7/29/2004 SECOND HOURLY **Maths 21a, O. Knill, Summer 2004**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points)

T F (F) The acceleration of $\vec{r}(t) = (\cos(t), \sin(t), t)$ is $(\cos(t), \sin(t), 0)$.

Solution:
Sign error.

T (T) F There are functions $f(x, y)$ which have no critical point.

Solution:
True. Every nonconstant linear function for example.

T F (F) If $f_x(x, y) = f_y(x, y) = 0$ for all (x, y) then $f(x, y) = 0$ for all (x, y) .

Solution:
False, f could be constant.

T F (F) $(0, 0)$ is a local maximum of the function $f(x, y) = x^2 - y^2 + x^4 + y^4$.

Solution:
$(0, 0)$ is a saddle point.

T (T) F If $f(x, y)$ has a local maximum at the point $(0, 0)$ with discriminant $D > 0$ then $g(x, y) = f(x, y) - x^4 + y^3$ has a local maximum at the point $(0, 0)$ too.

Solution:
Adding $x^4 + y^3$ does not change the first and second derivatives.

T (T) F The value of the function $f(x, y) = \sqrt{1 + 3x + 5y}$ at $(-0.002, 0.01)$ can by linear approximation be estimated as $1 - (3/2) \cdot 0.002 + (5/2) \cdot 0.01$.

Solution:
Use formula for $L(x, y)$.

T F

The curve $\vec{r}(t) = (x(t), y(t)) = (t^3, t^3)$ is a line in the plane

Solution:
True.

T F

The directional derivative $D_{\vec{v}} f$ is a vector normal to $\vec{v}$.

Solution:
The directional derivative is a number, not a vector.

T F

The function $\sin(x - t)y$ satisfies the partial differential equation $f_{tt} = (f_{xx} + f_{yy})$.

Solution:
This is a solution of the two dimensional wave equation.

T F

The gradient of f at a point (x_0, y_0, z_0) is tangent to the level surface of f which contains (x_0, y_0, z_0).

Solution:
It is a basic and important fact that ∇f is **perpendicular** to the level surface.

T F

The sign of the Lagrange multiplier λ tells you whether the extrema under constraint is a local maximum or local minimum.

Solution:
No, while changing the sign of f indeed changes the sign of the Lagrange multipliers and also interchanges local maxima and local minima, the change of the sign of g does not change the nature of the critical point but changes the sign of λ.

T F

If $D_{\vec{v}} f(1, 1, 1) = 0$ for all vectors $\vec{v}$, then $(1, 1, 1)$ is a critical point.

Solution:
Especially, $D_{\nabla f}(f) = |\nabla f|^2 = 0$ so that $\nabla f = (0, 0, 0)$.

T F

The vector $\nabla f(1, 1, 1)$ is perpendicular to the surface $f(x, y, z) = x^2 + y^2 + 2z^2 = 4$ at the point $(1, 1, 1)$.

Solution:
This is a basic property of gradients.

T F

The function $u(x, t) = x^3 + t^3$ satisfies the wave equation $u_{tt} = u_{xx}$.

Solution:
Just differentiate.

T F

For any curve $\vec{r}(t)$, the vectors $\vec{r}\,''(t)$ and $\vec{r}\,'(t)$ are always perpendicular to each other.

Solution:
This is most of the time wrong.

T F

Every critical point (x, y) for which D is not zero is either a local maximum, a local minimum or a saddle point.

Solution:
This is the second derivative test.

T F

The function $f(x, y) = e^y x^2 \sin(y^2)$ satisfies the partial differential equation $f_{xxyyyxyy} = 0$.

Solution:
By Cavalieri, we can have all three x derivatives at the beginning.

T F

If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) < 0$ then $(0, 0)$ can not be a local minimum.

Solution:
If $f_{xx}(0, 0) < 0$ then on the x-axes the function $g(x) = f(x, 0)$ has a local maximum. This means that there are points close to $(0, 0)$ where the value of f is larger.

T F In the second derivative test, one can replace the condition $D > 0, f_{xx} > 0$ with $D > 0, f_{yy} > 0$ to check whether a point is a local minimum.

