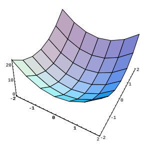
CRITICAL POINTS — Maths21a, O. Knill

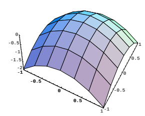
CRITICAL POINTS. A point (x_0, y_0) in a region G is called a **critical point** of $f(x,y)$ if $\nabla f(x_0,y_0) = (0,0)$. Remarks. Critical points are also called **stationary points**. Critical points are candidates for extrema because at critical points, the directional derivative is zero. It is usually assumed that f is differentiable.

EXAMPLE 1. $f(x,y) = x^4 + y^4 - 4xy + 2$. The gradient is $\nabla f(x,y) = (4(x^3-y), 4(y^3-x))$ with critical points $(0,0), (1,1), (-1,-1)$.
EXAMPLE 2. $f(x,y) = \sin(x^2+y) + y$. The gradient is $\nabla f(x,y) = (2x\cos(x^2+y), \cos(x^2+y)+1)$. For a critical points, we must have $x = 0$ and $\cos(y) + 1 = 0$ which means $\pi + k2\pi$. The critical points are at $(0,\pi), (0,3\pi), \dots$.
EXAMPLE 3. ("volcano") $f(x,y) = (x^2+y^2)e^{-x^2-y^2}$. The gradient $\nabla F = (2x - 2x(x^2+y^2), 2y - 2y(x^2+y^2))e^{-x^2-y^2}$ vanishes at $(0,0)$ and on the circle $x^2+y^2 = 1$. There are ∞ many critical points.
EXAMPLE 4 ("pendulum") $f(x,y) = -g\cos(x) + y^2/2$ is the energy of the pendulum. The gradient $\nabla F = (y, -g\sin(x))$ is $(0,0)$ for $x = 0, \pi, 2\pi, \dots, y = 0$. These points are equilibrium points, where the pendulum is at rest.
EXAMPLE 5 ("Volterra Lodka") $f(x,y) = a\log(y) - by + c\log(x) - dx$. (This function is left invariant by the flow of the Volterra Lodka differential equation $\dot{x} = ax - bxy, \dot{y} = -cy + dxy$ which you might have seen in Math1b.) The point $(c/d, a/b)$ is a critical point.

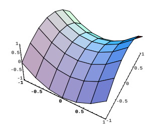
TYPICAL EXAMPLES.



$f(x,y) = x^2 + y^2$

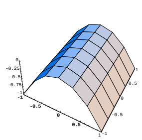


$f(x,y) = -x^2 - y^2$

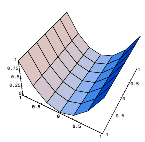


$f(x,y) = x^2 - y^2$

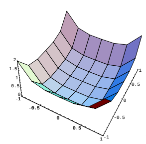
EXAMPLES WITH DISCRIMINANT $D = \det(H) = 0$.



$f(x,y) = x^2$

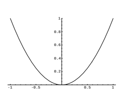


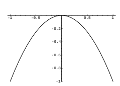
$f(x,y) = -x^2$

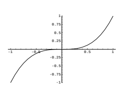


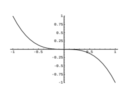
$f(x,y) = x^4 + y^4$

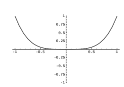
CLASSIFICATION OF CRITICAL POINTS IN 1 DIMENSION.
$f'(x) = 0, f''(x) > 0$, local minimum, $f''(x) < 0$ local maximum, $f'' = 0$ undetermined.











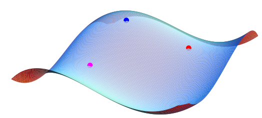
CLASSIFICATION OF CRITICAL POINTS: SECOND DERIVATIVE TEST. Let $f(x,y)$ be a function of two variables with a critical point (x_0, y_0). Define $D = f_{xx}f_{yy} - f_{xy}^2$, called the **discriminant** or **Hessian**. (Remark: With the **Hessian matrix** $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ we can write $D = \det(H)$ as a **determinant**.

