8/17/2004 FINAL EXAM **Maths 21a, O. Knill, Summer 2004**

Name:

- Start by printing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- Problems 1-3 do not require any justifications. For the rest of the problems you have to show your work.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points)

T F The distance of the point $(3,1,6)$ and the point $(1,1,1)$ is $\sqrt{29}$.

Solution:
True. By definition.

T F $\int_0^2 \int_0^{2\pi} (r^2/2) \; d\theta \; dr$ computes the area of a disc of radius 2 in the plane.

Solution:
False. The correct formula is $\int_0^{2\pi} \int_0^2 r \; dr \; d\theta$

T F The identity $\vec{v} \cdot \vec{w} \geq |\vec{v}| \cdot |\vec{w}|$ is always true.

Solution:
False. The inequality is the other way round.

T F The velocity and acceleration vectors of a curve $\vec{r}(t) = (x(t), y(t))$ are always perpendicular.

Solution:
False. This is already false for a line $r(t) = (t^2, t^2)$, where the velocity and acceleration are parallel.

T F A circle of radius 5 has a smaller curvature than a circle of radius 1.

Solution:
True. The curvature of a circle of radius r is equal to $1/r$.

T F The curve $\vec{r}(t) = (-\sin(t), \cos(t))$ for $t \in [0, 2\pi]$ is a circle.

Solution:
True. Indeed, one can check that $\sin^2(t) + \cos^2(t) = 1$.

T | F (marked: F)

The function $\sin(x-t)$ is a solution of the Burger equation $u_t = uu_x + u_{xx}$.

Solution:
False, $uu_x = \sin(x-t)\cos(x-t)$, $u_t = -\cos(x-t)$ and $u_{xx} = -\sin(x-t)$. For $t=0$, the two sides are different.

T | F (marked: F)

The length of the curve $\vec{r}(t) = (t^3, t^2)$ on $[1,2]$ is $\int_1^2 (9t^4 + 4t^2)\, dt$.

Solution:
False. The derivatives have to be squared. The correct answer would be $\int_1^2 \sqrt{9t^4 + 4t^2}\, dt$.

T | F (marked: F)

Let (x_0, y_0) be the maximum of $f(x,y)$ under the constraint $g(x,y) = 1$. Then the gradient of g at (x_0, y_0) is perpendicular to the gradient of f at (x_0, y_0).

Solution:
The gradients are **parallel**, not perpendicular.

T | F (marked: F)

The directional derivative $D_{\vec{v}} f(x_0, y_0, z_0)$ of $f(x,y,z) = x^2 + y^2 - z^2$ into the direction $\vec{v} = (0,0,1)$ is negative at every point (x_0, y_0, z_0).

Solution:
This directional derivative is $f_z = -2z$. For $z > 0$, this is negative, for $z < 0$, this is positive.

T | F (marked: T)

If a vector field $F(x,y)$ is not conservative, we always can find a curve C for which the line integral $\int_C F \cdot dr$ is positive.

Solution:
Indeed. There must exist then a curve for which the line integral is not zero. If this line integral is positive, we have found our curve, if it is negative, we reverse the direction of the curve.

T | F (marked: T)

If C is a closed level curve of a function $f(x,y)$ and $F = (f_x, f_y)$ is the gradient field of f, then $\int_C F \cdot dr = 0$.

Solution:
The gradient field is perpendicular to the level curves.

T | F (marked: F)

The divergence of the gradient of any function $f(x,y,z)$ is always zero.

Solution:
No, just take a simple example like $f(x,y,z) = x^2$, where $\mathrm{div}(\mathrm{grad}(f)) = 2$.

T | F (marked: F)

The function $f(x,y) = x^2 y^3$ has no critical points.

Solution:
$\nabla f(x,y) = (2xy^3, 3x^2y^2)$, which vanishes at $(0,0)$.

