This is part 3 (of 3) of the weekly homework. It is due July 27 at the beginning of class.

SUMMARY.

- Extremize $f(x,y)$ under the constraint $g(x,y) = c$: Solve $g(x,y) = c, \nabla f(x,y) = \lambda \nabla g(x,y)$ with **Lagrange multiplier** λ, These are 3 equations for 3 unknowns x, y, λ:

$$\begin{aligned} f_x(x,y) &= \lambda g_x(x,y) \\ f_y(x,y) &= \lambda g_y(x,y) \\ g(x,y) &= c \end{aligned}$$

In three dimensions, the Lagrange equations form 4 equations for 4 unknowns.

$$\begin{aligned} f_x(x,y,z) &= \lambda g_x(x,y,z) \\ f_y(x,y,z) &= \lambda g_y(x,y,z) \\ f_z(x,y,z) &= \lambda g_z(x,y,z) \\ g(x,y,z) &= c \end{aligned}$$

Homework Problems

1) (4 points) Find the extrema of the function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ on the circle $g(x,y) = x^2+y^2 = 4$ using the method of Lagrange multipliers.

Solution:
The Lagrange equations are

$$\begin{aligned} (2x - 2x^3 - 4xy^2)e^{-x^2-y^2} &= \lambda 2x \\ (4y - 4y^3 - 2x^2y)e^{-x^2-y^2} &= \lambda 2y \\ x^2+y^2 &= 4\ . \end{aligned}$$

Case 1: If $x = 0$, then the first equation is ok and we get from the third equation $y = \pm 2$.
Case 2: If $y = 0$, then the second equation is ok and we get from the third edquation $x = \pm 2$.
Case 3: If $x = 0$ and $y = 0$, then the first two equations are ok, but clashes with the third. Forget this case.
Case 4: If both x and y are not zero we can divide the first equation by $2x$ and the second by $2y$. We also replace $-x^2 - y^2$ by -4

$$\begin{aligned} (1 - x^2 - 2y^2)e^{-4} &= \lambda \\ (2 - 2y^2 - x^2)e^{-4} &= \lambda \\ x^2+y^2 &= 4\ . \end{aligned}$$

But setting the first two equations equal leads to a contradiction. Also this case 4) has no solutions. We end up with the four solutions $(2,0), (-2,0), (0,2), (0,-2)$. The minimal values are $f(\pm 2, 0) = 4e^{-4}$, the maximal values are $f(0, \pm 2) = 8e^{-4}$.

2) (4 points) Find the extrema of the same function $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$ as in the previous problem but now on the entire disc $\{x^2+y^2 \leq 4\}$ of radius 2.

Solution:
In the last homework we have seen that the equation $\nabla f = ((2x - 2x^3)e^{-x^2-y^2}, (4y - 4y^3)e^{-x^2-y^2}) = (0,0)$ has the solutions $x = 0, y = 0$, $x = 0, y = \pm 1$, $y = 0, x = \pm 1$. Together with the previous problem, we can now make a list of all the candidates for extrema

1. Extrema inside

point	$f =$
(0,0)	0
(1,0)	1/e
(-1,0)	1/e
(0,1)	2/e
(0,-1)	2/e

2. Extrema on the boundary

point	$f =$
(2,0)	$4/e^4$
(-2,0)	$4/e^4$
(0,2)	$8/e^4$
(0,-2)	$8/e^4$

We see that the origin is the minimum and the points $(0, \pm 1)$ are both the maxima.

3) (4 points) Find the points (x, y, z) on the surface $g(x, y, z) = xy^2 - z^3 - 2 = 0$ that are closest to the origin $(0, 0, 0)$.

Solution:
Instead of extremizing the distance $\sqrt{x^2+y^2+z^2}$ we extremize the function $f(x,y) = x^2+y^2+z^2$. We have the Lagrange equations

$$\begin{aligned} 2x &= \lambda y^2 \\ 2y &= \lambda 2xy \\ 2z &= -\lambda 3z^2 \\ xy^2 &= z^3+2 \end{aligned}$$

1. Case: $z = 0$. We can then not have $y = 0$ nor $x = 0$ and end up with

$$\begin{aligned} 2x &= \lambda y^2 \\ 2 &= \lambda 2x \\ xy^2 &= 2 \end{aligned}$$

which gives $2 = \lambda^2 y^2, x^2 = \lambda$ and so that $x = 1, y = \pm\sqrt{2}$. 2. Case: $x = 0$. Gives $y = 0$ and $z = -2^{1/3}$.
3. Case: $y = 0$. Gives $x = 0$ and $z = -2^{1/3}$.
4. Case: all x, y, z are nonzero. Then

$$\begin{aligned} 2x &= \lambda y^2 \\ 1 &= \lambda x \\ 2 &= -\lambda 3z \\ xy^2 &= z^3+2 \end{aligned}$$

Eliminating $\lambda = 1/x$ gives

$$\begin{aligned} 2x^2 &= y^2 \\ 2 &= -3z/x \\ xy^2 &= z^3+2 \end{aligned}$$

Solving for $y^2 = 2x^2$ and $z = -2x/3$ from the first two equations and plugging this into the third gives $2x^3 = -8x^3/27+2$ gives $x = 3/31^{(1/3)}$ and so $y = \pm\sqrt{2}(3/31^{(1/3)})$.
The distance from the point $(0,0,-2^{(1/3)})$ to the origin is $2^(1/3) \sim 1.25$ the distance from the points $(3/31^{1/3}(1, \pm\sqrt{2}, -2/3)$ to the origin is $31^{1/6} \sim 1.7723$. The first one is the minimum.

