CURVES — Maths21a

PARAMETRIC PLANE CURVES. If $x(t)$, $y(t)$ are functions of one variable, defined on the **parameter interval** $I = [a, b]$, then $\vec{r}(t) = \langle f(t), g(t) \rangle$ is a **parametric curve** in the plane. The functions $x(t), y(t)$ are called **coordinate functions**.

PARAMETRIC SPACE CURVES. If $x(t), y(t), z(t)$ are functions of one variables, then $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a **space curve**.
Always think of the **parameter** t as **time**. For every fixed t, we have a point $(x(t), y(t), z(t))$ in space. As t varies, we move along the curve.

EXAMPLE 1. If $x(t) = t$, $y(t) = t^2 + 1$, we can write $y(x) = x^2 + 1$ and the curve is a **graph**.

EXAMPLE 2. If $x(t) = \cos(t), y(t) = \sin(t)$, then $\vec{r}(t)$ follows a **circle**.

EXAMPLE 3. If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $\vec{r}(t)$ describes a **spiral**.

EXAMPLE 4. If $x(t) = \cos(2t), y(t) = \sin(2t), z(t) = 2t$, then we have the same curve as in example 3 but we traverse it **faster**. The **parameterization** changed.

EXAMPLE 5. If $x(t) = \cos(-t), y(t) = \sin(-t), z(t) = -t$, then we have the same curve as in example 3 but we traverse it in the **opposite direction**.

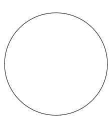
EXAMPLE 6. If $P = (a, b, c)$ and $Q = (u, v, w)$ are points in space, then $\vec{r}(t) = \langle a + t(u-a), b + t(v-b), c + t(w-c) \rangle$ defined on $t \in [0, 1]$ is a **line segment** connecting P with Q.

ELIMINATION: Sometimes it is possible to eliminate the parameter t and write the curve using equations (one equation in the plane or two equations in space).

EXAMPLE: (circle) If $x(t) = \cos(t), y(t) = \sin(t)$, then $x(t)^2 + y(t)^2 = 1$.

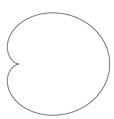
EXAMPLE: (spiral) If $x(t) = \cos(t), y(t) = \sin(t), z(t) = t$, then $x = \cos(z), y = \sin(z)$. The spiral is the intersection of two graphs $x = \cos(z)$ and $y = \sin(z)$.

CIRCLE



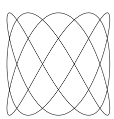
$(\cos(t), 3\sin(t))$

HEART



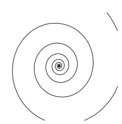
$(1 + \cos(t))(\cos(t), \sin(t))$

LISSAJOUS



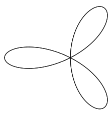
$(\cos(3t), \sin(5t))$

SPIRAL



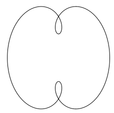
$e^{t/10}(\cos(t), \sin(t))$

TRIFOLIUM



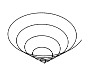
$-\cos(3t)(\cos(t), \sin(t))$

EPICYCLE



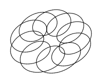
$(\cos(t) + \cos(3t)/2, \sin(t) + \sin(3t)/2)$

SPRING (3D)



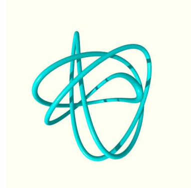
$(\cos(t), \sin(t), t)$

TORAL KNOT (3D)



$(\cos(t) + \cos(9t)/2, \sin(t) + \cos(9t)/2, \sin(9t)/2)$

WHERE DO CURVES APPEAR? Objects like particles, celestial bodies, or quantities change in time. Their motion is described by curves. Examples are the motion of a star moving in a galaxy, or data changing in time like (DJIA(t),NASDAQ(t),SP500(t))



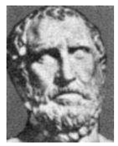
Strings or knots are closed curves in space.
Molecules like RNA or proteins can be modeled as curves.
Computer graphics: surfaces are represented by mesh of curves.
Typography: fonts represented by Bezier curves.
Space time A curve in space-time describes the motion of particles.
Topology Examples: space filling curves, boundaries of surfaces or knots.

