8/12/2008 FINAL EXAM PRACTICE II **Maths 21a, O. Knill, Summer 2008**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please try to write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work. Answers without derivation are not given credit.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
15		10
Total:		160

Problem 1) (20 points)

1) T F If $\vec{u} + \vec{v} + \vec{w} = \vec{0}$ then $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$.

2) T F $\int_0^5 \int_0^\pi r \, d\theta \, dr$ is half the area of a disc radius 5 in the plane.

3) T F If a vector field $\vec{F}(x, y)$ satisfies $\text{curl}(F)(x, y) = 0$ for all points (x, y) in the plane, then $\vec{F}$ is conservative.

4) T F If the acceleration of a parameterized curve $\vec{r}(t) = (x(t), y(t), z(t))$ is zero then the curve $\vec{r}(t)$ is a line.

5) T F A circle of radius 1/2 has a smaller curvature than a circle of radius 1.

6) T F The curve $\vec{r}(t) = (-\sin(t), \cos(t))$ for $t \in [0, \pi]$ is half a circle.

7) T F The function $u(t, x) = \sin(x + t)$ is a solution of the partial differential equation $u_{tx} + u = 0$

8) T F The length of a curve $\vec{r}(t)$ in space parameterized on $a \leq t \leq b$ is the value of the integral $\int_1^2 |\vec{T}\,'(t)| \, dt$, where $\vec{T}(t)$ is the unit tangent vector.

9) T F Let (x_0, y_0) be the maximum of $f(x, y)$ under the constraint $g(x, y) = 1$. Then the gradient of g at (x_0, y_0) is parallel to the gradient of f at (x_0, y_0).

10) T F At a point which is not a critical point, the directional derivative $D_{\vec{v}} f(x_0, y_0, z_0)$ can take both the negative and the positive sign.

11) T F If a nonzero vector field $\vec{F}(x, y)$ is a gradient field, we always can find a curve C for which the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive.

12) T F If C is a closed level curve of a function $f(x, y)$ and $\vec{F} = (f_x, f_y)$ is the gradient field of f, then $\int_C \vec{F} \cdot d\vec{r} = 0$.

13) T F The divergence of a gradient vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ is always zero.

14) T F The line integral of the vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ along a line segment from $(0, 0, 0)$ to $(1, 1, 1)$ is 1.

15) T F If $\vec{F}(x, y) = (x^2 - y, x)$ and $C : \vec{r}(t) = \langle \sqrt{\cos(t)}, \sqrt{\sin(t)} \rangle$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C F \cdot \, ds$ is twice the area of R.

16) T F The flux of the vector field $\vec{F}(x, y, z) = \langle 0, y, 0 \rangle$ through the boundary S of a solid sphere E is equal to the volume the sphere.

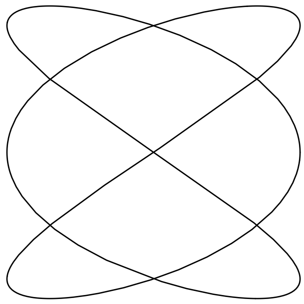
17) T F The quadratic surface $-x^2 + y^2 + z^2 = 5$ is a one-sheeted hyperboloid.

18) T F If $\vec{F}$ is a vector field in space and S is the boundary of a sphere then the flux of $\vec{\text{curl}}(\vec{F})$ through S is 0.

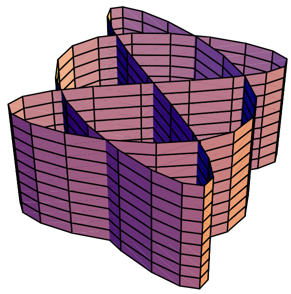
19) T F If $\text{div}(\vec{F})(x, y, z) = 0$ for all (x, y, z) and S is a torus surface, then the flux of $\vec{F}$ through S is zero.

20) T F In spherical coordinates, the equation $\rho \cos(\phi) = \rho \cos(\theta) \sin(\phi)$ defines a plane.