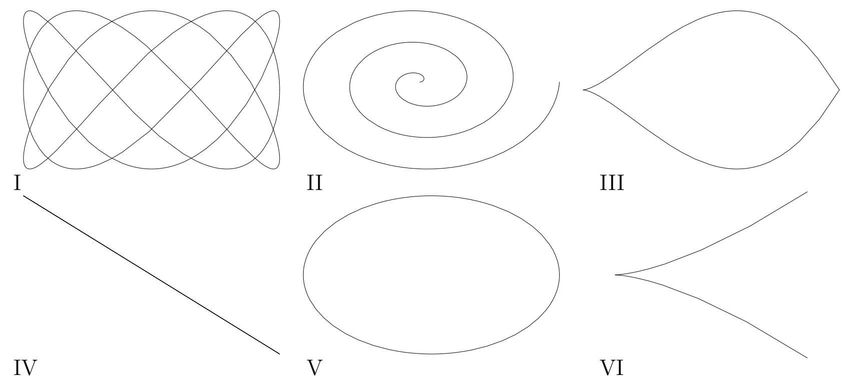
Solution:
True. If $f_{xx}f_{yy} - f_{xy}^2 > 0$, then f_{xx} and f_{yy} must have the same signs.

T F The length of the curve $\vec{r}(t) = (0,0,1) + t(0,3,4)$ with $t \in [0,2]$ is equal to 10.

Solution:
The speed is 5. The path has length $\int_0^2 5\, dt = 10$.

Problem 2)	(10 points)

Match the equations with the curves. No justifications are needed.



Enter I,II,..., until VI here	Equation
	$r(t) = (\cos(t), \sin(t))$
	$r(t) = (\cos(3t), \sin(5t))$
	$r(t) = (t\cos(3t), t\sin(3t))$
	$r(t) = (t^2, t^3 - t^5)$
	$r(t) = (\cos(t)^2, \sin(t)^2)$
	$r(t) = (t^2, t^3)$

Solution:
V I II III IV VI

Problem 3)	(10 points)

Tell from each of the 10 following objects, whether they are a vector or a scalar. No justifications are needed. Each correct answer is 1 point.

object	vector	scalar
velocity		
speed		
acceleration		
arc length		
curvature		
discriminant		
directional derivative		
gradient		
partial derivative		

Solution:
Only velocity, accelaration and the gradient are vectors.

Problem 4) (10 points)

Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Solution:
$\nabla f(x,y) = (x^4 - x, (y^2 - 1)) = (0,0)$ so that the critical points are $(0,1), (0,-1), (1,1), (1,-1)$. We have $D = (4x^3 - 1)2y$ and $f_{xx} = 4x^3 - 1$.

Point	D	f_{xx}	type
$(0,1)$	$D=-2$	-	saddle
$(0,-1)$	$D=2$	-1	local max
$(1,1)$	$D=6$	3	local min
$(1,-1)$	$D=-6$	-	saddle

Problem 5) (10 points)

Use the technique of linear approximation to estimate $f(0.003, -0.0001, \pi/2 + 0.01)$ for

$$f(x,y,z) = \cos(xy+z) + x + 2z \; .$$

Solution:
$L(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0)$
$f(x_0,y_0,z_0) = \cos(\pi/2) + \pi = \pi$
$a = f_x(x_0,y_0,z_0) = -0\sin(\pi/2) + 1 = 1$
$b = f_y(x_0,y_0,z_0) = -0\sin(\pi/2) = 0$
$c = f_z(x_0,y_0,z_0) = -\sin(\pi/2) + 2 = 1$
$L(x,y) = \pi + 0.003 \cdot 1 + -0.0001 \cdot 0 + 0.01 \cdot 1 = \pi + 0.013.$

Problem 6) (10 points)

