STOKES THEOREM Maths21a, O. Knill

REMINDERS. The **curl** of a vector field F is

$$\text{curl}(P,Q,R) = \nabla \times F = (R_y - Q_z, P_z - R_x, Q_x - P_y) .$$

The flux integral of a vector field F through a surface $S = \vec{r}(R)$ is defined as

$$\int\int_R F(r(u,v)) \cdot (r_u \times r_v) \; dudv$$

The line integral of a vector field F along a curve $C = r([a,b])$ is given as

$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \; dt .$$

The picture shows a tornado near Cordell, Oklahoma. Date: May 22, 1981. Photo Credit: NOAA Photo Library, NOAA Central Library. The tornado points into the direction of the field curl(F), where F is the velocity of the air. Twisters cause annually about 80 deaths in the US. The 17 Illinois twisters last Tuesday killed 8 people

STOKES THEOREM. Let S be a surface with boundary curve C and let F be a vector field. Then

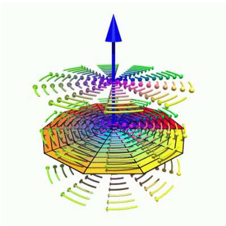
$$\int\int_S \text{curl}(F) \; \cdot dS = \int_C F \cdot dr .$$

Note: the orientation of C is such that if you walk along the surface (head into the direction of the normal $r_u \times r_v$), then the surface to your left.

EXAMPLE. Let $F(x,y,z) = (-y,x,0)$ and let S be the upper semi hemisphere, then curl$(F)(x,y,z) = (0,0,2)$. The surface is parameterized by $r(u,v) = (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$ on $R = [0,2\pi] \times [0,\pi/2]$ and $r_u \times r_v = \sin(v) r(u,v)$ so that curl$(F)(x,y,z) \cdot r_u \times r_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi} \int_0^{\pi/2} \sin(2v) \; dvdu = 2\pi$.

The boundary C of S is parameterized by $r(t) = (\cos(t), \sin(t), 0)$ so that $dr = r'(t)dt = (-\sin(t), \cos(t), 0)dt$ and $F(r(t)) \; r'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C F \cdot dr$ along the boundary is 2π.

SPECIAL CASE: GREEN'S THEOREM. If S is a surface in the $x-y$ plane and $F = (P,Q,0)$ has zero z component, then curl$(F) = (0,0,Q_x - P_y)$ and curl$(F) \cdot dS = (Q_x - P_y) \; dxdy$.



PROOF OF STOKES THEOREM.
For a surface which is flat, Stokes theorem can be seen with Green's theorem. If we put the coordinate axis so that the surface is in the xy-plane, then the vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of curl(F). The reason is that the third component $Q_x - P_y$ of curl$(F) = (R_y - Q_z, P_z - R_x, Q_x - P_y)$ is the two dimensional curl: $F(r(u,v)) \cdot (0,0,1) = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S F(r(u,v)) \cdot (0,0,1) \; dudv = \int_C F(r(t)) \; r'(t)dt$.

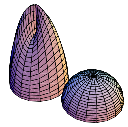
For a general surface, we approximate the surface by a mesh of small parallelepipeds. When summing up line integrals along all these parallelepipeds, the line integrals inside the surface cancel and only the integral along the boundary remain. On the other hand, the sum of the fluxes of the curl through boundary adds up to the flux through the surface.

DISCOVERY OF STOKES THEOREM **Stokes theorem was found by Ampère in 1825.** George Gabriel Stokes: (1819-1903) was probably inspired by work of Green and rediscovers the identity around 1840.

George Gabriel Stokes

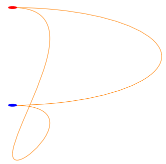
André Marie Ampere

EXAMPLE. Calculate the flux of the curl of $F(x,y,z) = (-y,x,0)$ through the surface parameterized by $r(u,v) = (\cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u + \pi/2))$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.



For every surface bounded by C the flux of curl(F) through through the surface is the same. The flux of the curl of a vector field through a surface S depends only on the boundary of S.

Compare this with the earlier statement:



For every curve between two points A, B the line integral of grad(f) along C is the same. The line integral of the gradient of a function of a curve C depends only on the end points of C.

BIOT-SAVARD LAW. A magnetic field B in absence of an electric field satisfies a Maxwell equation curl$(B) = (4\pi/c)j$, where j is the current and c is the speed of light. How do we get the magnetic field B, when the current is known? Stokes theorem can give the answer: take a closed path C which bounds a surface S. The line integral of B along C is the flux of curl(B) through the surface. By the Maxwell equation, this is proportional to the flux of j through that surface. Simple situation. Assume j is contained in a wire of thickness r which we align on the z-axis. To measure the magnetic field at distance $R > r$ from the wire, we take a curve $C : r(t) = (R\cos(t), R\sin(t), 0)$ which bounds a disc S and measure $2\pi RB = \int_C B \cdot ds = \int\int_S \text{curl}(B) \; dS = \int\int_S 4\pi/cj \; dS = 4\pi J/c$, where J is the total current passing through the wire. The magnetic field satisfies $B = 2J/(cR)$. This is called the Biot-Savard law.

THE DYNAMO, FARADEY'S LAW. The electric field E and the magnetic field B are linked by a Maxwell equation curl$(E) = -\frac{1}{c}\dot{B}$. Take a closed wire C which bounds a surface S and consider $\int\int_S B \cdot dS$, the flux of the magnetic field through S. Its change can be related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c \; \text{curl}(E) \cdot dS = -c\int_C E \; ds = U$, where U is the voltage measured at the cut-up wire. It means that if we change the flux of the magnetic field through the wire, then this induces a voltage. The flux can be changed by changing the amount of the magnetic field but also by changing the direction. If we turn around a magnet around the wire, we get an electric voltage. This happens in a power-generator like an alternator in a car. In practical implementations, the wire is turned inside a fixed magnet.