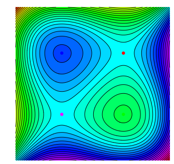
If $D > 0$ and $H_{11} > 0 \Rightarrow$ local minimum (bottom of valley)
If $D > 0$ and $H_{11} < 0 \Rightarrow$ local maximum (top of mountain).
If $D < 0 \Rightarrow$ saddle point (mountain pass).

In the case $D = 0$, we would need higher derivatives to determine the nature of the the critical point.

EXAMPLE. (A "napkin").
The function $f(x,y) = x^3/3 - x - (y^3/3 - y)$ has the gradient $\nabla f(x,y) = (x^2-1, -y^2+1)$. It is the zero vector at the 4 critical points $(1,1)$,$(-1,1)$,$(1,-1)$ and $(-1,-1)$. The Hessian matrix is $H = f''(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & -2y \end{bmatrix}$.

$H(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(-1,1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$	$H(1,-1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$H(-1,-1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$
$D = -4$ Saddle point	$D = 4, f_{xx} = -2$ Local maximum	$D = 4, f_{xx} = 2$ Local minimum	$D = -4$ Saddle point





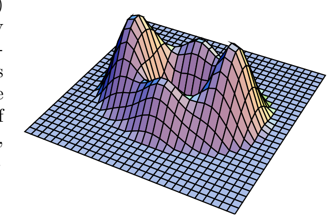
GLOBAL MAXIMA AND MINIMA. To determine the maximum or minimum of $f(x,y)$ on a domain, determine all critical points **in the interior the domain**, and compare their values with maxima or minima **at the boundary**.

(A point is in the interior of G, if there is a small disc around (x_0, y_0) contained in G. A point is at the boundary of G, if any disc aound (x_0, y_0) contains both points in G and in the complement).

EXAMPLE 5. Find the critical points of $f(x,y) = 2x^2 - x^3 - y^2$. With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3, 0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3, 0)$ is a local maximum.

WHY DO WE CARE ABOUT CRITICAL POINTS?
- Critical points are candidates for extrema like maxima or minima.
- Knowing all the critical points and their nature tells alot about the function.
- Critical points are physically relevant. Examples are configurations with lowest energy).

A CURIOUS OBSERVATION: (The island theorem) Let $f(x,y)$ be the height on an island. Assume there are only finitely many critical points on the island and all of them have nonzero determinant. Label each critical point with a $+1$ "charge" if it is a maximum or minium, and with -1 "charge" if it is a saddle point. Sum up all the charges and you will get 1, independent of the function. This property is an example of an "index theorem", a prototype for important theorems in physics and mathematics.



CRITICAL POINTS IN PHYSICS. (informal) Most physical laws are based on the principle that the equations are critical points of a functional (in general in infinite dimensions).
- **Newton equations**. (Classical mechanics) A particle of mass m moving in a field V along a path $\gamma : t \mapsto r(t)$ extremizes the integral $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\ dt$. Critical points γ satisfy the Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$.
- **Maxwell equations**. (Electromagnetism) The electromagnetic field (E, B) extremizes the Integral $S(E,B) = \frac{1}{8\pi}\int (E^2 - B^2)\ dV$ over space time. Critical points are described by the the Maxwell equations in vacuum.
- **Einstein equations** (General relativity) If g is a dot product which depends on space and time, and R is the "curvature" of the corresponding curved space time, then $S(g) = \int\int_R dV(g)$ is a function of g for which critical points g satisfy the Einstein equations in general relativity.

OTHER WAYS TO FIND CRITICAL POINTS. Some ideas: walk in the direction of the gradient until you reach a local maximum or walk backwards to reach a local minima. To find saddle points, consider the shortest path connecting two local minima and take the maximum along this path.