T | F (marked: T)

If $F(x,y) = (y, 2x)$ and $C : \vec{r}(t) = (\sqrt{\cos(t)}, \sqrt{\sin(t)})$ parameterizes the boundary of the region $R : x^4 + y^4 \leq 1$, then $\int_C F \cdot\, ds$ is the area of R.

Solution:
This is a direct consequence of Green's theorem and the fact that the two-dimensional curl $Q_x - P_y$ of $F = (P, Q)$ is equal to 1.

T | F (marked: T)

The flux of the vector field $F(x,y,z) = (0, y, 0)$ through the boundary S of a solid torus E is equal to the volume of the torus.

Solution:
It is the **volume** of the solid torus.

T | F (marked: T)

The quadratic surface $x^2 + y^2 + 4x - z^2 = -3$ is a one sheeted hyperboloid.

Solution:
Completion of the square gives the equation $(x+2)^2 + y^2 - z^2 = 1$.

T | F (marked: T)

If $\vec{F}$ is a vector field in space and S is the boundary of a solid torus, then the flux of $\mathrm{curl}(\vec{F})$ through S is 0.

Solution:
This is true by Stokes theorem.

T	F (highlighted)

If $\text{div}(\vec{F})(x,y,z) = 0$ for all (x,y,z) and S is a torus surface, then the flux of F through S is zero.

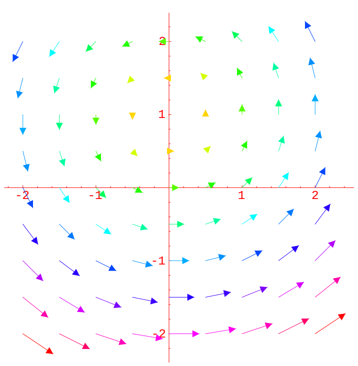
Solution:
This is a consequence of the divergence theorem.

T (highlighted)	F

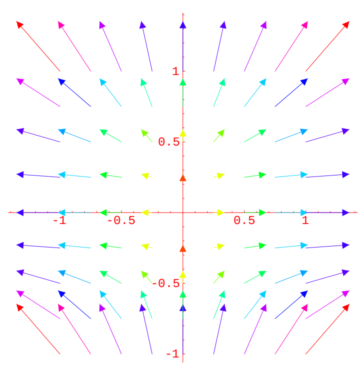
In spherical coordinates, the equation $\rho\cos(\phi) = \rho\sin(\theta)\sin(\phi)$ defines a plane.

Solution:
True. It is the plane $z = y$.

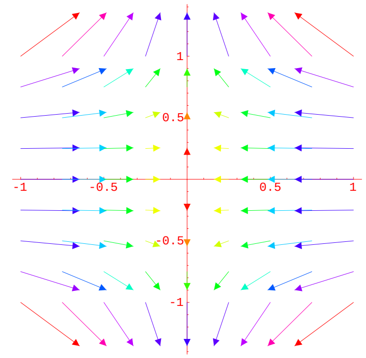
Problem 2) (10 points)



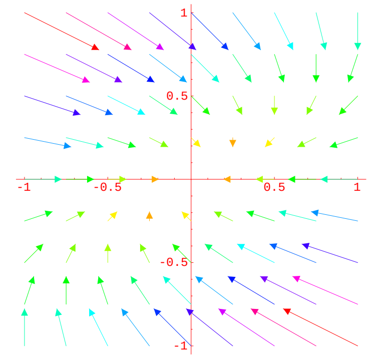
I



II



III



IV

Enter I,II,III,IV here	Vector field
	$F(x,y) = (1-y, x)$
	$F(x,y) = (x, y^2)$
	$F(x,y) = (-x, y^3)$
	$F(x,y) = (y-x, -y)$

Solution:

Enter I,II,III,IV here	Vector field
I	$F(x,y) = (1-y, x)$
II	$F(x,y) = (x, y^2)$
III	$F(x,y) = (-x, y^3)$
IV	$F(x,y) = (y-x, -y)$

