7/1/2008 PLANES/DISTANCES **O.Knill,Maths21a**

This is part 1 (of 3) of the homework which is due July 8 at the beginning of class.

SUMMARY.

- $\vec{r}(t,s) = \vec{OP} + t\vec{v} + s\vec{w}$ **parametric equation** for a **plane**. $P = (x_0, y_0, z_0)$ is a point $\vec{v}, \vec{w}$ are vectors.
- $ax + by + cz = d$, **implicit equation** for a **plane**.
- $\vec{r}(t) = \vec{OP} + t\vec{v}$ **parametric equation** for a line, P a point, $\vec{v}$ is a vector.
- $\frac{(x-x_0)}{a} = \frac{(y-y_0)}{b} = \frac{(z-z_0)}{c}$ **symmetric equation** for a line.
- Distance Point-Point $d(P,Q) = |\vec{PQ}|$.
- Distance Point-Plane $d(P,\Sigma) = |(\vec{PQ}) \cdot \vec{n}|/|\vec{n}|$.
- Distance Point-Line $d(P,L) = |(\vec{PQ}) \times \vec{u}|/|\vec{u}|$.
- Distance Line-Line $d(L,M) = |(\vec{PQ}) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|$.

Homework Problems

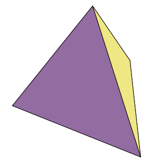
1) (4 points) Find the equation for the plane which contains the point $P = (1,2,3)$ and the line which passes through $Q = (3,4,4)$ and $R = (1,1,2)$.

Solution:
A normal vector $\vec{n} = (1,-2,2) = (a,b,c)$ of the plane $ax + by + cz = d$ is obtained as the cross product of $P - Q$ and $R - Q$ With $d = \vec{n} \cdot P = 3$, we have the equation $x - 2y + 2z = 3$.

2) (4 points)
a) (3) Find the distance between the point $(2,-1,2)$ and the plane $4x - 2y + z = 2$.
b) (1) If no absolute value is taken in the distance formula, what does the sign of the result say?

Solution:
a) The point $Q = (0,0,2)$ is on the plane. The scalar projection of $P - Q = (2,-1,0)$ onto the normal vector $(4,-2,1)$ of the plane is $10/\sqrt{21}$.
b) If $(P - Q) \cdot \vec{n}/|\vec{n}|$ is positive, then the point P is on the side into which the normal vector points.

3) (4 points) A regular tetrahedron has vertices at the points $P_1 = (0,0,3)$, $P_2 = (0,\sqrt{8},-1)$, $P_3 = (-\sqrt{6},-\sqrt{2},-1)$ and $P_4 = (\sqrt{6},-\sqrt{2},-1)$. Find the distance between two edges which do not intersect.



Solution:
The vector $\vec{v} = (P_2 - P_1) = (0, 2\sqrt{2}, -4)$ is parallel to the first edge, the vector $\vec{w} = (P_4 - P_3) = (2\sqrt{6}, 0, 0)$ is parallel to the second edge. The cross product of $\vec{v}$ and $\vec{w}$ is $\vec{n} = (0, -8\sqrt{6}, -8\sqrt{3})$. The distance between the two edges is the scalar projection of $P_3 - P_1$ onto $\vec{n}$. It is $(P_3 - P_1).\vec{n}/|\vec{n}| = 2\sqrt{3}$.

4) (4 points) a) Find a parametric equation for the line through the point $P = (3,1,2)$ that is perpendicular to the line $L : x = 1 + t, y = 1 - t, z = 2t$ and intersects this line in a point Q.

Solution:
b) The point $Q = (1,1,0)$ is on the line. The vector $\vec{v} = (1,-1,2)$ parallel to the line. We have $P - Q = (2,0,2)$. The vector $\vec{n} = \vec{v} \times (\vec{v} \times (P - Q)) = (-6,-6,0)$ is the direction from P to the normal intersection with the line. The line can be given by $\vec{r}(t) = (3 - 6t, 1 - 6t, 2)$.

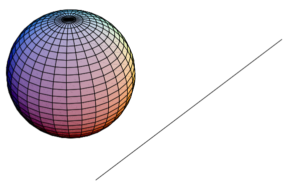
5) (4 points) Compute the distance of P to L in the previous problem and verify that it is equal to $d(P,Q)$.

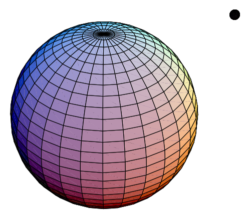
Solution:
The solution is $\sqrt{2}$. You can get the answer using the distance formula $|(P - Q) \times v|/|v|$.

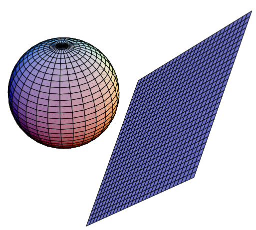
Challenge Problems

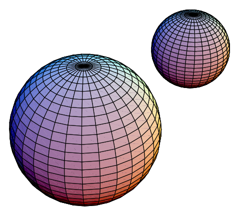
(Solutions to these problems are **not** turned in with the homework.)

1) Following some examples giving during lecture, can you write down distance formulas between
a) A point and a sphere.
b) A plane and a sphere.
c) Two spheres.
d) A line and a sphere.









2) How does one describe a three dimensional "hyper-plane" in four dimensional space? Find a parametric description and an implicit description.

3) Can you find a line and a two dimensional plane in $\mathbf{R}^4$ which are not parallel and which do not intersect? How would you compute the distance between a line and a two dimensional plane in $\mathbf{R}^4$?