DERIVATIVES. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a curve, then $\boxed{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle = \langle \dot{x}, \dot{y}, \dot{z} \rangle}$ is called the **velocity**. Its length $|\vec{r}\,'(t)|$ is called **speed** and $\vec{v}/|\vec{v}|$ is called **direction of motion**. The vector $\vec{r}\,''(t)$ is called the **acceleration**. The third derivative $\vec{r}\,'''$ is called the **jerk**.

The velocity vector $\vec{r}(t)$ is tangent to the curve at $\vec{r}(t)$.

EXAMPLE. If $\vec{r}(t) = \langle \cos(3t), \sin(2t), 2\sin(t) \rangle$, then $\vec{r}\,'(t) = \langle -3\sin(3t), 2\cos(2t), 2\cos(t) \rangle$, $\vec{r}\,''(t) = \langle -9\cos(3t), -4\sin(2t), -2\sin(t) \rangle$ and $\vec{r}\,'''(t) = \langle 27\sin(3t), 8\cos(2t), -2\cos(t) \rangle$.

WHAT IS MOTION? **The paradoxon of Zeno of Elea**: "When looking at a body at a specific time, the body is fixed. Being fixed at each instant, there is no motion". While one might wonder today a bit about Zeno's thoughts, there were philosophers like Kant, Hume or Hegel, who thought seriously about Zeno's challenges. Physicists continue to ponder about the question: "what is time and space?" Today, the derivative or rate of change is defined as a **limit** $(\vec{r}(t + dt) - \vec{r}(t))/dt$, where dt approaches zero. If the limit exists, the velocity is defined.



EXAMPLES OF VELOCITIES.

Person walking:	1.5 m/s
Signals in nerves:	40 m/s
Plane:	70-900 m/s
Sound in air:	Mach1=340 m/s
Speed of bullet:	1200-1500 m/s
Earth around the sun:	30'000 m/s
Sun around galaxy center:	200'000 m/s
Light in vacuum:	300'000'000 m/s

EXAMPLES OF ACCELERATIONS.

Train:	0.1-0.3 m/s^2
Car:	3-8 m/s^2
Space shuttle:	$\leq 3G = 30m/s^2$
Combat plane (F16) (blackout):	9G=90 m/s^2
Ejection from F16:	14G=140 m/s^2.
Free fall:	1G = 9.81 m/s^2
Electron in vacuum tube:	10^{15} m/s^2

DIFFERENTIATION RULES.
The rules in one dimensions $(f + g)' = f' + g'$ $(cf)' = cf'$, $(fg)' = f'g + fg'$ (Leibniz), $(f(g))' = f'(g)g'$ (chain rule) generalize for vector-valued functions: $(\vec{v} + \vec{w})' = \vec{v}\,' + \vec{w}\,'$, $(c\vec{v})' = c\vec{v}\,'$, $(\vec{v} \cdot \vec{w})' = \vec{v}\,' \cdot \vec{w} + \vec{v} \cdot \vec{w}\,'$ $(\vec{v} \times \vec{w})' = \vec{v}\,' \times \vec{w} + \vec{v} \times \vec{w}\,'$ (Leibniz), $(\vec{v}(f(t)))' = \vec{v}\,'(f(t))f'(t)$ (chain rule). The Leibniz rule for the triple dot product $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is $d/dt[\vec{u}, \vec{v}, \vec{w}] = [\vec{u}\,', \vec{v}, \vec{w}] + [\vec{u}, \vec{v}\,', \vec{w}] + [\vec{u}, \vec{v}, \vec{w}\,']$ (see homework).

INTEGRATION. If $\vec{r}\,'(t)$ and $\vec{r}(0)$ is known, we can figure out $\vec{r}(t)$ by **integration** $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(s)\, ds$.

Assume we know the acceleration $\vec{a}(t) = \vec{r}\,''(t)$ as well as initial velocity and position $\vec{r}\,'(0)$ and $\vec{r}(0)$. Then $\vec{r}(t) = \vec{r}(0) + t\vec{r}\,'(0) + \vec{R}(t)$, where $\vec{R}(t) = \int_0^t \vec{v}(s)\, ds$ and $\vec{v}(t) = \int_0^t \vec{a}(s)\, ds$.

EXAMPLE. Shooting a ball. If $\vec{r}\,''(t) = \langle 0, 0, -10 \rangle$, $\vec{r}\,'(0) = \langle 0, 1000, 2 \rangle$, $\vec{r}(0) = \langle 0, 0, h \rangle$, then $\vec{r}(t) = \langle 0, 1000t, h + 2t - 10t^2/2 \rangle$.