6/24/2008 COORDINATES/DISTANCE **O.Knill,Maths21a**

This is part 1 (of 3) of the weekly homework. It is due Tuesday, July 1 at the beginning of class.

SUMMARY.

- $P = (x, y, z)$ **point**, x, y, z **coordinates**, $O = (0, 0, 0)$ **origin**.
- the plane is divided into **4 quadrants**, space into **8 octants**.
- $d((x, y, z), (u, v, w)) = \sqrt{(x-u)^2 + (y-v)^2 + (z-w)^2}$ **distance**.
- $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ **sphere** with **center** (a, b, c) and **radius** r.
- **completion of the square**: $x^2 + ax = b \Leftrightarrow (x + \frac{a}{2})^2 = x^2 + ax + \frac{a^2}{4} = b + \frac{a^2}{4}$

Homework Problems

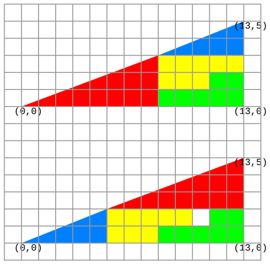
1) (4 points) Describe and sketch the set of points $P = (x, y, z)$ in $\mathbf{R}^3$ represented by

a) $9y^2 + 4z^2 = 81$
b) $x/7 - y/11 - z/13 = 1$
c) $(y-3)^2 = 25$
d) $d(P, (1, 0, 0)) + d(P, (0, 1, 0) = 10.$
e) $xy = 10$
f) $x^2 + (y-2)^2 = 0$
g) $x^2 - y^2 = 0$
h) $x^2 = y$.

Solution:
a) An elliptical cylinder with the x axis as a center width 3 and height 2.
b) A plane through the points $(7, 0, 0), (0, -11, 0), (0, 0, -13)$.
c) The union of two planes $y = 8$ and $y = -2$.
d) An ellipsoid with focal points (1,0,0) and (0,1,0).
e) A hyperbolic cylinder. f) The z-axis through $(0, 2, 0)$.
g) A union of two planes which are perpendicular to each other and intersect in the z-axis. The $z-$trace forms 45 degree angles with the x and y axes.
h) A cylindrical paraboloid.

2) (4 points)
a) Find the distance from the point $P = (3, 2, 5)$
i) to the y-axis.
ii) to the xz-coordinate plane.

b) A famous magicians trick is to let a square disappear as the picture shows. What is going on? Hint: look at distances.



Solution:
a) (i) A general point (x, y, z) has distance $\sqrt{x^2 + z^2}$ from the y-axis. The point P has distance $\sqrt{38}$ from the z-axis.
(ii) A general point $P = (x, y, z)$ has distance y from the xz-plane. In this case, 5.
b) Solution: The triangle is not a triangle. because $\sqrt{25+4} + \sqrt{64+9}$ is not equal to $\sqrt{169+25}$.

3) (4 points)

a) Find an equation of the sphere with center $(-1, 8, -5)$ and radius 6.
b) Describe the traces of this surface, its intersection with each of the coordinate planes.

Solution:
a) $(x+1)^2 + (y-8)^2 + (z+5)^2 = 36$.
b) $z = 0$: intersection with xy-plane: $(x+1)^2 + (y-8)^2 = 36 - 25 = 11$ is a circle.
$y = 0$: intersection with xz-plane: $(x+1)^2 + (z+4)^2 = -28$ is empty.
$x = 0$: intersection with yz-plane: $(y-8)^2 + (z+5)^2 = 0$ is a point.

4) (4 points) a) Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 4$.
A set in the plane of the form $(x-u)^2/a^2 + (y-v)^2/b^2 = 1$ is called an ellipse. The point (u, v) is called the center of the ellipse, the constants a, b are called the lengths of the semi-axes.
b) Find the center and semi-axes of the ellipse $9x^2 + 30x + y^2 = 0$.

Solution:
a) Completion of the square gives $(x^2-2x+1)+(y^2+4y+4)+z^2 = (x-1)^2+(y+2)^2+(2z) = 4+1+4 = 9$, so that the sphere is centered at the point $(1, -2, 0)$ and has radius $r = \sqrt{9} = 3$.
b) Completion of the sphere gives $(3x+5)^2 + y^2 = 25$ which can be brought into the form $(x - (-5/3))^2/(5/3)^2 + y^2/5^2 = 1$. The center is $(-5/3, 0)$ and the semi-axes are $5/3$ and 5.

