Extended hour to hour syllabus

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Maths 21a, Summer 2004

1. Week: Geometry and Space

29. June: Space, coordinates, distance

Coordinates for describing space was promoted by Descartes in the 16'th century at about the time, when Harvard College was founded. A fundamental notion is the **distance** between two points. In order to get a feel about space, we will look at some geometric objects defined through coordinates. We will focus on **circles** and **spheres** and learn how to find the midpoint and radius of a sphere given as a quadratic expression in x, y, z. This method is called **completion of the square**.

30. June: Vectors, dot product, projections

Two points define an object which we call a **vector**. Vectors can be attached everywhere in space but are identified if they have the same length and direction. Vectors can describe for example velocities, forces or color. We learn first how to compute with vectors, use **addition**, **subtraction** and **scaling** both graphically as well as algebraically. The **dot product** is a product between two vectors which results in a scalar. Using the dot product, we can compute length, angles or projections.

1. July: Cross product, lines

The **cross product** is a product between two vectors which results in a new vector perpendicular to both. The product can be used for many things. It is useful for example to compute areas, it can be used to compute the distance between a point and a line. It will also be important for constructions like to get a plane through three points or to find the line which is in the intersection of two planes. In general, there are different ways to describe a geometric object. For lines, we will see the parametric description, as well as an implicit description which we will identify later as the intersection between two planes.

2. Week: Functions and Surfaces

6. July: Planes, distance formulas

The simplest equations are **linear equations**. They describes **planes**. We will learn how to describe planes using linear equations and how to construct them for example, from a line and a point or from three points. As an application of the tools, we will look at some **distance formulas** like the distance from a point to a plane, or the distance between two lines.

7. July: Functions, graphs, quadrics

Functions of several variables play an essential role in this course. The graph of functions of two variables define **graphs** $z - f(x, y) = 0$. We will also look at surfaces of the form $g(x, y, z) = 0$, where g is a function which only involves quadratic terms. These are called **quadrics**. Important quadrics are **spheres**, **ellipsoids**, **cones**, **cylinders** as well as various **hyperboloids**.

8. July: Implicit and parametric surfaces

Surfaces can be described in two fundamental way. Implicitly or parametrically. The first form is $g(x, y, z) = 0$ like $x^2 + y^2 + z^2 - 1 = 0$ the second form is $r(u, v) = (x(u, v), y(u, v), z(u, v))$ like $r(u, v) = (r\cos(u)\sin(v), r\sin(u)\sin(v), r\cos(v))$ In many cases, it is possible to go from one form to the other like for the sphere, the plane, graphs of functions of two variables or surfaces of revolution. Using a computer, one can **visualize** surfaces very well. Computer algebra systems with graphical capabilities are for the mathematician what the telescope is for the astronomer or the microscope for the biologist. With a bit of patience you find your own surface which nobody has seen before.

3. Week: Curves and Partial Derivatives

13. July: Curves, velocity, acceleration, chain rule

Curves are one dimensional objects. Both in the plane as well as in space, they can take many different forms. A special case are closed curves in space which are called **knots**. By differentiation, one obtains **velocity** and **acceleration** which are both vectors. The **chain rule** tells us how a function changes along a curve.

14. July: Arc-length, curvature, partial derivatives

There is a formula for the **length** of a curve. Lengths can be computed by evaluating a one-dimensional integral. The curvature of a curve is a quantity telling how much a curve is bent. Finally, we will see partial derivatives as well as see some **partial differential equations** abbreviated as PDE's.

15. July: First midterm (on week 1-2)

4. Week: Extrema and Lagrange Multipliers

20. July: Gradient, linearization, tangents

The **gradient** of a function is an important tool to describe the geometry of surfaces. Fundamental is the property that the gradient vector ∇g is perpendicular to the implicit surface $g = c$. This allows us to compute **tangent planes** and **tangent lines** as well as to approximate a linear function by a linear function near a point. Many physical laws are actually just linearization of more complicated nonlinear laws.

21. July: Extrema, second derivative test

A central application of multi-variable calculus is to **extremize** functions of two variables. One first identifies **critical points**, points where the gradient vanishes. The nature of these critical points can be established using the **second derivative test**. There will be three fundamentally different cases: **local maxima**, **local minima** as well as **saddle points**.

22. July: Extrema with constraints

The topic with maybe the most applications both in science or economics is to extremize a function $f(x, y)$ in the presence of a **constraint** $g(x, y) = 0$. A necessary condition for a critical point is that the gradients of f and g are parallel. This leads to equations called the **Lagrange equation**.

5. Week: Double Integrals and Surface Integrals

27. July: Double integrals, type I,II regions

Integration in two dimensions is first done on rectangles, then on regions bound by graphs of functions. Similar than in one dimension, there is a **Riemann sum approximation** of the integral. This allows us to prove results like **Fubinis theorem** on the change of the integration order. An application of double integration is the computation of **area**.

28. July: Polar coordinates, surface area

Many regions can be described better in **polar coordinates**. Examples are so called **roses** which trace flower-like shapes in the plane but are graphs in polar coordinates. Changing coordinates comes with an integration factor which can be explained also after introducing the surface area.

29. July: Second midterm (on week 3-4)

Triple Integrals and Line Integrals

3. August: Triple integrals, cylindrical coordinates

Triple integrals allow the computation of volumes, moment of inertias or centers of masses of solids. First introduced for cubes it is then extended to more general regions bound by graphs of functions of two variables. Some regions can be described better in **cylindrical coordinates**, the analogue of polar coordinates in space.

4. August: Spherical coordinates, vector fields

Spherical coordinates allow an even more elegant computation of triple integrals for certain regions like cones or spheres. Next, we will introduce **vector fields**. They occur as force fields or velocity fields or mechanics and are closely related to the field of ordinary differential equations. Vector fields will occupy us until the end of the course.

5. August: Line integrals, fundamental thm of lineintegrals

Line integrals are one dimensional integrals along a curve in the presence of a vector field. If the vector field is a force field, then the line integral has the interpretation work done, when walking along the path. For a class of vector fields which we call **conservative vector fields** one can compute the line integral easily using an identity called the fundamental theorem of line integrals.

Exterior Derivatives and Integral Theorems

10. August: Curl and Green theorem

Greens theorem relates a line integral along a closed curve with a double integral of a derivative of the vector field in the region enclosed by the curve. The theorem is useful for example to compute areas. It also allows an easy computation of line integrals in certain cases. We will see a derivative of the vector field which is called the **"curl"**. It is a scalar field which measures the vorticity of the vector field in the plane.

11. August: Curl and Stokes theorem

Stokes theorem is Greens theorem lifted into three dimensions, where the region is replaced by a surface. Again, one can replace the line integral along the boundary of the surface by an integral of the "curl" of the field over the surface. This integral is a **flux integral**. The curl of a vector field in three dimensions is a vector field itself. The three components give the vorticity of the vector field in the x,y and z direction.

12. August: Div and Gauss theorem

Finally, the **divergence** of a vector field inside a solid is related to the flux of the vector field through the boundary of the surface using the **divergence theorem** which is sometimes also called **Gauss theorem**. The divergence theorem relates the "local expansion rate" of a vector field with the flux through a closed surface and is useful for example to compute the gravitational field inside a solid.

17. August: Final exam (on week 1-7) 1:30 PM