4) (4 points) Let a, b, c be non-negative constants and let F be the function $F(x,y,z) = -x\log(x) - y\log(y) - z\log(z) - ax - by - cz$. Find the maxima and minima of F on $x > 0, y > 0, z > 0$ under the constraint $x+y+z = 1$.

Solution:
The Lagrange equations are

$$\begin{aligned} -\log(x) - 1 - a &= \lambda \\ -\log(x) - 1 - b &= \lambda \\ -\log(x) - 1 - c &= \lambda \\ x + y + z &= 1 \end{aligned}$$

From the first three equations, we get

$$\begin{aligned} x = e^{-(1+a+\lambda)} &= e^{-1-\lambda}e^{-a} \\ y = e^{-(1+b+\lambda)} &= e^{-1-\lambda}e^{-b} \\ z = e^{-(1+c+\lambda)} &= e^{-1-\lambda}e^{-c} \end{aligned}$$

Plugging this into the fourth equation gives $e^{-1-\lambda}(e^{-a} + e^{-b} + e^{-c}) = 1$ so that $e^{-1-\lambda} = (e^{-a} + e^{-b} + e^{-c})^{-1}$ and

$$\begin{aligned} x &= e^{-a}/(e^{-a} + e^{-b} + e^{-c}) \\ y &= e^{-b}/(e^{-a} + e^{-b} + e^{-c}) \\ z &= e^{-c}/(e^{-a} + e^{-b} + e^{-c}) \end{aligned}$$

5) (4 points) Which pyramid of height h over a square $[-a, a] \times [-a, a] = \{(x, y) \mid -a \leq x \leq a, -a \leq y \leq a$ and surface area 4 has maximal volume?

P.S. Solving the Lagrange equations are a bit tricky here and most people have to make several attempts to crack them. If you should be short on time to hack the equations, feel free to ask Ben or Oliver.

Solution:
The area is $4a\sqrt{h^2+a^2}+4a^2=4$, the volume is $V=4ha^2/3$. We have the mathematical problem to extremize $f(x,y)=yx^2$ over the constraint $g(x,y)=x\sqrt{y^2+x^2}+x^2=1$. (We do not drag along the factor 4/3).
The Lagrange system is

$$\begin{aligned} 2xy &= \lambda(\sqrt{y^2+x^2}+x^2/\sqrt{y^2+x^2}+2x) \\ x^2 &= \lambda yx/\sqrt{y^2+x^2} \\ 1 &= x\sqrt{y^2+x^2}+x^2 \end{aligned}$$

There are different possibilities to solve this a bit tricky system. Since $x \neq 0$, we can divide the second equation by x. replace it by $x\sqrt{x^2+y^2}=\lambda y$ or its square. The third equation becomes therefore $1=\lambda y+x^2$. The square roots of the first equation can then also be replaced by $\lambda y/x$.

$$\begin{aligned} 2xy &= \lambda(\lambda y/x+x^3/(y\lambda)+2x) \\ x^2(y^2+x^2) &= \lambda^2y^2 \\ 1 &= \lambda y+x^2 \ . \end{aligned}$$

Until now, we were just trying to simplify, not yet eliminiating. To eliminate λ, we multiply the first equation by λ and replace λy by $(1-x^2)$. We end up with two equations:

$$\begin{aligned} 2xy^2 &= (1-x^2)((1-x^2)/x+x^3/(1-x^2)+2x) \\ x^2(y^2+x^2) &= (1-x^2)^2 \end{aligned}$$

Te second equation is equivalent to $x^2y^2=1-2x^2$ so that the second equation gives $y^2=(1-2x^2)/x^2$. This can be plugged into the first equation to obtain

$$2x(1-2x^2)=x(1-x^2)^2+x^5+2x^3(1-x^2)$$

which simplifies to $x=4x^3$. The solution is therefore $a=1/2$, $h=\sqrt{2}$. Whao!

Remark. This problem was on the edge what is possible to solve by hand. A two dimensional version of the problem, where one asks for the equilateral triangle of height h and ground side $2a$ which has maximal area if the circumfence $2a+2\sqrt{a^2+h^2}=2$ is fixed. The solutions of the Lagrange equations are then $a=1/3, h=1/\sqrt{3}$.

Remarks

(You don't need to read these remarks to do the problems.)

Remark to problem 4) This problem appears in thermodynamics and is relevant in biology or chemistry. If x, y, z are the probabilities that a system is in state X, Y, Z and a, b, c are the energies for these states. Then $-x\log(x) - y\log(y) - z\log(z)$ is called the **entropy** of the system and $E = ax + by + cz$ is the **energy**. The number $F(x, y, z)$ is called the **free energy**. If energy is fixed, nature tries to maximize entropy. Otherwise it tries to **minimize the free energy** $F = S - E$. If we extremize F under the constraint of having total probability $G(x, y, z) = x + y + z = 1$, we obtain the so called **Gibbs distribution**.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) What does it mean that the Lagrange multiplier λ is zero in a constrained optimization problem?
2) Extend the Lagrange method to arbitrary dimensions. Find the equations to find the extrema of a function $f(x_1, ..., x_n)$ under the constraint $g(x_1, ..., x_n) = c$.
3) Let $I = -\sum_{i=1}^{n} p_i \log(p_i)$ be the entropy of a probability distribution $(p_1, \dots, p_n)$. Show that among all probability distributions, the one where $p_i = 1/n$ is the one which maximizes entropy.
4) Find the box of dimensions x, y, z, w in four dimensional space which has maximal "hyper volume" $V = xyzw$ under the constraint that the "surface volume" $A = 2xy + 2xz + 2xw + 2yz + 2yw + 2zw$ is fixed equal to 12.