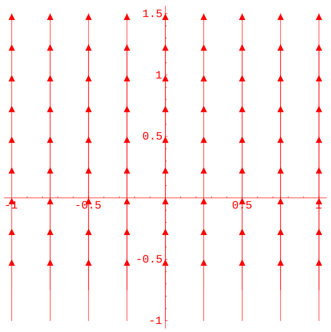
Problem 3) (10 points)

In this problem, vector fields F are written as $F = (P, Q)$. We use abbreviations $\text{curl}(F) = Q_x - P_y$. When stating $\text{curl}(F) = 0$, we mean that $\text{curl}(F)(x, y) = 0$ vanishes for **all** (x, y). Similarly, we say $\text{div}(F) = P_x + Q_y = 0$ if $\text{div}(F)(x, y) = (P_x(x, y) + Q_x(x, y)) = 0$ for all x, y.

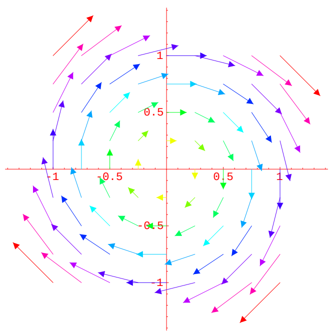
Check the box which match the formulas of the vectorfields with the corresponding picture I,II,III or IV and mark also the places, indicating the vanishing of curl(F).

Vectorfield	I	II	III	IV	curl(F) = 0	div(F) = 0
$\mathbf{F}(x, y) = (2, x)$						
$\mathbf{F}(x, y) = (y, -x)$						
$\mathbf{F}(x, y) = (0, 5)$						
$\mathbf{F}(x, y) = (-x, -y)$						

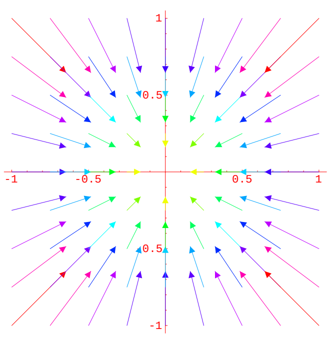
I



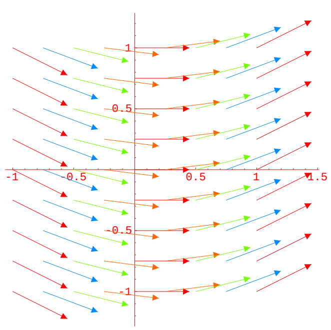
II



III



IV



Solution:

Vectorfield	I	II	III	IV	curl(F) = 0	div(F) = 0
$\mathbf{F}(x, y) = (2, x)$				X		X
$\mathbf{F}(x, y) = (y, -x)$		X				X
$\mathbf{F}(x, y) = (0, 5)$	X				X	X
$\mathbf{F}(x, y) = (-x, -y)$			X		X	

Problem 4) (10 points)

Find the distance of the point $P = (2, 3, 2)$ to the plane containing the points $A = (0, 0, 1)$, $B = (1, 3, 1)$ and $C = (-2, 2, 1)$.

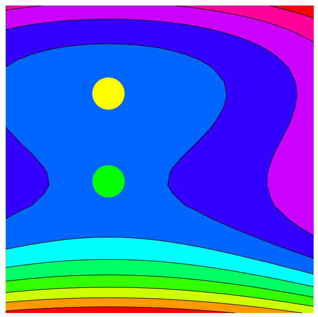
Solution:
The vectors $\vec{v} = B - A = (1, 3, 0)$ and $\vec{w} = C - A = (-2, 2, 0)$ are in the plane. Their cross product is $\vec{n} = (0, 0, 8)$. The distance is the length of the scalar projection of $P - A$ onto $\vec{n}$ which is $(2, 3, 1) \cdot \vec{n}/|\vec{n}| = \boxed{1}$.

