This is part 1 (of 3) of the weekly homework. It is due Monday August 16 at the review.

SUMMARY.

- $F(x,y) = (P(x,y), Q(x,y))$ **vector field in the plane**.
- $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)dt$ **line integral** of F along curve $C : t \mapsto r(t)$.
- Example: $C : r(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$ (circle), $F(x,y) = (-y, x)$. $\int_C F \cdot dr = \int_F (r(t)) \cdot r'(t)\, dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\, dt = \int_0^{2\pi} 1\, dt = 2\pi$.
- $\mathrm{curl}(P,Q) = Q_x - P_y$. **Curl** for 2D vector fields.
- C **positively oriented boundary** of the region D:
(the region is "to the left" when you follow the boundary).
- $\boxed{\int_C F \cdot dr = \int \int_D \mathrm{curl}(F)\, dA}$ **Greens theorem**.
Written out: $\int_C F(r(t)) \cdot r'(t)\, dt = \int \int_D (Q_x - P_y)\, dxdy$.
- $\int_C x\, dy$ **area** of D, C is the positively oriented boundary of D:
Example: C unit circle: $x(t) = \cos(t), dy = \cos(t)dt$, Area $= \int_0^{2\pi} \cos^2(t)\, dt = \pi$.

Homework Problems

1) (4 points) Calculate the line integral $\int_C 2(y + x\sin(y), x^2\cos(y) - 3y^2)\, dr$ along a triangle C with edges $(0,0)$, $(1,0)$ and $(1,1)$ using Green's theorem.

Solution:
$\mathrm{curl}(F)(x,y) = 4x\cos(y) - 2 - 2x\cos(y) = 2x\cos(y) - 2$. By Green's theorem, we have to integrate this function over the region R enclosed by the triangle:

$$\int_0^1 \int_0^x 2x\cos(y) - 2\, dydx = \int_0^1 (2x\sin(x) - 2x)\, dx = -1 - 2\cos(1) + 2\sin(1)\ .$$

2) (4 points) Evaluate the line integral of the vector field $F(x,y) = (xy^2, x^2)$ along the rectangle with vertices $(0,0), (2,0), (2,3), (0,3)$ in two ways. Do this by calculating the line integral as well as using Greens theorem.

Solution:
Integrating $\mathrm{curl}(F) = 2x - 2xy$ over the rectangle gives

$$\int_0^2 \int_0^3 2x - 2xy\, dydx = \int_0^2 6x - 9x\, dx = -12/2 = -6$$

The line integral is a sum of four line integrals: $\int_0^2 (t0^2, t^2) \cdot (1,0)\, dt = 0$ and $\int_0^3 (2t^2, 4) \cdot (0,1)\, dt = 12$ and $-\int_0^2 (9t, t^2) \cdot (1,0)\, dt = -18$ as well as $\int_0^3 (0t^2, 0) \cdot (0,1)\, dt = 0$. The sum is also -6.

3) (4 points) Find the area of the region bounded by the hypocycloid $\vec{r}(t) = (\cos^3(t), \sin^3(t))$ using Green's theorem. The curve is parameterized by $t \in [0, 2\pi]$.

Solution:
Take a vector field $F(x,y) = (0,x)$ which has the curl 1. Then by Green the area is the line integral $\int_0^{2\pi}(0,\cos^3(t))\cdot(-3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t))\ dt = 3\int_0^{2\pi}\cos^4(t)\sin^2(t)\ dt = 3\int_0^{2\pi}\sin^2(2t)/4(\cos(2t)+1)/2 = 3/8\pi$.

4) (4 points) Let $F(x,y) = (-y/(x^2+y^2), x/(x^2+y^2))$. Let $C : \vec{r}(t) = (\cos(t), \sin(t)), t \in [0, 2\pi]$.
a) What is $\int_C F \cdot dr$?
b) Show that $\text{curl}(F) = 0$ everywhere for $(x,y) \neq (0,0)$.
c) Let $f(x,y) = \arctan(y/x)$. Verify that $\nabla f = F$.
d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property?

Hint: is f really a continuous potential?

Solution:
a) 2π.
b) Direct differentiation $Q_x(x,y) - P_y(x,y) = 0$.
c) Use $\arctan'(x) = 1/(1+x^2)$.
d) The function $f(x,y)$ is not continuous everywhere. Also the vector field $F(x,y)$ is not smooth everywhere. There is a singularity at $(0,0)$.

