CURVES — Maths21a

PARAMETRIC PLANE CURVES. If $x(t)$, $y(t)$ are functions of one variable, defined on the **parameter interval** $I = [a, b]$, then $\vec{r}(t) = \langle f(t), g(t) \rangle$ is called a **parametrisation of a curve** in the plane. The functions $x(t), y(t)$ are called **coordinate functions**. The image of r is the actual curve.

PARAMETRIC SPACE CURVES. If $x(t), y(t), z(t)$ are functions of one variables, then $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a **space curve**.
Always think of the **parameter** t as **time**. For every fixed t, we have a point $(x(t), y(t), z(t))$ in space. As t varies, we move along the curve.

EXAMPLE 1. If $x(t) = t$, $y(t) = t^2 + 1$, we can write $y(x) = x^2 + 1$ and the curve is a **graph**.

EXAMPLE 2. If $x(t) = \cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows a **circle**.

EXAMPLE 3. If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $\vec{r}(t)$ describes a **spiral**.

EXAMPLE 4. If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in example 3 but we traverse it **faster**. The **parameterization** changed.

EXAMPLE 5. If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve as in example 3 but we traverse it in the **opposite direction**.

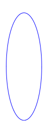
EXAMPLE 6. If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a+t(u-a), b+t(v-b), c+t(w-c) \rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q.

ELIMINATION: Sometimes it is possible to eliminate the parameter t and write the curve using equations (one equation in the plane or two equations in space).

EXAMPLE: (circle) If $x(t) = \cos(t), y(t) = \sin(t)$, then $x(t)^2 + y(t)^2 = 1$.

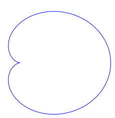
EXAMPLE: (spiral) If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $x = \cos(z), y = \sin(z)$. The spiral is the intersection of two graphs $x = \cos(z)$ and $y = \sin(z)$.

CIRCLE



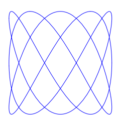
$(\cos(t), 3\sin(t))$

HEART



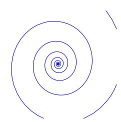
$(1 + \cos(t))(\cos(t), \sin(t))$

LISSAJOUS



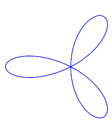
$(\cos(3t), \sin(5t))$

SPIRAL



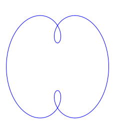
$e^{t/10}(\cos(t), \sin(t))$

TRIFOLIUM



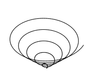
$-\cos(3t)(\cos(t), \sin(t))$

EPICYCLE



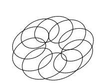
$(\cos(t) + \cos(3t)/2, \sin(t) + \sin(3t)/2)$

SPRING (3D)



$(\cos(t), \sin(t), t)$

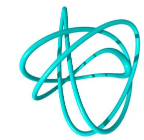
TORAL KNOT (3D)



$(\cos(t) + \cos(9t)/2, \sin(t) + \cos(9t)/2, \sin(9t)/2)$

WHERE DO CURVES APPEAR? Objects like particles, celestial bodies, or quantities change in time. Their motion is described by curves. Examples are the motion of a star moving in a galaxy, or data changing in time like (DJIA(t),NASDAQ(t),SP500(t))

Strings or knots are closed curves in space.
Molecules like RNA or proteins can be modeled as curves.
Computer graphics: surfaces are represented by mesh of curves.
Typography: fonts represented by Bezier curves.
Space time A curve in space-time describes the motion of particles.
Topology Examples: space filling curves, boundaries of surfaces or knots.



DERIVATIVES. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\boxed{\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle}$ is called the **velocity**. Its length $|\vec{r}'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}''(t)$ is called the **acceleration**. The third derivative $\vec{r}'''$ is called the **jerk**.

$\boxed{\text{The velocity vector } \vec{r}(t) \text{ is tangent to the curve at } \vec{r}(t).}$

EXAMPLE. If $\vec{r}(t) = \langle \cos(3t), \sin(2t), 2\sin(t) \rangle$, then $\vec{r}'(t) = \langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$, $\vec{r}''(t) = \langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$ and $\vec{r}'''(t) = \langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$.

THE PARADOX OF ZENO OF ELEA: "When we look at a body at a specific time, the body is fixed. Being fixed at each instant, there is no motion". While one might wonder today about Zeno's way to think, there were philosophers like Kant, Hume or Hegel, who contemplated seriously about Zeno's challenges. Physicists still continue to ponder about the question: "what is time and space?" Today, the derivative or rate of change is defined as a **limit** $(\vec{r}(t + dt) - \vec{r}(t))/dt$, where dt approaches zero. If the limit exists, velocity is defined.

EXAMPLES OF VELOCITIES.

Person walking:	1.5 m/s
Signals in nerves:	40 m/s
Plane:	70-900 m/s
Sound in air:	Mach1=340 m/s
Speed of bullet:	1200-1500 m/s
Earth around the sun:	30'000 m/s
Sun around galaxy center:	200'000 m/s
Light in vacuum:	300'000'000 m/s

EXAMPLES OF ACCELERATIONS.

Train:	0.1-0.3 m/s^2
Car:	3-8 m/s^2
Space shuttle:	$\leq 3G = 30 m/s^2$
Combat plane (F16) (blackout):	9G=90 m/s^2
Ejection from F16:	14G=140 m/s^2.
Free fall:	1G = 9.81 m/s^2
Electron in vacuum tube:	10^{15} m/s^2

DIFFERENTIATION RULES.
The rules in one dimensions $(f + g)' = f' + g'$ $(cf)' = cf'$, $(fg)' = f'g + fg'$ (Leibniz), $(f(g))' = f'(g)g'$ (chain rule) generalize for vector-valued functions: $(\vec{v} + \vec{w})' = \vec{v}' + \vec{w}'$, $(c\vec{v})' = c\vec{v}'$, $(\vec{v} \cdot \vec{w})' = \vec{v}' \cdot \vec{w} + \vec{v} \cdot \vec{w}'$ $(\vec{v} \times \vec{w})' = \vec{v}' \times \vec{w} + \vec{v} \times \vec{w}'$ (Leibniz), $(\vec{v}(f(t)))' = \vec{v}'(f(t))f'(t)$ (chain rule). The Leibniz rule for the triple dot product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is $d/dt[\vec{u}, \vec{v}, \vec{w}] = [\vec{u}', \vec{v}, \vec{w}] + [\vec{u}, \vec{v}', \vec{w}] + [\vec{u}, \vec{v}, \vec{w}']$ (see homework).

INTEGRATION. If $\vec{r}'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}''(t)$ as well as initial velocity and position $\vec{r}'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

EXAMPLE. Shooting a ball. If $\vec{r}''(t) = \langle 0, 0, -10 \rangle$, $\vec{r}'(0) = \langle 0, 1000, 2 \rangle$, $\vec{r}(0) = \langle 0, 0, h \rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2 \rangle$.