Problem 2) (10 points)



I



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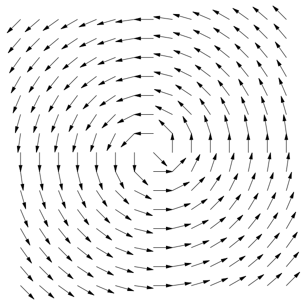

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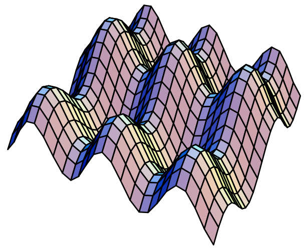

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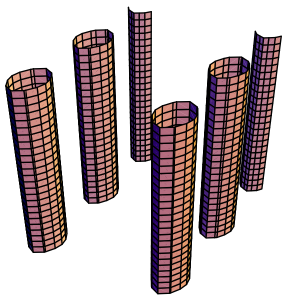

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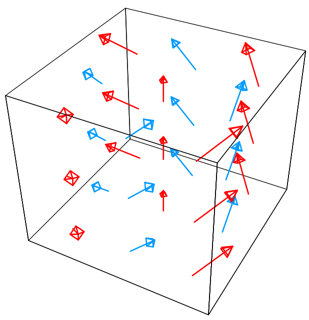

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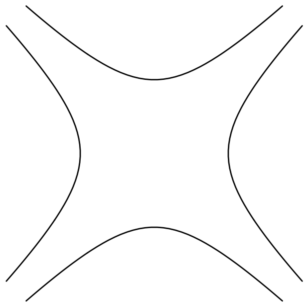
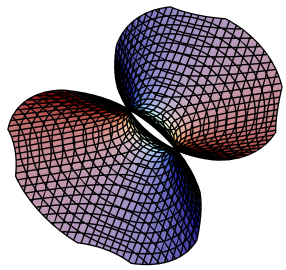


VII



VIII

Enter I,II,III,IV,V,VI,VII,VIII here	Equation
	$x^2 - y^2 + z^2 = 1$
	$\vec{r}(t) = \langle \cos(3t), \sin(2t) \rangle$
	$z = f(x,y) = \cos(3x) + \sin(2y)$
	$\vec{F}(x,y) = \langle -y/\sqrt{x^2+y^2}, x/\sqrt{x^2+y^2} \rangle$
	$\cos(3x) + \sin(2y) = 1$
	$\vec{F}(x,y,z) = \langle -y, x, 1 \rangle$
	$\vec{r}(u,v) = \langle \cos(3u), \sin(2u), v \rangle$
	$\{(x,y) \in \mathbf{R}^2 \mid \|x^2 - y^2\| = 1 \}$

Furthermore, fill in the peoples names, Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

Formula	Name of the theorem
$\int_C \vec{F} \cdot \vec{dr} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x,y) = f_{yx}(x,y)$	
$\int_C \vec{F} \cdot dr = \int \int_R \text{curl}(\vec{F}) \ dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)) \ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F) \ dV$	
$\int_a^b \int_c^d f(x,y) \ dxdy = \int_c^d \int_a^b f(x,y) \ dydx$	

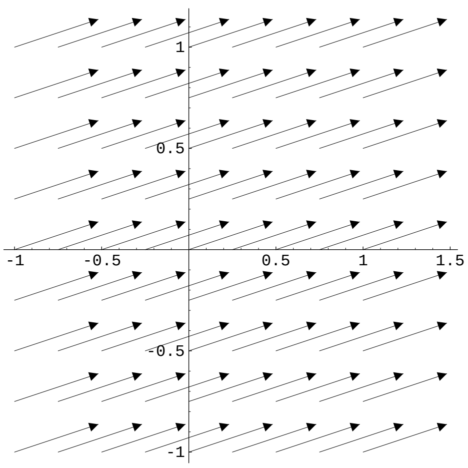
Problem 3) (10 points)

In this problem, vector fields $\vec{F}$ are written as $\vec{F} = \langle P, Q \rangle$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x, y) = 0$ vanishes for **all** (x, y). Similarly, we say $\text{div}(F)$ if $\text{div}(F)(x, y) = P_x(x, y) + Q_y(x, y) = 0$ for all x, y.

Check the box which match the formulas of the vector fields with the corresponding picture I,II,III or IV and mark also the places, indicating the vanishing of curl(F).