Problem 5) (10 points)

Find all the critical points of the function $f(x, y) = y^3 - 3y^2 + 4x + x^2 - 3$ and classify them by telling whether they are local maxima, local minima or saddle points.

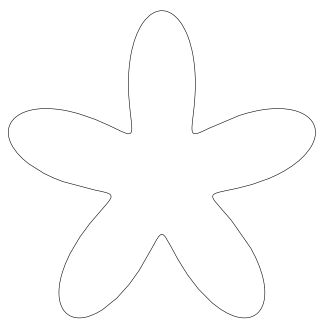
Solution:
The critical points are $P = (-2, 0)$ and $Q = (-2, 2)$. The Discriminant at P is $D = -12$ so that P is a saddle point. The Hessian at Q is 12 and $f_{xx} = 2$ which is a local minimum.

$P = (-2, 0)$	D=-12	Saddle point
$Q = (-2, 2)$	D=12, $f_{xx} = 2$	local minimum



Problem 6) (10 points)

Find the area $\int\int_R 1\ dxdy$ of the "ginger-man figure" R, enclosed by the polar curve $r(\theta) = 2 + \sin(5\theta)$, where $\theta \in [0, 2\pi]$.

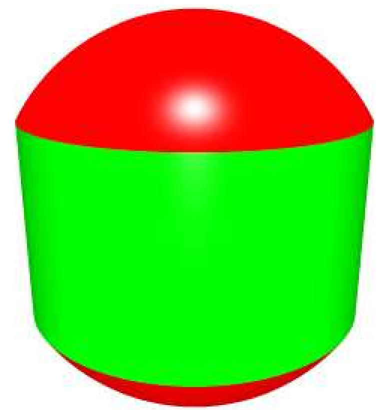


Solution:
$\int_0^{2\pi}\int_0^{2+\sin(5\theta)} r\ drd\theta = \int_0^{2\pi}(\sin^2(5\theta)+4+4\sin(5\theta))/2\ d\theta$ which is equal to $(\pi+8\pi)/2 = 9\pi/2$ $\boxed{= 4.5\pi}$.

Problem 7) (10 points)

Find the volume of the intersection of the solid cylinder $x^2 + y^2 \leq 3/4$ with the solid ball enclosed by the sphere $x^2 + y^2 + z^2 \leq 1$.

Hint. Draw a picture and chose your coordinate system carefully.



Solution:
Use cylindrical coordinates:

$$\int_0^{\sqrt{3}/2}\int_0^{2\pi} 2\sqrt{1-r^2}r\ d\theta dr = -2\pi\frac{2}{3}(1-r^2)^{3/2}|_0^{\sqrt{3}/2} = -2\pi\frac{2}{3}((1/4)^{3/2}-1) = 7\pi/6\ .$$

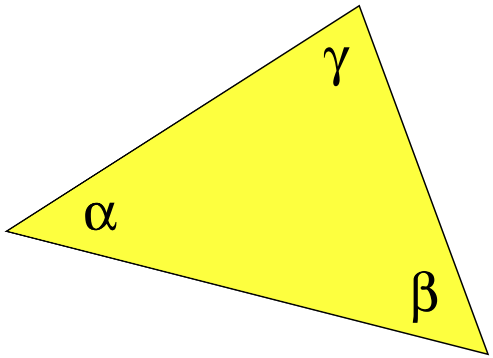
The result is $\boxed{7\pi/6}$.

Problem 8) (10 points)

What is the shape of the triangle with angles α, β, γ for which

$$f(\alpha, \beta, \gamma) = \log\left(\sin(\alpha)\sin(\beta)\sin(\gamma)\right)$$

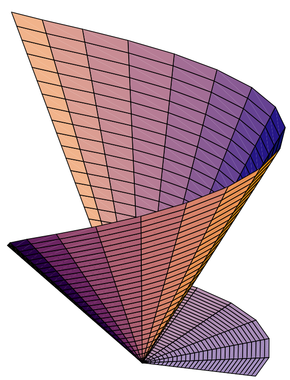
is maximal? Note that the angles of a triangle must satisfy some relation.



