7/17/2008 EXTREMA WITH CONSTRAINTS **O.Knill,Maths21a**

This is part 3 (of 3) of the weekly homework. It is due July 22 at the beginning of class.

SUMMARY.

- Extremize $f(x,y)$ under the constraint $g(x,y)=c$: Solve $g(x,y)=c, \nabla f(x,y) = \lambda \nabla g(x,y)$ with **Lagrange multiplier** λ, These are 3 equations for 3 unknowns x,y,λ:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= c \end{aligned}$$

In three dimensions, the Lagrange equations form 4 equations for 4 unknowns.

$$\begin{aligned} f_x(x,y,z) &= \lambda g_x(x,y,z) \\ f_y(x,y,z) &= \lambda g_y(x,y,z) \\ f_z(x,y,z) &= \lambda g_z(x,y,z) \\ g(x,y,z) &= c \end{aligned}$$

Homework Problems

1) (4 points) Find the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the circle $g(x,y) = x^2+y^2 = 4$ using the method of Lagrange multipliers.

2) (4 points) Find the extrema of the same function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ as in the previous problem but now on the entire disc $\{x^2+y^2 \leq 4\}$ of radius 2.

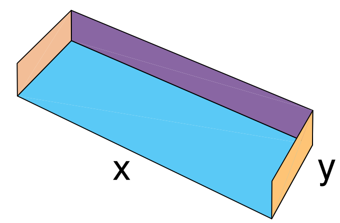
3) (4 points) Find the points (x,y,z) on the surface $g(x,y,z) = xy^2 - z^3 - 2 = 0$ that are closest to the origin $(0,0,0)$.

4) (4 points) Let a,b,c be non-negative constants and let F be the function $F(x,y,z) = -x\log(x) - y\log(y) - z\log(z) - ax - by - cz$. Find the maxima and minima of F on $x>0, y>0, z>0$ under the constraint $x+y+z=1$.

5) (4 points) Minimize the material cost of an office tray

$$f(x,y) = xy + x + 2y$$

of length x, width y and height 1 under the constraint that the volume $g(x,y) = xy$ is constant and equal to 4.



Remarks: (You don't need to read these remarks to do the problems.)

Remark to problem 4) This problem appears in thermodynamics and is relevant in biology or chemistry. If x,y,z are the probabilities that a system is in state X,Y,Z and a,b,c are the energies for these states. Then $-x\log(x) - y\log(y) - z\log(z)$ is called the **entropy** of the system and $E = ax+by+cz$ is the **energy**. The number $F(x,y,z)$ is called the **free energy**. If energy is fixed, nature tries to maximize entropy. Otherwise it tries to **minimize the free energy** $F = S - E$. If we extremize F under the constraint of having total probability $G(x,y,z) = x+y+z = 1$, we obtain the so called **Gibbs distribution**.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) What does it mean that the Lagrange multiplier λ is zero in a constrained optimization problem?

2) Extend the Lagrange method to arbitrary dimensions. Find the equations to find the extrema of a function $f(x_1,...,x_n)$ under the constraint $g(x_1,...,x_n) = c$.

3) Let $I = -\sum_{i=1}^n p_i \log(p_i)$ be the entropy of a probability distribution $(p_1,\dots,p_n)$. Show that among all probability distributions, the one where $p_i = 1/n$ is the one which maximizes entropy.

4) (4 points) Which pyramid of height h over a square $[-a,a]\times[-a,a] = \{(x,y) \mid -a \leq x \leq a, -a \leq y \leq a$ and surface area 4 has maximal volume?