7/31/2008 SPHERICAL INTEGRATION /VECTOR FIELDS O.Knill,Maths21a

This is part 2 (of 3) of the weekly homework. It is due August 5'th in class.

SUMMARY.

- **Spherical coordinates** $\int\int\int_R g(\rho,\theta,\phi)\ \rho^2 \sin(\phi)\ d\rho d\theta d\phi$.
- $F(x,y) = (P(x,y,z), Q(x,y,z))$ **vector field in the plane**.
- $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ **vector field in space**.
- $F(x,y,z) = \nabla f(x,y,z)$ **gradient field**.
- $F(x,y)$ vector field. **Flow line**: curve $\vec{r}(t)$ such that $\vec{r}\,'(t)$ is parallel to $F(r(t))$. Example: $\vec{r}(t) = (a\cos(t), a\sin(t))$ are flow lines to $F(x,y) = (-y,x)$.

Homework Problems

1) (4 points) Integrate the function $f(x,y,z) = e^{(x^2+y^2+z^2)^{3/2}}$ over the solid which lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$, which is in the first octant and which is above the cone $x^2+y^2 = z^2$.

Solution:
Because we are in the first octant, $\theta \in [0,\pi/2]$ The ϕ angle is between 0 and $\pi/4$, the radius varies between 1 and 2.
$\int_1^2 \int_0^{\pi/2} \int_0^{\pi/4} e^{\rho^3}\ \rho^2 \sin(\phi)\ d\phi d\theta d\rho = (\sqrt{2}/2)(\pi/2)e^{\rho^3}/3|_1^2 = \boxed{(\sqrt{2}\pi/4)(e^8-e)/3}$

2) (4 points) In electrostatics, the vector field $F(x,y) = (x/r^3, y/r^3)$ appears, where $r = \sqrt{x^2+y^2}$ is the distance to the charge. Is this vector field a gradient field? If yes, give a function $f(x,y)$ such that $F = \nabla f$. If not, give a reason why it is not a gradient field.

3) a) (2 points) Draw the gradient vector field of $f(x,y) = \sin(x+y)$.

b) (2 points) Draw the gradient vector field of $f(x,y) = (x-1)^2 + (y-2)^2$.

Hint: In both cases, draw first a contour map of f and use a property of gradients to draw the vector field $F(x,y) = \nabla f$.

Solution:
a) The level curves of $f(x,y)$ are the same then the level curves of the function $g(x,y) = x+y$ because $\sin(x+y) = c$ means $x+y = \arcsin(c) = C$. The level curves are therefore straight lines.
$\nabla f(x,y) = (\cos(x+y), \cos(x+y))$ is a vector field which is perpendicular to the level curves. It vanishes at places, where $\sin(x+y) = 1$ or $\sin(x+y) = -1$.
b) The contour map consists of circles centered at $(1,2)$. The vector field is perpendicular to those circles.

4) (2 points)
a) Is the vector field $F(x,y) = (P(x,y), Q(x,y)) = (xy, x^2)$ a gradient field?

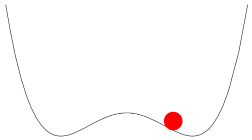
b) Is the vector field $F(x,y) = (P(x,y), Q(x,y)) = (\sin(x)+y, \cos(y)+x)$ a gradient field?

In both cases, give the potential if it exists and if there is no gradient, give a reason, why it is not a gradient field.

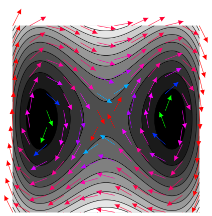
Solution:
a) No, because $Q_x = 2x$ and $P_y = x$, this can not be a gradient field $(P,Q) = (f_x, f_y)$.
b) Yes, the function is $f(x,y) = -\cos(x) + \sin(y) + xy$.

5) (4 points) A ball rolls on the graph of the function $f(x) = x^4 - x^2$. Its position at time t is $x(t)$. The ball feels the acceleration $x''(t) = -f'(x)$. (The sign is chosen as given because for a positive slope $f'(x) > 0$ the ball feels a negative acceleration and for a negative slope $f'(x) < 0$, the ball is accelerated.) If the motion of the ball is described in coordinates $(x(t), y(t))$, where $y(t) = x'(t)$ is the velocity of the ball, the corresponding vector field is $F(x,y) = (y, -f'(x))$. Because $r'(t) = (x'(t), y'(t)) = (y(t), -f'(x(t)))$, the flow lines $(x(t), y(t))$ describe the position and velocity of the ball at time t. Draw the vector field $F(x,y)$, draw a few typical flow lines in the plane and match these curves with the corresponding motion of the ball.

Hint. It helps to look at the places, where the vector field is zero. These are called "equilibrium points" and correspond to situations, where the ball does not move.



Solution:
The vector field looks similar as with the pendulum (see handout). The point $(1/\sqrt{2}, 0)$ in the phase space is a point, where the vector field is zero. Around this, points, there are flow lines which are circular. They correspond to the ball oscillating in the right well. A similar picture is around $(-1/\sqrt{2}, 0)$, which corresponds to the case, when the ball bounces around in the left well, not having enough energy to leave. There are big circular flow lines, which correspond to the situation when the ball bounces from the very left to the right and back, having more energy than necessary to take the "bump" in the middle. There is one special flow line which looks like a figure 8. It corresponds to the situation, when the ball has just enough energy to reach the top of the hill in the middle. Whereever you release a ball with that energy, it will end up in the point $(0, 0)$. To the orientation of the arrows. On the upper half plane, the velocity y is positive. This means that the ball moves to the right in that case. Therefore, the arrows point to the right on the upper half plane. On the lower half plane, where the velocity is negative, the ball moves to the left and the arrows point to the left.



Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) An electric charge produces the electic field $F = \nabla f$, where f is the function you have found in the homework. Assume you have three positive electric charges located on a equilateral triangle with side length 1. Without writing down any formula, sketch the field F in the plane spanned by the charges. The field is the sum of the fields of the three charges.

2) The torus can be realized as a square, on which opposite sides are identified. The "Packman" world is an example of a torus. If you leave the screen on the right hand side, you reappear on the left hand side. If you leave at the bottom, you reappear on the top. Investigate, whether the fundamental theorem of line integrals holds on the torus.