Vectorfield	I	II	III	IV	$\text{curl}(F) = 0$	$\text{div}(F) = 0$
$\vec{F}(x, y) = \langle 1, x \rangle$						
$\vec{F}(x, y) = \langle 3y, -3x \rangle$						
$\vec{F}(x, y) = \langle 7, 2 \rangle$						
$\vec{F}(x, y) = \langle x, y \rangle$						

I



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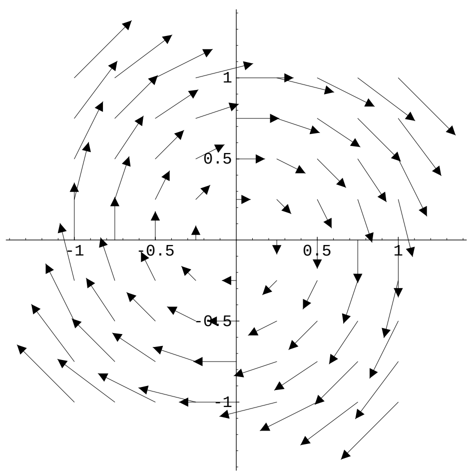

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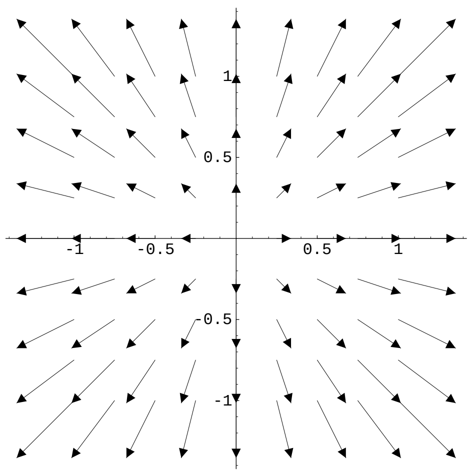

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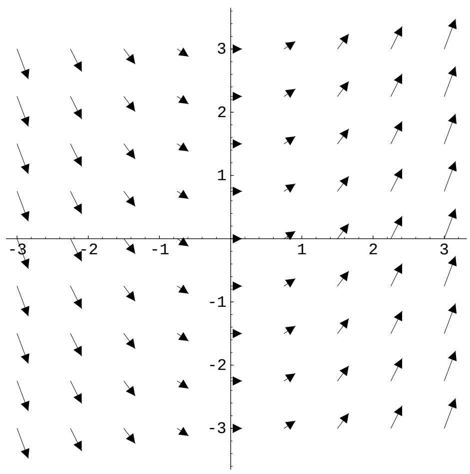


Problem 4) (10 points)

a) (5 points) What is the area of the triangle A, B, P, where $A = (1,1,1), B = (1,2,3)$ and $P = (3,2,4)$?

b) (5 points) Find the distance between the point the point P and the line L passing through the points A with B.

Problem 5) (10 points)

The height of the ground near the Simplon pass in Switzerland is given by the function

$$f(x,y) = -x - \frac{y^3}{3} - \frac{y^2}{2} + \frac{x^2}{2} .$$

There is a lake in that area as you can see in the photo.

a) (7 points) Find and classify all the critical points of f and tell from each of them, whether it is a local maximum, a local minimum or a saddle point.

b) (3 points) For any pair of two different critical points A, B found in a) let $C_{a,b}$ be the line segment connecting the points, evaluate the line integral $\int_{C_{a,b}} \nabla f \; d\vec{r}$.

Photo of the lake in the Swiss alps near the Simplon mountain pass.

Problem 6) (10 points)

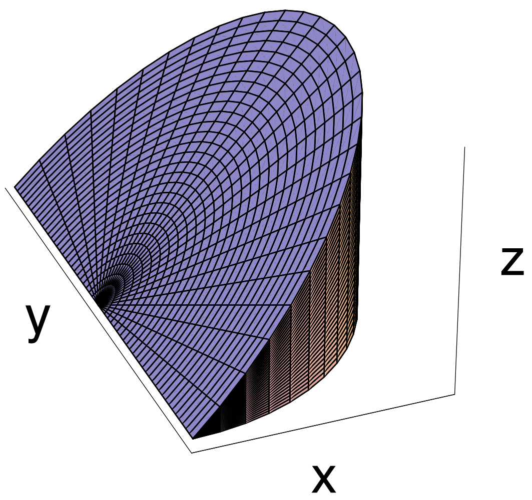
a) (4 points) Find the linearization $L(x,y,z)$ of $f(x,y,z) = 2 + z - \sin(-x - 3y)$ at the point $P = (0, \pi, 2)$.

b) (4 points) Find the equation of the tangent plane at that point $P = (0, \pi, 2)$.

c) (2 points) Estimate $f(0.001, \pi, 2.02)$ using the linearization.