Solution:
The Lagrange equations are $\cot(\alpha) = \lambda, \cot(\beta) = \lambda, \cot(\gamma) = \lambda$. Because α, β, γ are all in $[0, \pi]$, we conclude that all are the same. From the last equation follows $\boxed{\alpha = \beta = \gamma = \pi/3}$ and $\sin(\alpha)\sin(\beta)\sin(\gamma) = (\sqrt{3}/2)^3$.

Problem 9) (10 points)

A surface S is parametrized as $\vec{r}(t, s) = (s\cos(t), s\sin(t), st)$ where $t \in [0, 3\pi]$ and $s \in [0, 1]$. Find the surface area of the surface S.



Solution:
$r_t = (-s\sin(t), s\cos(t), s)$ and $r_s = (\cos(t), \sin(t), t)$. We have $|r_t \times r_s| = s\sqrt{(2 + t^2)}$. We obtain the integral $\boxed{\frac{1}{2}\int_0^{2\pi} \sqrt{2 + t^2}\, dt}$ (full credit). It can be solved explicitly: using integration by parts with $du = 1, v = \sqrt{2 + t^2})$ so that $\int \sqrt{2 + t^2}\ dt = t\sqrt{2 + t^2}/2 + \text{arcsinh}(t/\sqrt{2}$. The definite integral is $\boxed{3\pi\sqrt{2 + 9\pi^2}/4 + \text{arcsinh}(3\pi/\sqrt{2}}$.

Problem 10) (10 points)

Find the equation of the tangent plane to the hyperbolic paraboloid $x^2 - y^2 = z$ at the point $(2, 1, 3)$.

Solution:
The gradient of $g(x, y, z) = x^2 - y^2 - z = 0$ is $(2x, -2y, -1)$ which is $(4, -2, -1)$ at the given point. The plane has the equation $4x - 2y - z = d$ where $d = 4 \cdot 2 - 2 \cdot 1 - 3 = 3$. The result is $\boxed{4x - 2y - z = 3}$.

Problem 11) (10 points)

Use Green's theorem to find the line integral of the vectorfield $F(x, y) = (-y + e^{xy}y, e^{xy}x)$ along the boundary C of the triangle R with vertices $A = (0, 0), B = (1, 0)$ and $C = (1, 1)$. (The path C connects the points in the order $A \to B \to C \to A$.)

Solution:
The curl is 1 so that by Green, the line integral is the area of R which is $\boxed{1/2}$.

Problem 12) (10 points)

Let S be the upper half (=the part contained in $\{\ z > 0\ \}$) of the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$$

oriented so that the normal vectors on S point upwards and let $F = (-y, x, 3)$ be a vectorfield. Find the flux of $\text{curl}(F)$ through S using Stokes theorem.

Solution:
We have to compute the line integral of F along the boundary which is the ellipse $x^2/9 + y^2/4$ in the xy-plane. This ellipse is parametrized by $\vec{r}(t) = (3\cos(t), 2\sin(t), 0)$ so that $F(r(t)) \cdot \vec{r}'(t) = (-2\sin(t), 3\cos(t), 3) \cdot (-3\sin(t), 2\cos(t), 0) = 6\sin^2(t) + 6\cos^2(t) = 6$ and the line integral is $\boxed{12\pi}$.

Problem 13)	(10 points)

Use the divergence theorem to compute the flux of the vector field

$$F(x, y, z) = (3x + 5y^2z, x, y^3 + \sin(xy^3))$$

through the boundary S of the rectangular cube $E = \{1 \leq x \leq 3, 2 \leq y \leq 5, -1 \leq z \leq 1\}$.

Solution:
The divergence is 3 so that the flux integral is 3 times the volume of the box which is $\boxed{36}$.