5) a) Verify that if C is the line segment connecting the point (x_1,y_1) to the point (x_2,y_2), and F is the vector field $F(x,y) = (-y,x)$ then $\int_C F \cdot dr = (x_1y_2 - x_2y_1)$.

b) Use a) to verify that if $(x_1,y_1), ..., (x_n,y_n)$ are the vertices of a polygon in the plane, then $A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + ... + (x_ny_1 - x_1y_n)]$ is the area of the polygon.

Solution:
Parametrize the segment by $\vec{r}(t) = (x_1,y_1) + t(x_2-x_1, y_2-y_1)$ and get $F(\vec{r}(t)) = (-y_1 - t(y_2-y_1), x_1 + t(x_2-x_1)$ and $\vec{r}'(t) = (x_2-x_1, y_2-y_1)$. So, $\int_0^1 F(r(t)) \cdot r'(t)\ dt = (y_2-y_1)(x_1 + (x_2-x_1)^2/2 + (x_2-x_1)(y_1 + (y_2-y_1)^2/2$.
b) Sum the result in a) from $i = 1$ to $i = n$.

Remarks

(You don't need to read these remarks to do the problems.)

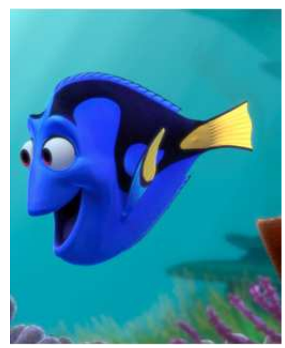
To problem 4: This vector field is important in fluid dynamics. It models a single **vortex**. The curl of F is zero everywhere except at the origin. Physisists would say that that curl(F) is a "delta function" located at the origin. In fluid dynamics, one can model fluids using a finite set of vortices. This homework problem is also crucial if you want to solve Nash's problem.

To problem 5: This formula is actually used in applications like for example in ray tracing which is a CPU time intensive task. The software has to compute the light ray paths bouncing around in a virtual world, compute reflections or refractions. Tracing an image can take from a few seconds to days. A single frame in movies like "Toy Story" took several hours to render. To get the large number of frames needed for a movie companies like "Pixar" (recently again visible with "Finding Nemo") use "computer farms" a huge number of workstations.

To compute the normal vector to a polygon, an area formula is used which is derived from Green's theorem. This formula is also used to compute normals to a surface. Computing the normal from three points on the surface only is error prone. It is often better to consider a polygon $P_i = (x_i, y_i, z_i)$ on the surface. What is the normal to such a polygon? Note that the points P_i are not necessarily on a plane. The xy projection of the polygon gives a polygon (x_i, y_i) in the plane which has the area $1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k)$ a formula derived from Green's theorem.

The normal vector to a not necessarily planar polygon $P_i = (x_i, y_i, z_i)$ in space is defined as

$$n = \begin{bmatrix} 1/2 \sum_k (y_{k+1} + y_k)(z_{k+1} - z_k) \\ 1/2 \sum_k (z_{k+1} + z_k)(x_{k+1} - x_k) \\ 1/2 \sum_k (x_{k+1} + x_k)(y_{k+1} - y_k) \end{bmatrix} .$$



Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) The planimeter calculates the area: the **planimeter vector field** $F(x,y) = (P(x,y), Q(x,y))$ is defined by attaching a unit vector orthogonal to the vector $(x-a, y-b)$ at (x,y), where (a,b) is the "knee" of the planimeter. The wheel rotation is the line integral of F along the boundary of R. By **Green's theorem**, this integral is the double integral of curl(F) over R. The planimeter vector field is explicitly given by $F(x,y) = (P(x,y), Q(x,y)) = (-(y - b(x,y)), (x - a(x,y)))$. Furthermore, $\text{curl}(F) = Q_x - P_y$ is equal to $2 + (-a_x - b_y)$ which is 2 plus the curl of the vector field $G(x,y) = (b(x,y), -a(x,y))$. Show that $\text{curl}(G) = -1$. For more information see http://www.math.duke.edu/education/ccp/materials/mvcalc/green/

2) Let D be a region bounded by a simple closed path C in the plane. Use Green's theorem to prove that the coordinates of the center of mass are $(\int_C x^2 dy/(2A), -\int_C y^2 dx)/(2A)$, where A is the area of D.