$.