Find the equation $ax + by + cz = d$ for the tangent plane to the level surface of

$$f(x,y,z) = \cos(xy+z) + x + 2z$$

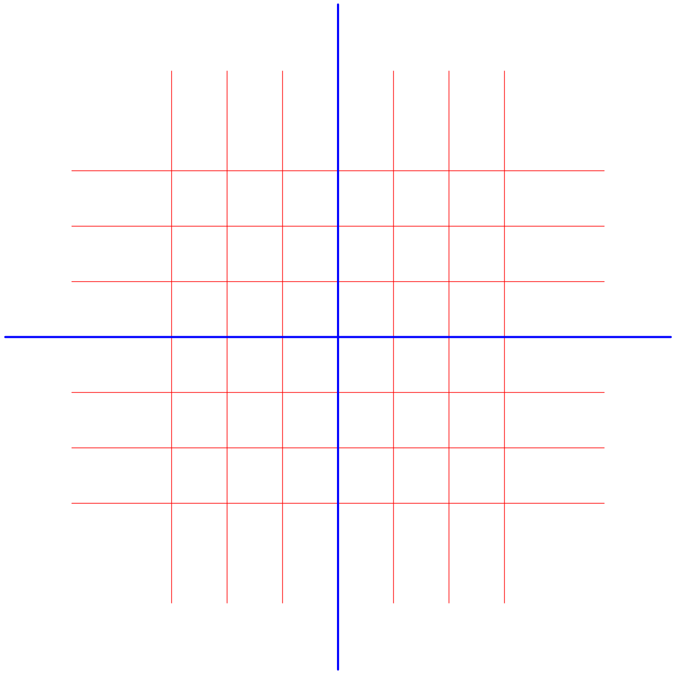
(same function as in last problem) which contains the point $(0,0,\pi/2)$.

Solution:
We have $\nabla f(0,0,\pi/2) = (1,0,1)$ so that the plane is $x + z = \pi/2$

Problem 7) (10 points)

Consider the function $f(x,y) = x^2 - y^2$ of two variables.

a) Sketch in the diagram below the level curves $f(x,y) = c$ for values $c = 0,1,3$.
b) Find and draw the gradient vector $\nabla f(x,y)$ at the point $(2,1)$.
c) Find the directional derivative $D_u f$ of $f(x,y)$ at the point $(2,1)$ into the direction $\vec{u} = (3,4)/5$.



Solution:
a) The level curves to $c = 0$ are lines $x = y, x = -y$. The level curves to $c = 1, c = 3$ are hyperbola, opend to the right and to the left.
b) $\nabla f(x, y) = (2x, -2y)$, so that $\nabla f(1, 1) = (4, -2)$.
c) $\nabla f(1, 1) \cdot (3, 4)/5 = 12/5 - 8/5 = 4/5$.

Problem 8)	(10 points)

A drawer of length x, width y and height z is open on the top and has the volume 32. For which dimensions are the material costs minimal?

Hint. The problem is to minimize $f(x, y, z) = xy + 2yz + 2xz$ under the constraint $g(x, y, z) = xyz = 32$.

Solution:
To minimize $f(x, y, z) = xy + 2yz + 2xz$ under the constraint $g(x, y, z) = xyz = 32$ we solve Lagrange equations

$$\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2y + 2x &= \lambda xy \\ xyz &= 32 \end{aligned}$$

Subtracting 2) from the 1) gives $(y - x) = \lambda(y - x)z$. If $y - x$ is not zero, then $\lambda z = 1$ which in 1) would give $y + 2z = y$ or $z = 0$ contradicting 4). Therefore, $y - x = 0$ or $y = x$. Equation 3) gives $4x = \lambda x^2$. Again $x = 0$ would contradict the 4) so that we can divide by x and get $4 = \lambda x$. Equation 2) gives $x + 2z = 4z$ or $x = 2z$. We have now $x = y = 2z$. Using 4) gives $xyz = 4z^3 = 32$ so that $z = 2$ and $x = 4$, $y = 4$.

Problem 9)	(10 points)

A stone follows a path $\vec{r}(t) = (x(t), y(t), z(t))$. It has the initial velocity $\vec{r}\,'(0) = (3, 0, 3)$ and is launched at the point $(5, 10, 14)$. It is subject to the gravitational force which means $\vec{r}\,''(t) = (0, 0, -10)$. Where will the stone hit the ground $z = 0$?

Solution:
$r(t) = (5, 10, 14) + (3, 0, 3)t + (0, 0, -10t^2/2)$. $14 + 3t - 5t^2 = 0$ gives $t = 2$. At time $t = 2$, the stone is at $(11, 10, 0)$.