Problem 7) (10 points)

Find the volume of the wedge shaped solid that lies above the xy-plane and below the plane $z = x$ and within the solid cylinder $x^2 + y^2 \leq 9$.

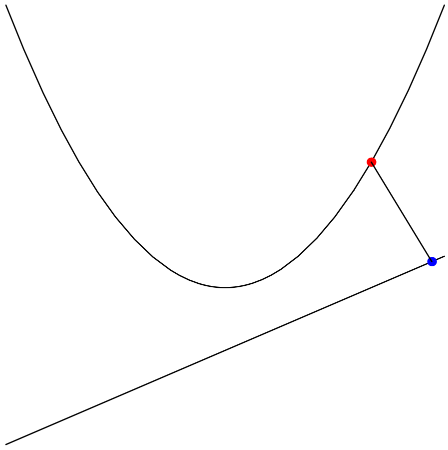


Problem 8) (10 points)

The distance from a point (x, y) to the line $y = x$ in the plane is given by $f(x, y) = (y - x)/\sqrt{2}$. Use the Lagrange method to find the point (x, y) on the parabola

$$g(x, y) = x^2 - y = -2$$

which is closest to the line.



Problem 9) (10 points)

a) (5 points) A ribbon of a girl is modeled as a surface S which is parameterized by $\vec{r}(t,s) = (s\cos(t), \sin(t), t)$, where $t \in [0, 2\pi]$ and $s \in [0, 1]$. Find the surface area of this ribbon S.

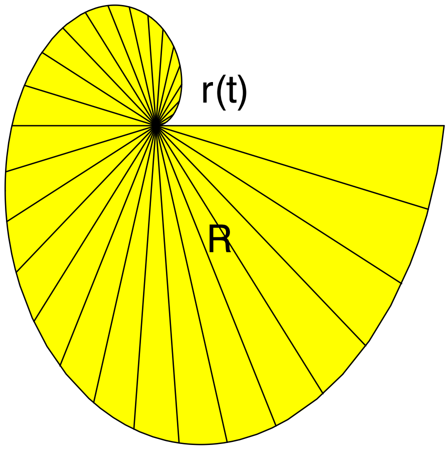
b) (5 points) Part of the boundary of the ribbon is obtained when fixing $s = 1$. It is a curve in space. Find the arc length of this curve $\vec{r}(t)$, parametrized from $t = 0$ to 2π.

Painting: "Young Girl with Blue Ribbon" by the French painter Jean-Baptiste Greuze (1725-1805)

Problem 10) (10 points)

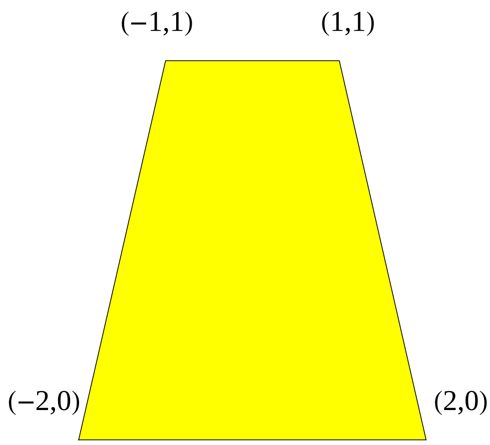
A region R in the xy-plane is given in polar coordinates by $0 \leq r(\theta) \leq \theta$ for $\theta \in [0, 2\pi]$. You see the region in the picture to the right. Its boundary is called the **Archimedes spiral**. It can be found on the tomb of Jacob Bernoulli. Evaluate the double integral

$$\iint_R \frac{e^{-x^2-y^2}}{(2\pi - \sqrt{x^2+y^2})} \, dx \, dy \; .$$



Problem 11) (10 points)

Find the line integral of the vector field $\vec{F}(x,y) = \langle 3y, 8x \rangle$ along the boundary of the trapezoid with vertices $(-2,0), (2,0), (1,1), (-1,1)$.