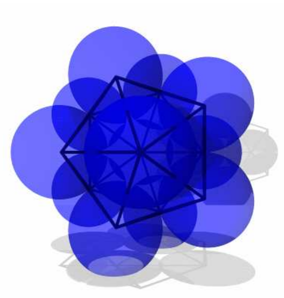
5) (4 points) Four spheres of equal radius are located in space so that any pair of two spheres touch. The centers of three spheres are known to be $(\sqrt{2}, 0, 0), (0, \sqrt{2}, 0), (0, 0, \sqrt{2})$ and the forth sphere with coordinates $(-a, -a, -a)$ is located in the octant $\{x < 0, y < 0, z < 0\}$. Find the radius of the spheres as well as the center of the fourth sphere.

Solution:
The distance between two of the three spheres is 2. The fourth sphere has to have the coordinates $P = (-a, -a, -a)$. In order that this sphere touches the first spheres, we must have $(a + \sqrt{2})^2 + a^2 + a^2 = 4$. Solving the quadratic equation gives $a = \sqrt{2}/3$.

Remarks

(You don't need to read these remarks to do the problems.)

Remark to problem 5) Mathematicians call the touching of sphere also "kissing". An old mathematical problem is to determine the **"kissing number"** of a sphere. How many spheres of radius 1 can you arrange around a given sphere S of radius 1 such that any of them touches the sphere S. Newton correctly believed that the kissing number was 12. Proofs that a 13'th sphere can not be squeezed in were only given in the 19'th and 20'th century. Packing 12 spheres around a central one can be realized if they kiss the central sphere S at the vertices of a **icosahedron**.
An other famous problem is the **Kepler problem** which asks about the densest sphere packing in space. A solution of this problem has only been announced a few years ago. The packing of 4 spheres you have looked at in the homework is a first small part of the densest packing. Can you guess, how it should continue?



Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Show that the set of points in the plane for which the difference of the distance from two points $(-1,0)$ and $(1,0)$ is constant 1 forms a hyperbola $x^2/a^2 - y^2/b^2 = c$.
Hint: Start with $\sqrt{(x-1)^2+y^2} - \sqrt{(x+1)^2+y^2} = 1$ and bring it into the form of a hyperbola.

2) The principle of GPS (Global positioning system) is based on a triangularization problem. Let us look at this problem in the plane. We have three known points A, B, C and want to compute our own position P. We do not know the distances $|A-P|, |B-P|, |C-P|$ to the three points but we know the differences between the distances. Can we figure out our position from this information? Explain, informally why the real GPS with 4 known satellite positions in space allows to compute the position of the GPS unit.

3) An other distance in the plane is defined by $d((x,y),(u,v)) = |x-u| + |y-v|$. It is called the **taxi metric** or **Manhattan metric** because a taxi driver in a town like Manhattan, where all streets are parallel either to the x or y axis experiences this distance between two points. How does an ellipse look like in this metric? You can assume that the ellipse is defined as the set of points (x,y) which have the property that the sum of the distances to $(-1,0)$ and $(0,1)$ is 4.

4) In the cube $[-2,2] \times [-2,2] \times [-2,2]$ of all points $\{(x,y,z) \mid |x| < 2, |y| < 2, |z| < 2\}$ are packed 8 spheres centered at $(\pm 1, \pm 1, \pm 1)$. What is the radius of the largest sphere centered at the origin $(0,0,0)$ which touches all 8 spheres? One can ask this problem in any dimensions. In 4 dimensions for example, there is a sphere at the center which touches all spheres centered at $(\pm 1, \pm 1, \pm 1, \pm 1)$. It is a surprising fact that for sufficiently large dimensions, the middle sphere does not fit into the cube $[-2,2] \times ... \times [-2,2]$ any more. Where is the threshold?

5) In how many regions do the three planes $x+y=0, x+z=0, y+z=0$ divide the cube $\{0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$?