Problem 12) (10 points)

Let $\vec{F}$ be the vector field $\vec{F}(x,y,z) = \langle -z + x^{(x^x)}, 5 + y^{(y^y)}, y + z^{(z^z)} \rangle$.
Let C be the curve given by the parameterization $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$, for $0 \leq t \leq 2\pi$.
Compute the line integral of $\vec{F}$ along C.

Hint. You might want to consider a surface contained in the xz-plane which is enclosed by the curve.

Problem 13) (10 points)

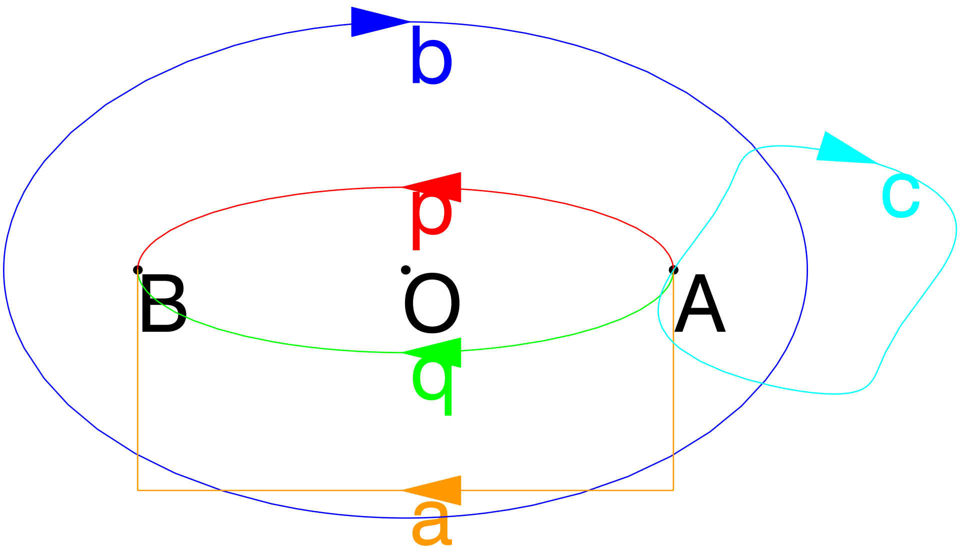
What is the flux of the vector field

$$\vec{F}(x,y,z) = \langle 3x + \cos(z^2 \sin(z)), x, \sin(y^3 + \cos(\sin(xy^3)))\rangle$$

through the boundary S of the solid cylinder $E = \{x^2+y^2 \leq 1,\ 0 \leq z \leq 10\ \}$. The surface of the cylinder is oriented so that the normal vector points outwards.

Problem 14) (10 points)

Suppose $\vec{F}$ is an irrotational vector field in the plane (that is, its curl is everywhere zero) that is not defined at the origin $O = (0,0)$. Suppose the line integral of $\vec{F}$ along the path p from A to B is 5 and the line integral of $\vec{F}$ along the path q from A to B is -4. Find the line integral of $\vec{F}$ along the following three paths:



a) (3 points) The path a from A to B going clockwise below the origin.

b) (4 points) The closed path b encircling the origin in a clockwise direction.

c) (3 points) The closed path c starting at A and ending in A without encircling the origin.

Problem 15) (10 points)

Let S be the graph of the function $f(x,y) = 2 - x^2 - y^2$ which lies above the disk $\{(x,y) \mid x^2 + y^2 \leq 1\}$ in the xy-plane. The surface S is oriented so that the normal vector points upwards. Compute the flux $\int\int_S \vec{F} \cdot d\vec{S}$ of the vectorfield

$$\vec{F} = (-4x + \frac{x^2 + y^2 - 1}{1 + 3y^2}, 3y, 7 - z - \frac{2xz}{1 + 3y^2})$$

through S using the divergence theorem.